A STEADY MIXING FLOW WITH NO-SLIP BOUNDARIES

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A steady mixing (even Bernoulli) smooth volume-preserving vector field in a bounded container in $\mathbb{R}^3$ with smooth no-slip boundary is constructed. An interesting feature is that it is structurally stable in the class of C^3 volume-preserving vector fields on the given domain of $\mathbb{R}^3$ with smooth no-slip boundary, thus if one could think how to drive it then it would be physically realisable. It is pointed out, however, that no flow with no-slip boundaries can mix faster than $1/t^2$ in time t.

1. Introduction

The possibility that the motion of ideal particles in a steady or time-periodic fluid flow could be chaotic was proposed by Arnol'd,[4] studied by people like Hénon[19] and Zel'dovich, and was part of the standard training at Princeton Plasma Physics Lab in 1978. It was found in convection[9] by 1983, but did not come to the attention of the fluid mechanics community at large until the article of Aref,[2] who christened the phenomenon "chaotic advection". The subsequent development of the subject has been reviewed in Ref. 3.

The ultimate in chaotic advection would be a mixing flow, in the ergodic theorist's sense: a flow $\phi : \mathbb{R} \times M \to M, (t, x) \mapsto \phi_t(x)$ on a manifold M preserving finite volume μ is *(strongly) mixing* if for all measurable $A, B \subset M$, then

$$\mu(\phi_t(A) \cap B) \to \mu(A)\mu(B)/\mu(M) \text{ as } t \to \infty$$

(no molecular diffusion is involved). Yet as far as I am aware, no-one has made an example of a fluid flow which is proved to be mixing. Most examples in the literature have, or are suspected to have, tiny unmixed "islands" (at fixed phase for a time-periodic 2D flow) or long thin invariant solid tori (for a steady 3D flow).

Furthermore, to be realistic for engineering purposes such a flow should be constructed in a container in $\mathbb{R}^3$ with no-slip boundary (an alternative for a physicist could be a flow in a gravitationally or surface-tension bounded ball, but let us restrict to the case of a no-slip container).

So, a further 23 years on from Ref. 2, this paper constructs a steady mixing volume-preserving flow in a bounded container in $\mathbb{R}^3$ with no-slip boundary. Interestingly, the tools have been available in the pure mathematics literature since 1975. The paper leaves open the question of how one might drive such a flow, but makes two further significant points.

Firstly, the flow can be proved structurally stable within the class of C^3 volume-preserving vector fields in the interior of the given container with no-slip boundary. Thus all "nearby" flows are topologically equivalent to the given one, and any such flow is mixing. This robustness gives the hope that such an example can be realised physically. The proof will be published elsewhere.

Secondly, it is proved that no C^2 volume-preserving vector field with C^2 no-slip boundaries mixes faster than $1/t^2$ in time t, in a sense to be made precise.

The paper concludes with a discussion of possible variants and additional results.

2. The construction

I begin from a steady vector field which I call **s** (Figure 1), proposed by Arnol'd[5] (who showed it to be irrotational Euler for a Riemannian metric to be recalled in (2)). It is the suspension vector field of the automorphism

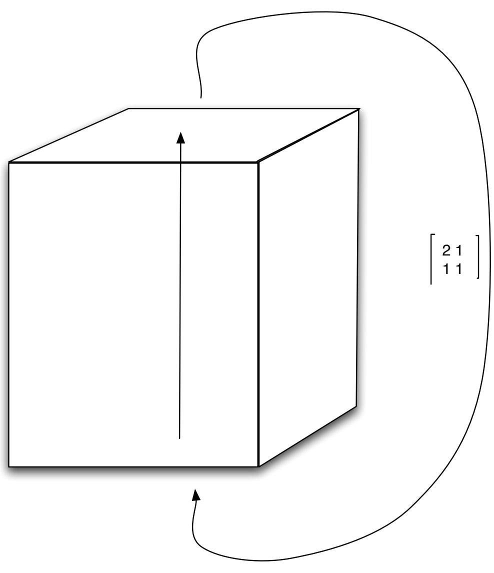

Fig. 1. The suspension flow **s**.

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. This means it is the vector field $(0,0,1)$ in components (x,y,z) on the quotient space $M = (\mathbb{T}^2 \times [0,1])/\alpha$ where

$$\alpha(\mathbf{x}, 1) = (A\mathbf{x}, 0)$$

for $\mathbf{x} = (x,y) \in \mathbb{T}^2$ (meaning that points $(\mathbf{x},1)$ and $(A\mathbf{x},0)$ are to be considered as the same). M is a C^∞ manifold and $\mathbf{s}$ is a C^∞ vector field on it. It preserves volume $dx \wedge dy \wedge dz$ and has exponentially contracting and backwards contracting subbundles $E^\pm$ leading (by direct sum with the vector field) to invariant foliations $\mathcal{F}^\pm$ by the "planes" $y = -\gamma x + c_+$ and $y = x/\gamma + c_-$, respectively, where $\gamma = (1+\sqrt{5})/2$ is golden ratio and $c_\pm$ denote arbitrary constants, as a result of which it is ergodic and C^1-structurally stable. It is not physically realisable, however, because the suspension manifold M can not be embedded in $\mathbb{R}^3$.

The orbit of $(0,0,0)$ is periodic and hyperbolic. Blow it up to a cylinder by the inverse of the mapping from $[0,\varepsilon) \times \mathbb{S}^1 \times [0,1] \to M$ ($\mathbb{S}^1$ is the unit circle with angular coordinate θ) defined by

$$(r,\theta,z) \mapsto (r\cos(\theta+\beta), r\sin(\theta+\beta), z)$$

for some $\varepsilon < 1/4$, where $\beta = \arctan(1/\gamma)$ (the inclusion of β is not essential but simplifies the next formulae). The identification α becomes $\alpha(r,\theta,1) = (r',\theta',0)$ with

$$\begin{aligned} r' &= r\sqrt{f(\theta)}, \\ f(\theta) &= \gamma^4\cos^2\theta + \gamma^{-4}\sin^2\theta, \\ \tan\theta' &= \gamma^{-4}\tan\theta, \end{aligned} \tag{1}$$

where θ' in the third equation is chosen from the same quadrant of the circle as θ; it defines a C^∞ map ψ of the circle (whose derivative ψ' is $1/f$). Denote the blowup manifold by N. It is a C^∞ manifold with boundary ∂N diffeomorphic to $\mathbb{T}^2$. Use coordinates (r,θ,z) near the boundary and (x,y,z) elsewhere, taking into account the identifications α and the horizontal integer translations. If one wants to be a stickler for rigour, one can make a cover of N by 10 charts, based on these two coordinate systems and the gluing map α.

The vector field $\mathbf{s}$ on M induces one on N that I call $\mathbf{t}$. It looks like $\mathbf{s}$ but the periodic orbit along $\mathbf{x} = \mathbf{0}$ is blown up into an invariant torus with coordinates (θ, z) (modulo gluing by ψ), representing horizontal directions of approach θ to points z of the periodic orbit. On this boundary torus the vector field has two attracting periodic orbits $\theta = 0, \pi$ and two repelling ones $\theta = \pm\pi/2$, separating four annuli on which all orbits come from a repelling one at large negative time and go to an attracting one at large positive time (this comes out of the gluing ψ). The vector field $\mathbf{t}$ preserves the volume form $dx \wedge dy \wedge dz$ in (x, y, z) coordinates and $r dr \wedge d\theta \wedge dz$ in (r, θ, z), and inherits invariant foliations from $\mathbf{s}$.

Next, for a C^3 function $g : N \to \mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ to be chosen later, let $\mathbf{u} = g\mathbf{t}$ on N. Note that g is bounded away from 0 and $+\infty$ because N is compact. The vector field $\mathbf{u}$ preserves volume form $\frac{1}{g} dx \wedge dy \wedge dz$ and has the same invariant foliations as $\mathbf{t}$.

Then, choose a C^3 function $\rho : N \to \mathbb{R}$ which is positive in $\mathring{N} = N \setminus \partial N$ (the interior of N) and asymptotic to distance to ∂N near ∂N, measured with some C^3 metric. Specifically, I choose Arnol'd's Riemannian metric

$$ds^2 = \gamma^{-4z} dx_+^2 + \gamma^{4z} dx_-^2 + dz^2, \tag{2}$$

where

$$\begin{aligned} dx_- &= \cos\beta \; dx + \sin\beta \; dy, \\ dx_+ &= -\sin\beta \; dx + \cos\beta \; dy, \end{aligned} \tag{3}$$

and require

$$\rho \sim r\sqrt{\gamma^{-4z}\sin^2\theta + \gamma^{4z}\cos^2\theta} \text{ as } r \to 0.$$

Let $\mathbf{v} = \rho\mathbf{u}$. Then $\mathbf{v}$ is C^3, preserves volume form

$$\omega_g = \frac{r}{\rho g} dr \wedge d\theta \wedge dz = \frac{1}{\rho g} dx \wedge dy \wedge dz \tag{4}$$

in the two coordinate systems, and is zero on ∂N.

A remarkable fact is that N is C^∞-diffeomorphic to the exterior of a figure-eight knot in the 3-sphere $\mathbb{S}^3$ ("exterior" means the closure of the complement of a closed tubular neighbourhood). Although this is stated in many places,[11,15,16,27,28,38–40] I have found it hard to locate a proof in the literature, partly because the interest in many of these references focusses on the additional fact that it can be endowed with a hyperbolic metric. Even then, most topologists are happy with existence rather than an explicit diffeomorphism. In an Appendix I briefly survey the proofs of which I am aware.

As the final step, I transfer the example from $\mathbb{S}^3$ into $\mathbb{R}^3$: choose the figure-eight knot to pass through the "North pole" $(0, 0, 0, 1)$ of $\mathbb{S}^3$ (considered as the unit sphere in $\mathbb{R}^4$) and map the rest of $\mathbb{S}^3$ stereographically to the plane tangent to the "South pole", i.e. $(x, y, z, w) \mapsto \frac{2(x,y,z)}{1-w}$. The result is a C^∞ diffeomorphism h from N to the closure of a bounded domain Ω of $\mathbb{R}^3$ which looks like Figure 2. Think of it as an apple through the core of which a worm has eaten a tubular hole in the form of a figure-eight knot. The domain Ω is the remaining flesh of the apple.

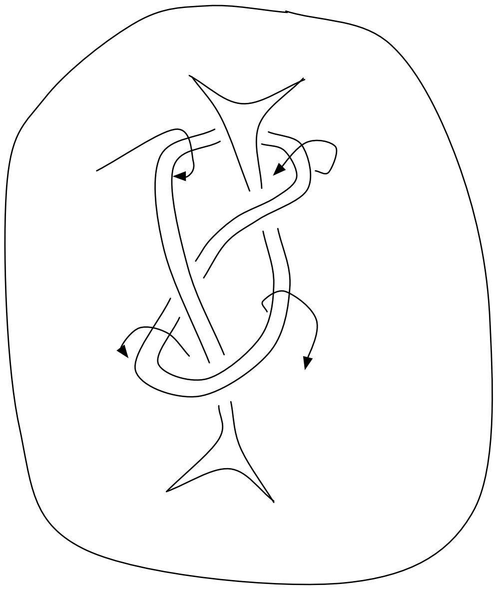

Fig. 2. The domain Ω for the vector field $\mathbf{w}$ and four orbit segments.

The desired vector field is $\mathbf{w} = h^*\mathbf{v}$, the image of $\mathbf{v}$ under h. It is C^3, vanishes on the boundary of Ω and preserves volume $h^*\omega_g$. To make it preserve a pre-ordained C^3 volume form *vol* on Ω (e.g. the Euclidean volume from $\mathbb{R}^3$), it suffices to choose the function $g = \frac{h^*\omega_1}{vol}$, where ω_1 is the special case of (4) with $g = 1$ (since all volume forms at a point are multiples of a given one, this ratio makes sense; also it is a C^3 positive function as required).

To give some idea of what the vector field $\mathbf{w}$ looks like on Ω, Fig. 2 also indicates orbit segments approaching or departing from the four periodic orbits of the *skin friction field* $\frac{\partial \mathbf{w}}{\partial r}$ on the boundary (r being distance from the boundary): they alternately attract and repel along the boundary and repel and attract from the interior. The fact that the periodic orbits go the "short" way around the boundary is a consequence of a nice argument explained to me by Luisa Paoluzzi which I summarise in the Appendix (see also Ref. 38).

Fig. 3 shows a slightly different view in which the bottom lobe of the knot has been rotated round the back to enable visualisation of the image of the cross-section $z = 0$ in N by h. It is based on Fig.11 of Ref. 39. Convince

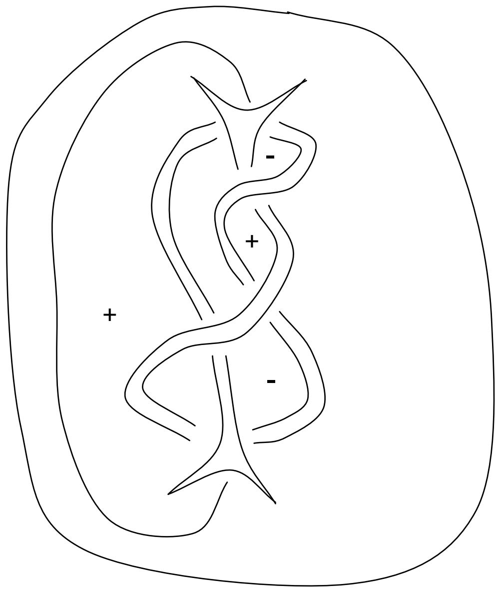

Fig. 3. The domain Ω and a cross-section to the vector field $\mathbf{w}$, with direction of flow indicated by $\pm$.

yourself that the cross-section is indeed diffeomorphic to a torus minus a round open disc, a space I'll denote by $\mathbb{T}_O$, and that it can be swept round in Ω, following a given co-orientation and keeping the boundary on $\partial\Omega$, and that the action on the surface induced by sweeping once round is homotopic to A', the blowup of the toral automorphism A. Ω is said to *fibre over the circle*, with fibre (or "Seifert spanning surface") $\mathbb{T}_O$ and monodromy A'. More pictures of this can be found in Refs 15,27.

A similar construction was used in Ref. 11 to make an example of a flow in $\mathbb{R}^3$ where the possible knot and link types of periodic orbits could be shown to be very rich (indeed Ref. 16 proved it contains all knots and links, and the same for any flow transverse to the fibration). Their vector field, however, is not volume-preserving. It was obtained from $\mathbf{s}$ by a DA ("derived from Anosov") construction, perturbing the gluing map α near the fixed point $(0,0)$ to replace it by a repelling fixed point and two saddles and then excising the repelling orbit.

3. Mixing

The point of the example $\mathbf{w}$ is the following theorem.

Theorem 3.1. *All vector fields topologically equivalent to* $\mathbf{w}$ *on* Ω *within the class of vector fields on* Ω *preserving given volume form vol,* C^3 *on* $\bar{\Omega}$ *and vanishing on the boundary are mixing.*

Proof. The first return map ψ to the cross-section $\{z = 0\}$ minus its boundary is mixing for the area form given by the flux of *vol* under $\mathbf{w}$, by the standard Hopf argument using the existence of the invariant foliations for ψ (e.g. see Ref. 12 for a nice exposition). By Anosov's alternative[1] (rediscovered in Ref. 32), the only obstacle to the flow being mixing would be if the return time function $\tau : \mathring{\mathbb{T}}_O \to \mathbb{R}^+$ for ψ were a constant plus a coboundary. A "coboundary" for a map ψ is a function $\tau : \mathring{\mathbb{T}}_O \to \mathbb{R}$ of the form $\tau(\mathbf{x}) = \sigma(\psi(\mathbf{x})) - \sigma(\mathbf{x})$ for some function σ, so its sum along an orbit of ψ telescopes. This is a somewhat exceptional situation. Indeed, in our case the return time goes to infinity at the boundary, so can not be a constant plus a coboundary.[24] $\square$

Actually, from mixing and a general argument of Ref. 31, it follows that the flow is Bernoulli.

A nice feature of the example which makes it potentially physically realisable is that it is robust.

Theorem 3.2. $\mathbf{w}$ *is structurally stable within the above class of vector fields.*

A vector field is *structurally stable* if all small perturbations are topologically equivalent to it. Since the proof involves many technicalities, it will be published elsewhere.

How fast does the example mix? To answer this requires first a discussion about how to define rate of mixing.

A standard way to define the rate of mixing of a flow ϕ on a manifold M preserving a volume form μ is to choose a class F of functions $f : M \to \mathbb{R}$ and ask how fast the correlation $C_{fg}(t) = \int f(\phi_t(x))g(x)d\mu(x)$ for $f, g \in F$ decays to the product of the means of f and g, in comparison to the product of the sizes of f and g using a notion of size appropriate to the function class (or f, g can come from different function spaces). The answer depends strongly on the chosen class of functions, however. For example, if F is L^2 then there is no uniform decay estimate: g could be chosen to be $f \circ \phi_T$ for some large T and then $C_{fg}(T) = \|f\|_{L_2}\|g\|_{L_2}$. For some mixing systems, exponential decay can be proved for Hölder continuous functions, but the decay rate depends in general on the Hölder exponent α.

An alternative is to use a metric on a space of probability measures on M and ask how fast the push-forward of an initial measure converges to μ. A natural metric is the total variation metric, but for a volume-preserving flow this metric is invariant, so gives no information about mixing. A better one is the transportation metric

$$D(p,q) = \inf\{\int d(x,y)L(dx,dy) : L \in P_{p,q}\},$$

where $P_{p,q}$ is the set of probability measures on $\Omega \times \Omega$ with marginals p, q on the first and second factors. It is the minimum average distance that mass from one measure has to be moved to turn it into the other measure.

A nice result of Ref. 21 is that

$$D(p,q) = \sup_f \frac{p(f) - q(f)}{\|f\|_{Lip}}$$

over non-constant Lipschitz functions f, where $p(f)$ is the expectation of f in measure p and $\|f\|_{Lip}$ is the smallest Lipschitz constant for f. So the two views come close when Hölder is specialised to Lipschitz ($\alpha = 1$). In particular, given an initial measure ν absolutely continuous with respect to μ, it can be written as $g\mu$ for a function $g \in L^1(\mu)$. Then $(\phi_t^*\nu)(f) - \mu(f) = C_{fg}(t)$, so $D(\phi_t^*(\nu), \mu) = \sup_f \frac{C_{fg}(t)}{\|f\|_{Lip}}$ and any upper bound on the correlation function proportional to $\|f\|_{Lip}$ gives a corresponding upper bound on the transportation distance. It is not clear to me, however, whether lower bounds transfer so easily, because to obtain an accurate lower bound for the transportation distance one may have to change the choice of f as time progresses.

In any case, I choose to use transportation metric.

Theorem 3.3. *No C^2 volume-preserving vector field with compact no-slip C^2 boundary mixes faster than $1/t^2$ in time t.*

Proof. Let $\mathbf{v}$ be a C^2 volume-preserving vector field with no-slip boundary, ρ a C^2 positive function asymptotic to distance to the boundary, and $\mathbf{u} = \mathbf{v}/\rho$. Then a simple calculation shows that $\mathbf{u}$ is tangent to the boundary. Let

$C = \sup \frac{\partial u_r}{\partial r}$ in a neighbourhood $r \le r_1$ of the boundary. Then $|u_r| \le Cr$ for $r \le r_1$. Thus $|v_r| \le Cr^2$ for $r \le r_1$. It follows that fluid from outside $r \le r_1$ can get to at most distance $1/(1/r_1 - Ct)$ of the boundary in time t. Take an initial "dye" density 1 in $r \le r_1$ and 0 outside. Then the subset $r < 1/(1/r_1 - Ct)$ remains of density 1. It is of thickness of order $1/t$, so has volume of order $1/t$ and the average distance that dye must be moved to achieve the average density is at least half the thickness. Thus the transportation distance to the uniformly mixed state is at least of order $1/t^2$. □

The fact that some flows with no-slip boundaries mix like a power law was noted numerically in Ref. 18, albeit with molecular diffusion added and a different notion of mixing rate.

An open question is to determine an upper bound on the transportation distance as a function of time. This would require some study of the return-time function to a cross-section, among other things.

If one switches attention to correlation functions, there is some literature on systems with power law decay, e.g. Ref. 13 for upper and Ref. 36 for lower bounds. It seems likely to me that the correlation of many pairs of function decays like $1/t$ for our flow. This would give rise to anomalous diffusion. Corresponding to the coordinate z of $\mathbf{s}$ is a quantity one can continue to denote by z which measures how many times (plus fractional part) trajectories have crossed the cross-section of Fig. 3. Then one can examine the deviation from the mean rate of increase of z with time. If the autocorrelation function for $\dot{z}$ is integrable then the deviation would spread like normal diffusion, but if its integral is infinite then the deviation should spread anomalously. One way to obtain a handle on this would be to use the fact that the flow has a Markov partition and compute the large deviation rate function for the increment in z (cf. Ref. 26).

4. Discussion

At the physical level, there remains the question of how to drive the flow. It suffices to compute $\mathbf{w}.\nabla\mathbf{w} - \nu\Delta\mathbf{w}$, where ν is the kinematic viscosity, subtract off its gradient part, and apply a body force equal to the remainder. It might not be easy to implement, however.

One can contrast results of Ref. 14 making an Euler flow on $\mathbb{S}^3$ containing all knots and links. Being an Euler flow it requires no forcing at all, but the catches are that it also requires zero viscosity, the Riemannian structure could not be specified in advance, and it is not claimed to be mixing:

indeed the knots and links are supported on a proper subset.

I believe it is possible to make a similar construction of a flow with stress-free boundaries, by using symplectic polar blowup instead. This ought to be C^2 structurally stable. To obtain mixing, however, one would need to ensure that the speed function is nontrivial.

One can ask whether the flow is a fast dynamo. The dynamics of a magnetic field in a steady conducting fluid flow may have a positive growth rate. The flow is said to be a *fast dynamo* if the growth rate has a positive lower bound as the magnetic diffusivity goes to zero (in principle this depends on the Riemannian metric assumed for the magnetic diffusion) (see survey in Ch.V of Ref. 6). Arnol'd[7,8] proved that **s** is a fast dynamo with respect to metric (2). It would be interesting to investigate whether **w** is a fast dynamo. To make this problem well posed one has to specify what the magnetic field does outside Ω.

One can ask whether there are alternative constructions of robust mixing fluid flows. I believe one would be the "pigtail stirrer". Start from **s** on M but quotient by $\sigma(x,y,z) = (-x,-y,z)$ and blowup the orbits of both $(0,0,0)$ and $(\frac{1}{2},\frac{1}{2},0)$ to tori. This gives a vector field in a solid torus minus a tubular neighbourhood of a knot which goes three times round the solid torus making the closure of a pigtail braid (as sketched in Ref. 25 for example). The monodromy goes back to Lattès.[22] The analysis is slightly different from the example **w**, because the blown-up orbits are 1-prong singularities rather than regular orbits, but I think the same structural stability result should be possible. Furthermore, this example opens the possibility to make the outer boundary axisymmetric and to rotate it about its axis, so that the no-slip condition gives a non-zero field on the outer boundary. Equivalently (though different for the fluid dynamics), one could rotate the 3-braid and examine the flow in the rotating frame.

Another starting point is geodesic flow on the unit tangent bundle of a surface of negative curvature, which is mixing Anosov. Birkhoff showed that blowup of 6 periodic orbits of the genus 2 case produces a suspension of a hyperbolic toral automorphism with 12 points blown up,[10] and I expect this can be mapped into $\mathbb{R}^3$.

What if one abandons the structural stability requirement but just asks for robust mixing, i.e. all nearby volume-preserving flows are also mixing? I believe this can be achieved by what I call a "baker's flow" by analogy with the well known baker's map. It is a volume-preserving flow in a container whose boundary is a surface of genus 2. The 2D stable manifold of a reattachment point on the boundary separates the volume into orbits which go round one loop from ones which go round the other loop. These two sets glue together again along the 2D unstable manifold of a separation point on the boundary. If the two manifolds are designed to intersect transversely, the eigenvalues of the skin-friction field satisfy certain inequalities at the separation and reattachment points, and the flow round the loops rotates trajectories suitably, then the return map to a transverse section in the middle is a nonlinear version of the baker's map. The system is a volume-preserving analogue of the Lorenz system. The flows are not structurally stable, but are probably robustly mixing (just as for the Lorenz system in the good parameter regime[24]).

Lastly, one can ask about time-periodic 2D flows. I think it might be possible to make a codimension-3 submanifold of C^2 area-preserving maps of the torus, isotopic to the identity (so realisable by time-periodic flows), looking perhaps a bit like Zeldovich's alternating sine-flow, which are mixing and topologically conjugate. The idea is to start from a pseudo-Anosov example (maybe a variant of Ref. 26), then smooth it and show topological conjugacy for all small smooth perturbations preserving the singular orbits.

Appendix

Here I survey what I have found about the diffeomorphism between the blow-up of the suspension manifold and the exterior of a figure-eight knot.

The starting point is to notice that they have isomorphic fundamental groups, with isomorphism respecting the subgroup for the boundary. Then a result of Ref. 37 applies to give a homeomorphism (alternative proofs are in Ref. 30 using Ref. 29, and Cor 6.5 of Ref. 41). Stallings' paper worries me, however, because he ends by saying that it is not clear whether fibred manifolds with isotopic monodromy are homeomorphic. All of these proofs involve various cutting and gluing operations that make it difficult to see an explicit homeomorphism and they do not address the question of smoothness (but Ref. 23 redoes it in the differentiable category).

More explicit are three approaches which involve viewing the manifold as a quotient of hyperbolic 3-space $\mathbb{H}^3$ by a discrete group of isometries[20,33,38] (see also Ref. 40).

Another strategy[17] (also described in 10.J of Ref. 35) is to notice that the figure-eight knot has a $\mathbb{Z}_2$ symmetry by a half-rotation about some unknot (this was used also by Ref. 11). Quotienting by the symmetry reduces it to the closure of the pigtail braid relative to the unknot symmetry axis. Since any braid-closure is fibred, so is the figure-eight knot, and the monodromy can be seen to act like A', the blowup of $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ to $\mathring{\mathbb{T}}_O$.

With an explicit diffeomorphism it would be easy to verify the claim of Section 2 about the homotopy class of the periodic orbits on the boundary, but the following argument of Luisa Paoluzzi answers the question anyway. Choose as base point for the fundamental group $\pi_1(N)$ the point at $z = 0$ on ∂N with $\theta = 0$. Choose the following generators for $\pi_1(N)$: a translates by $(1, 0, 0)$ passing over the second tube, b translates by $(0, 1, 0)$ passing to the left of the second tube, c translates by $(0, 0, 1)$ up the periodic orbit at $r = \theta = 0$ (and glues by A). Then a generating set of relations is $c^{-1}ac = a^2b, c^{-1}bc = ab$. Also, going once round the tube anticlockwise in the plane $z = 0$ is achieved by $\kappa = b^{-1}a^{-1}ba$. The preimage under the diffeomorphism $h : \mathring{N} \to \Omega$ of the homotopy class of a closed curve going the short way around the knot in Ω, cutting the Seifert surface positively, is $c\kappa^n$ for some integer n. We want to show $n = 0$. The quotient of $\pi_1(N)$ by $c\kappa^n$ is trivial, since it is equivalent to reinserting the knot and its tubular neighbourhood into $\mathbb{S}^3$. The quotient of $\pi_1(N)$ by c is indeed trivial (the relations then imply $a = a^2b, b = ab$, so $a = b = e$, the identity). In contrast, one can argue that for any $n \neq 0$, the quotient of $\pi_1(N)$ by $c\kappa^n$ is non-trivial.

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References

1. Anosov DV, Geodesic flows on closed Riemannian manifolds of negative curvature, Trudy Mat Inst Steklov 90 (1967) 210
2. Aref H, Stirring by chaotic advection, J Fluid Mech 143 (1984) 1–21.
3. Aref H, The development of chaotic advection, Phys Fluids 14 (2002) 1315–25.

4. Arnol'd VI, Sur la topologie des écoulements stationnaires des fluides parfaites, Comptes Rendus Acad Sci Paris 261 (1965) 17–20.
5. Arnol'd VI, Notes on the three-dimensional flow pattern of a perfect fluid in the presence of a small perturbation of the initial velocity field, J Appl Math Mech 36 (1972) 236–242.
6. Arnol'd VI, Khesin BA, Topological methods in hydrodynamics (Springer, 1998).
7. Arnol'd VI, Korkina EI, The growth of a magnetic field in a three-dimensional steady incompressible flow, Moscow Univ Math Bull 38 (1983) 50–4.
8. Anrol'd VI, Zel'dovich YaB, Ruzmaikin AA, Sokolov DD, A magnetic field in a stationary flow with stretching in Riemannian space, Sov Phys JETP 54 (1981) 1083–6.
9. Arter W, Ergodic streamlines in three-dimensional convection, Phys Lett A 97 (1983) 171–4.
10. Birkhoff GD, Dynamical systems with two degrees of freedom, Trans Am Math Soc 18 (1917) 199–300.
11. Birman JS, Williams RF, Knotted periodic orbits in dynamical systems II: knot holders for fibered knots, in Low-dimensional topology, ed Lomonaco SJ, Contemp Math 20 (Am Math Soc, 1983) 1–60.
12. Burns K, Pugh C, Shub M, Wilkinson A, Recent results about stable ergodicity, in: Smooth ergodic theory and its applications, eds Katok A, Llave R de la, Pesin Ya, Weiss, Proc Symp Pure Math 69 (2001) 327–66.
13. Chernov N, Zhang H-K, Billiards with polynomial mixing rates, Nonlin 18 (2005) 1527–53.
14. Etnyre J, Ghrist R, Contact topology and hydrodynamics III: knotted orbits, Trans Am Math Soc 352 (2000) 5781–94.
15. Francis GF, Drawing Seifert surfaces that fiber the figure-8 knot complement in $\mathbb{S}^3$ over $\mathbb{S}^1$, Am Math Month 90 (1983) 589–599; and Chapter 8, A topological picture book (Springer, 1987).
16. Ghrist RW, Branched two-manifolds supporting all links, Topology 36 (1997) 423–448.
17. Goldsmith DL, Symmetric fibered links, in: Knots, groups and 3-manifolds, ed Neuwirth LP, Ann Math Studies 84 (Princeton, 1975) 3–23.
18. Gouillart E, Kuncio N, Dauchot O, Dubrulle B, Roux S, Thiffeault J-L, Walls inhibit chaotic mixing, Phys Rev Lett, in press, 2007.
19. Hénon MR, Sur la topologie des lignes de courant dans un cas particulier, Comptes Rendus Acad Sci Paris 262 (1966) 312–
20. Jorgensen T, Compact 3-manifolds of constant negative curvature fibering over the circle, Ann Math 106 (1977) 61–72.
21. Kantorovich LV, Rubenstein G Sh, On a space of totally additive functions, Vestnik Leningrad Univ 13:7 (1958) 52–59.
22. Lattès S, Sur l'iteration des substitutions rationelles et les fonctions de Poincaré, Comptes Rendus Acad Sci Paris 16 (1918) 26–8.
23. Laudenbach F, Le theoreme de fibration de J.Stallings, Seminaire Rosenberg, Orsay (1969) M13.369
24. Luzzatto S, Melbourne I, Paccaut F, The Lorenz attractor is mixing, Com-

mun Math Phys 260 (2005) 393–401.
25. MacKay RS, Postscript: Knot types for 3-D vector fields, in: Topological Fluid Mechanics, eds Moffatt HK, Tsinober A, IUTAM Conf Proc, Aug 89 (CUP, 1990), 787.
26. MacKay RS, Cerbelli and Giona's map is pseudo-Anosov and nine consequences, J Nonlin Sci 16 (2006) 415–434.
27. Miller SM, Geodesic knots in the figure-eight knot complement, Exp Math 10 (2001) 419–436.
28. Milnor J, Hyperbolic geometry: the first 150 years, Bull Am Math Soc 6 (1982) 9–24.
29. Neuwirth L, The algebraic determination of the topological type of the complement of a knot, Proc Am Math Soc 12 (1961) 904–6.
30. Neuwirth L, On Stallings fibrations, Proc Am Math Soc 14 (1963) 380–1.
31. Ornstein DS, Weiss B, Statistical properties of chaotic systems, Bull Am Math Soc 24 (1991) 11-116.
32. Plante JF, Anosov flows, Am J Math 94 (1972) 729–754.
33. Riley R, A quadratic parabolic group, Math Proc Camb Phil Soc 77 (1975) 281–8.
34. Robbin JW, On the existence theorem for differential equations, Proc Am Math Soc 19 (1968) 1005–6.
35. Rolfson D, Knots and links (Publish or Perish, 1976).
36. Sarig O, Subexponential decay of correlations, Invent Math 150 (2002) 629–653.
37. Stallings J, On fibering certain 3-manifolds, in: Topology of 3-manifolds and related topics, ed Fort MK (Prentice Hall, 1962), 95–100.
38. Thurston WP, The geometry and topology of 3-manifolds, Chs 3 and 4 (preprint, 1978)
39. Thurston WP, Three dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull Am Math Soc 6 (1982) 357–381.
40. Thurston WP, Three-dimensional geometry and topology, vol 1, ed Levy S (Princeton U Press, 1997).
41. Waldhausen F, On irreducible 3-manifolds which are sufficiently large, Ann Math 87 (1968) 56–88.