Reminder: no lecture Fri 11th Dec

extra lecture Fri 18th Dec 09:15 if desired

<u>Thm</u> X "=" $\prod_{s\in S} X_s$, Y "=" $\prod_{s\in S} Y_s$, X_s, Y_s Banach spaces, with sup-norm. $L: X\to Y$ linear φ-exp. local, bdd inverse, $y\in Y$ (C,λ)-exp. localised around $o\in S$ (i.e. $|y_s|\leq C\lambda^{d(s,o)}$)

$\Rightarrow$ $x = L^{-1}y$ is (WC,μ)-exp. localised around o

for some W,μ functions of $\|L^{-1}\|,\lambda,\varphi$

<u>Proof</u>: Let $z\in[1,\zeta)$, $z\lambda<1$,

$\tilde{y}_r = y_r z^{d(r,o)}$, $\tilde{x}_r = x_r z^{d(r,o)}$

Then $\|\tilde{y}\|<\infty$. Want to bound $\|\tilde{x}\|$

$$Lx=y \Rightarrow \sum_s L_{rs} z^{d(r,o)-d(s,o)}\tilde{x}_s = \tilde{y}_r$$

So $(L+L^\circ)\tilde{x}=\tilde{y}$ with $L^\circ_{rs} = L_{rs}\left(z^{d(r,o)-d(s,o)}-1\right)$

Let $\tilde{L}_{rs} = L_{rs} z^{d(r,s)}$ $\quad \|\tilde{L}\|\leq\varphi(z)<\infty$

$d(r,o)-d(s,o)\leq d(r,s)$ so

$|L^\circ_{rs}| \leq |\tilde{L}_{rs}-L_{rs}|$ so $\|L^\circ\|\leq\|\tilde{L}-L\|$

For $z=1$ $\tilde{L}=L$. $\tilde{L}(z)$ depends continuously on z, because given $1\leq z_1<z_2<\zeta$

$$\left(\tilde{L}(z_2)-\tilde{L}(z_1)\right)_{rs} = L_{rs}\left(z_2^{d(r,s)}-z_1^{d(r,s)}\right)$$

So $$\left|\left(\tilde{L}(z_2)-\tilde{L}(z_1)\right)_{rs}\right| \leq |L_{rs}|\, d(r,s)\, z_2^{d(r,s)}\log\frac{z_2}{z_1}$$

$$\leq \frac{1}{e}\frac{\log z_2/z_1}{\log z_3/z_2}\underbrace{z_3^{d(r,s)}|L_{rs}|} \text{ for any } z_3>z_2$$

So $$\|\tilde{L}(z_2)-\tilde{L}(z_1)\| \leq \frac{1}{e}\frac{\log z_2/z_1}{\log z_3/z_2}\|\tilde{L}(z_3)\|$$

$$\leq \frac{1}{e}\frac{\log z_2/z_1}{\log z_3/z_2}\varphi(z_3).$$

In particular $\|\tilde{L}(z)-L\|\leq\beta(z)=\min_{z_3}\frac{1}{e}\frac{\log z}{\log z_3/z}\varphi(z_3)$

$\to 0$ as $z\searrow 1$, so $<\|L^{-1}\|^{-1}$ for z near enough to 1. Hence

$\|(L+L^\circ)^{-1}\|\leq\left(\|L^{-1}\|^{-1}-\|\tilde{L}-L\|\right)^{-1}$ for z near 1.

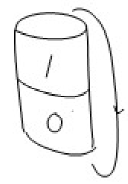
So $$\|\tilde{x}\| \leq \frac{\|\tilde{y}\|}{\|L^{-1}\|^{-1}-\|\tilde{L}-L\|}. \quad \square$$

Apply this to u. hyp splitting for CML taking $\mathbb{S} = S \times \mathbb{Z}$ by considering $\mathcal{D}F$ exp. local wrt $d\big((s_1,t_1),(s_2,t_2)\big) = d(s_1,s_2) + |t_2 - t_1|$ and repeat construction of splitting & see that resulting projections $P^{\pm} : TM_x \circlearrowleft$ are exp. local wrt $d_{\mathbb{S}}$.

As for T only, get robustness of u. hyp orbits in $S \times T$ wrt exp. local perturbations in space. Also S-T shadowing.

Similarly get Markov partitions for u hyp sets of CML (cf. Pesin & Sinai). A coding $\underline{\underline{x}} = \underline{\underline{x}}(\underline{\underline{\sigma}})$ of the orbits $\underline{\underline{x}} = (x_s^t)_{s \in S}^{t \in T = \mathbb{Z}}$ of the u hyp set by S-T symbol tables $\underline{\underline{\sigma}}$ from some allowed set Σ s.t. every allowed table occurs & $\exists!\, \underline{\underline{x}}$ for each $\underline{\underline{\sigma}}$.

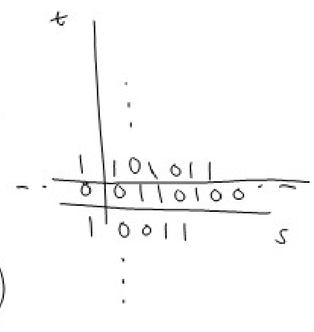
Simplest case $\Sigma = \left(\prod_{s \in S} \Sigma_s\right)^{\mathbb{Z}}$ with each Σ_s finite e.g. uncoupled lattice of solenoid maps $\Sigma_s = \{0,1\}$ and $x_s^t = X_s(\ldots \sigma_s^{t-1} . \sigma_s^t \sigma_s^{t+1} \ldots)$ the unique pt whose orbit visits R_0, R_1 of $\mathbb{D}^2 \times \mathbb{T}^1$ in sequence $(\sigma_s^t)_{t \in \mathbb{Z}}$



Now make a C^1-small exp. local coupling. By the robustness of u hyp sets, can deform coding $\underline{\underline{x}} = \underline{\underline{x}}(\underline{\underline{\sigma}})$ but now x_s^t depends in general on all $\sigma_{s'}^{t'}$ (not just $\sigma_s^{t'}$) though exp. weakly on distant (s',t').

Same holds if uncoupled dynamics has a general finite graph of allowed transitions e.g. cat map

(symbols = {5 edges})



Also can use S-T shadowing thm to construct M ptn for locally max u hyp sets without assuming nearby uncoupled case.

<u>Natural measures on u hyp attractors for CML</u>

Those which arise by starting in distant past with a prob measure ν on $M \left(= \prod M_s\right)$ whose marginals on all finite subsets of S are abs cts and Hölder density.

They are the Gibbs phases for "energy" with contribution

$$\varphi_s^t(\underline{\underline{\sigma}}) = \operatorname{tr}\left[\log Df_-\left(\underset{\sim}{x}^t(\underline{\underline{\sigma}})\right)\right]_{ss}$$

for space-time site (s,t).

By u.hyp theory thus φ_s^t depends exp weakly

on $\sigma_{s'}^{t'}$ wrt $d((s',t'),(s,t))$

The connection with single dyn sys (SRB) is

$$\log|\det A| = \operatorname{tr}\log A = \sum_s \operatorname{tr}[\log A]_{ss}$$

+ cst from choice of branch of log

See Bricmont & Kupiainen 1996

Thus natural measures for u.hyp attractors of CML

$\Leftrightarrow$ eqm stat mech for a special class of spin systems on $S \times T$.

When $\dim(S \times T) \geq 2$ can expect to make examples with non-unique Gibbs phase if sufficiently coupled.

<u>Challenge</u>: Make a CML of solenoid maps with u.hyp attractor exhibiting non-unique S-T phase

cf. Bardet & Keller got close.

Finished! So <u>no</u> lecture on 18 Dec after all!

Happy Holidays! Come to S-T Phases workshop 6-8 Jan!