Ch 2 Deterministic Case

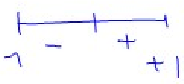
① Examples of CML with non-unique ST phase.

CML (coupled map lattice) is a map

$f : M \circlearrowleft \quad M = \prod_{s \in S} M_s \quad M_s$ finite-dimensional manifold e.g. $[-1,+1]$, S countable metric space

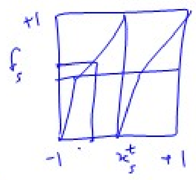
Given a PCA on $\Sigma = \prod_{s \in S} \Sigma_s$ with transition probs $p_s^{\underline{\sigma}}$ $s \in S$, $\underline{\sigma} \in \Sigma$, partition $[-1,+1]$ into $|\Sigma_s|$ equal intervals labelled by possible values of $\sigma_s \in \Sigma_s$, and define piecewise affine map $f : M \circlearrowleft$ with

$$\frac{\partial f_s(\underline{x})}{\partial x_s} = \frac{1}{p_s^{\underline{\sigma}(\underline{x})}(\sigma_s')}$$

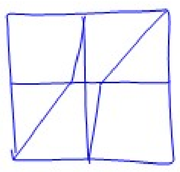


$$\frac{\partial f_s}{\partial x_r} = 0 \quad r \neq s \qquad \text{for } x_s \in \sigma_s, \; x_r \in \sigma_r$$

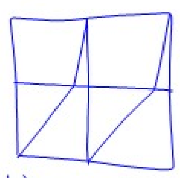
e.g. Toom's voter



if $x_{s+N}^t, x_{s+E}^t \in +$



if $+,-$ or $-,+$



if $-,-$

slopes $\frac{1}{\lambda}, \frac{1}{1-\lambda}$ $\quad \lambda \in (0, \frac{1}{2})$

Then for any initial prob on M with absolutely continuous marginals on finite subsets Λ of S and density h_Λ be Hölder continuous

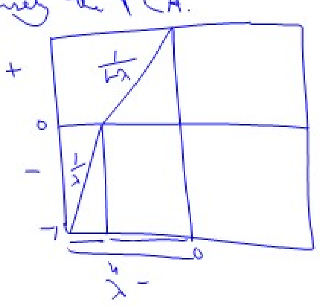
$$|\log h_\Lambda(x) - \log h_\Lambda(y)| \leq C \sum_{s \in \Lambda} |x_s - y_s|^\delta$$

in distant past, then $\underline{\sigma}(\underline{x})$ is distributed according to a phase of the PCA. So e.g. get CML with ferromagnetic phases.

Idea is that initial prob gets stretched and cut so becomes more uniform in each cylinder set (set with prescribed symbols) & action on uniform probs on cylinder sets is precisely the PCA.

Responds to challenge by Sinai & Bunimovich 1988

(Gielis & M, 2000)



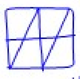
Similarly, can make CML with period-2 phases (simulate Toom with $\lambda \in (\frac{1}{2}, 1)$ anti-ferromagnetic, persistent infection phases ...

Can make invertible examples by replacing $[-1,+1]$ by solid torus $[-1,+1] \times D^2 /$ identify ends and use solenoid maps (distorted versions).

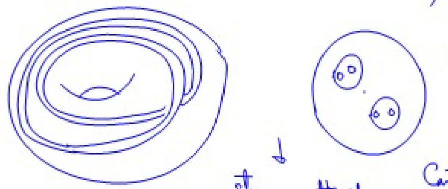
Recall standard solenoid map (Smale–Williams)

$$(x, w) \in S^1 \times D^2$$

$S^1 = \mathbb{R}/2\pi\mathbb{Z}$, $D^2 = \{w \in \mathbb{C} \mid |w| \leq 1\}$



$$\begin{cases} x' = 2x \\ w' = \lambda w + \mu e^{ix} \end{cases}$$ with $\lambda < \mu$, $\mu + \lambda \leq 1$



$x = \text{cst}$

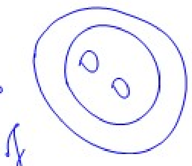
strange attractor

Cantor set of lines locally

Our distorted solenoid maps are $[-1,+1] \times D^2 \circlearrowleft$ depending on symbolic state of nbrs $[-1,0) \cup [0,+1]$ e.g. $+ +$ nbrs:

$$\begin{cases} x' = g_{++}(x) \\ w' = \lambda w + \mu e^{i\pi x} \end{cases}$$

This makes map one-to-one. To make it onto need to extend solenoid map outside $S^1 \times D^2$ to make backward orbits exist for all initial pts



But CML is not continuous because of jumps at changes of symbolic state of nbrs.

Bardet & Keller overcame this defect, even made C^∞. Main idea was to replace symbolic coupling by discrete diffusion

$$x_s^{t+1} = (1-\varepsilon) e(x_s^t) + \frac{\varepsilon}{2}\left(e(x_{s+N}^t) + e(x_{s+E}^t)\right)$$

but not clear how to make invertible

② <u>Statistical phases</u> for uniformly hyperbolic attractors of finite-dimensional deterministic dynamical systems i.e. T-phases (no space) "chaos". Only "complexity science" if think of states at successive times as the interdependent components. Necessary step towards S-T case.