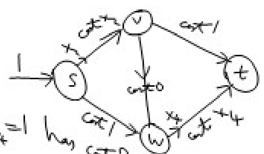
3. Avoiding Braess' paradox

Recall Braess' example (with the numbers of Roughgarden)



selfish solution $x_2 = x_4 = 1$ has cost 2 for everyone (all use shortcut)
whereas optimal soln $x_2 = x_4 = \frac{1}{2}$ (& no traffic in short cut) has cost $\frac{3}{2}$ for everyone

The paradox is that adding the short cut makes outcome of selfish routing worse for everyone.

Restrict attention to single commodity

One solution is to look for a subgraph H of G which minimises $d(H, r, c)$ = cost/unit of Nash flow on H (or at least an H which is close to minimising) and forbid the remaining edges (restrict to H that connect s to t)

First study severity of the problem.

<u>Thm</u> If (G, r, c) is single commodity & H is a subgraph of G then $d(G, r, c) \leqslant \alpha(\mathcal{C})\, d(H, r, c)$

<u>Proof</u>: Let $f, \tilde{f}$ be Nash for G & H

$C(f) = r\, d(G, r, c)$, $C(\tilde{f}) = r\, d(H, r, c)$

$\tilde{f}$ is feasible for G so $C(f) \leqslant \alpha(\mathcal{C})\, C(\tilde{f})$. □

But can sometimes can do better if $\alpha(\mathcal{C})$ large

<u>Defn</u>: $S \subset E$ <u>sparse</u> if no 2 edges in S share an endpoint & no edge in S has s or t as an endpt

<u>Thm</u>: (G, r, c) single commodity, H subgraph, S = {edges in G but not H}. If every sparse subset of S has $\leqslant k$ edges then $d(G, r, c) \leqslant (k+1)\, d(H, r, c)$

In particular, every sparse set has $\leqslant \left\lfloor \frac{n-2}{2} \right\rfloor$ edges (n = # vertices), hence $k+1 \leqslant \left\lfloor \frac{n}{2} \right\rfloor$

<u>Proof</u>: a lot of discrete maths see Roughgarden pp. 125–8 □

Also he shows this bound is optimal.

<u>Note</u>: Braess' paradox has been observed in Stuttgart & Winnipeg (the authorities closed some road & the traffic flow improved!)

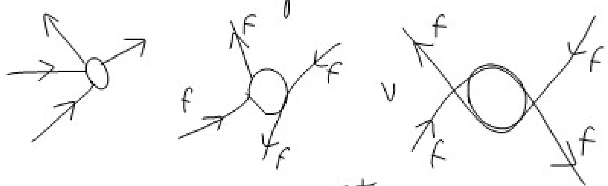
Finish with some possible research directions:

1. Taxes/incentives: we saw that adding taxes $f_e^* \, c_e'(f_e^*)$ to c_e where f^* is optimal for r makes f^* into a Nash flow.
   Can subtract any set of constants k_i for each commodity i if desired (eg to make total revenue = 0). Or can implement this via incentives λ_e on edges s.t.
   $$\sum_{e \in P} \lambda_e = k_i \quad \forall P \in \mathcal{P}_i$$

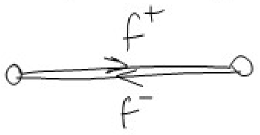
But there is a "fairness" question about k_i ?

But the optimal taxes depend on $(r_i, s_i, t_i)_i$ via f^*, so how to devise taxes that take this into account?

2. Include junction costs and interaction effects.

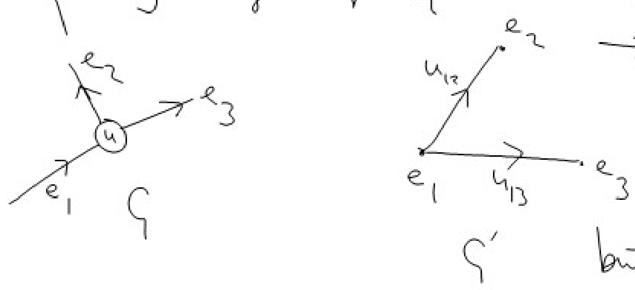


So junction cost depends on all flows on paths using vertex

2-way road 

$c_+(f^+, f^-)$

One trick (maybe): think of edges of G as nodes of a graph G' & vertices of G as providing edges of G'



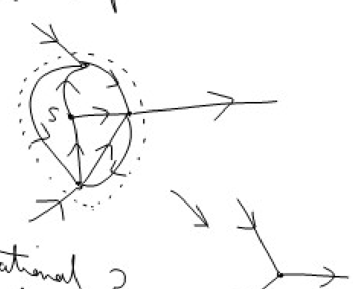
edge + jn costs on G → costs for flows on u_{12}, u_{13} etc., edges of G'

but they are fns of all flows using same vertex u

So can reduce to case of interacting flows only.

Analyse this extension (look at literature too in case something has been done)

3. Aggregation procedures: Could be computationally efficient (&/or insightful) to compute selfish flow hierarchically e.g. aggregate all of some subgraph into one "super-node" with effective jn costs



Qs: good choices of which bits to aggregate? good computational procedures?