We're analysing λ near 0 (low recovery rate, high infectivity) for Stavskaya PCA

We'll prove that $\delta_{\underline{0}} P^t \not\to \delta_{\underline{1}}$ as $t \to \infty$

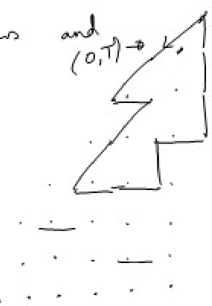
$\underline{0}$: all infected; $\underline{1}$: all healthy

In fact $P^{\underline{0}}\{ x_s^t = 0 \} \geqslant c(\lambda) > 0$ $\forall s, t$, $\lambda < \frac{1}{54}$

Recall we can generate prob. for space-time configs starting from $\underline{0}$ by putting "stoppers" independently with prob λ for each (s,t) & propagating infection deterministically up to a stopper

So $x_0^T = 1$ iff every path along ups & up-rights from $t = 0$ encounters a stopper

<u>iff</u> $\exists$ "fence" around $(0,T)$ starting at $\left(\frac{1}{2}, T+\frac{3}{4}\right)$ formed from stoppers and down-lefts and ups, e.g.

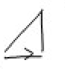


$(0,T) \to$

If a fence has k stoppers then it has precisely k ups & k down-lefts

Let $N_k = \#$ fences with k stoppers

$$\leqslant \binom{3k}{k\ k\ k} \leqslant 27^k$$

(a large overestimate because could exclude those which self-intersect & those which don't surround $(0,T)$ e.g. )

So $P^{\underline{0}}\{ x_0^T = 1 \} \leqslant \sum_{k \geqslant 1} N_k \lambda^k$

$$\leqslant \frac{27\lambda}{1-27\lambda} \quad \text{for } \lambda < \tfrac{1}{27}$$

$$< 1 \quad \text{for } \lambda < \tfrac{1}{54} \quad \text{indpt of } T$$

So $P^{\underline{0}}\{ x_0^T = 1 \} \not\to 1$ as $T \to \infty$, and

$$P^{\underline{0}}\{ x_0^T = 0 \} \geqslant c(\lambda) = \frac{1-54\lambda}{1-27\lambda} > 0 \ \forall T$$

So $\delta_{\underline{0}} P^t \not\to \delta_{\underline{1}}$ as $t \to \infty$

Where does $\delta_{\underline{0}} P^t$ go? or νP^t for arb. initial ν?

To analyse this we'll use "monotonicity" of Stavskaya PCA. [also called "attractivity")

Say $\underline{x} \leqslant \underline{x}' \in \{0,1\}^{\mathbb{Z}}$ if $\forall s \in \mathbb{Z}$ $x_s \leqslant x'_s$ $(0 < 1)$

Say $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ is <u>non-decreasing</u> if $\underline{x} \leqslant \underline{x}' \Rightarrow f(\underline{x}) \leqslant f(\underline{x}')$

Say probs $\rho < \rho'$ if $\forall$ non-decr f

$$\rho(f) \leqslant \rho'(f)$$

Say transition operator P is <u>monotone</u> if

$$\rho \leqslant \rho' \Rightarrow \rho P \leqslant \rho' P$$

equivalently f non-decr $\Rightarrow Pf$ non-decr.

Prove Stavskaya is monotone by joining ("coupling") processes from 2 initial conditions $\underline{x} \leq \underline{x}'$ in such a way that $\forall t > 0$ $\underline{x}^t \leq \underline{x}'^t$, so it follows that $(P^t f)(\underline{x}) \leq P^t f(\underline{x}')$ $\forall$ non-decr. f

Joining on $\{00, 01, 11\}^{\mathbb{Z}}$ without 10

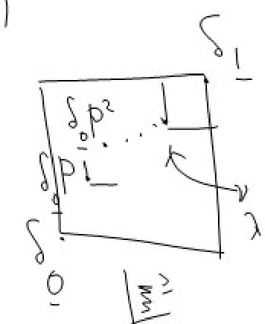
with transition rates

$$00 \to \begin{cases} 11 & P(0 \to 1 \mid \underline{x}) \\ 01 & P(0 \to 1 \mid \underline{x}') - P(0 \to 1 \mid \underline{x}) \\ 00 & P(0 \to 0 \mid \underline{x}') \end{cases}$$

$$01 \to \begin{cases} 11 & P(0 \to 1 \mid \underline{x}) \\ 01 & P(0 \to 0 \mid \underline{x}) - P(1 \to 0 \mid \underline{x}') \\ 00 & P(1 \to 0 \mid \underline{x}') \end{cases}$$

$$11 \to \begin{cases} 11 & P(1 \to 0 \mid \underline{x}) \\ 01 & P(1 \to 0 \mid \underline{x}) - P(1 \to 0 \mid \underline{x}') \\ 00 & P(1 \to 0 \mid \underline{x}') \end{cases}$$

Then $\delta_{\underline{0}} P \geq \delta_{\underline{0}}$



So $\delta_{\underline{0}} P^t$ is a non-decr seq

It is bdd above by $\delta_{\underline{1}}$

So converges to its supremum in $\leq$

(consider $\delta_{\underline{0}} P^t$ acting on arbitrary non-decr f)

Call the limit ν_λ : $\delta_{\underline{0}} P^t \to \nu_\lambda$

In addition,

- ν_λ non-decr wrt λ
  (prove by joining processes for $\lambda < \lambda'$ to make $\underline{x}^t(\lambda) \leq \underline{x}^t(\lambda')$)
- $\exists \lambda_c \in \left(\frac{1}{54}, \frac{1}{2}\right)$ s.t. $\nu_\lambda = \delta_{\underline{1}}$ $\forall \lambda \geq \lambda_c$
- If $\lambda < \lambda_c$ then $\nu_\lambda(\underline{1}) = 0$. $< \delta_{\underline{1}}$ $\forall \lambda < \lambda_c$