Recall metric on space of Borel probs on a product X of complete separable metric spaces X_s, $s \in S$

$$D(\rho, \rho') = \|\rho - \rho'\|_Z \quad \text{where } \|\mu\|_Z = \sup_{f \in F \setminus 0} \frac{\mu(f)}{\|f\|_F}$$

and $F = \{ f : X \to \mathbb{R}$; cts wrt product top, st.

$$\|f\|_F = \sum_{s \in S} \Delta_s(f) < \infty \} / {+\text{cst}}$$

$\Delta_s(f) =$ best Lip cst wrt x_s.

Also recall no lecture Tue 3 Nov. Next lecture is Fri 6 Nov.

This metric is good because e.g.

$\rho_\lambda =$ product of N (indpt) probs $(1-\lambda, \lambda)$ on $\{0,1\}$

then $\left\| \frac{d\rho}{d\lambda} \right\|_Z = 1$ whereas for

TV metric, KL divergence, Hellinger, Fisher info $\sim \sqrt{N}$

projective metric, ℓ_∞-transportation metric $\sim N$

Could divide these metrics by $\sqrt{N}$ or N, but then many localised changes have small distance & even all changes for some of the above (eg TV metric has diam $= 1$)!

(for more, see my paper "Robustness of Markov ..."

To bound a transition operator L (or their differences)

Can use $L : Z \circlearrowleft$ bdd linear induces $L : F \circlearrowleft$

by $\mu(Lf) = (\mu L)(f) \quad \forall \mu \in Z$, and

$\boxed{\|L\|_Z \leq \|L\|_F}$ because

$$\frac{|\mu L f|}{\|f\|_F} \leq \frac{\|L\|_F}{\|Lf\|_F} |\mu L f| \leq \|L\|_F \|\mu\|_Z$$

So $\|\mu L\| \leq \|L\|_F \|\mu\|$. □

<u>Remark</u>: Actually $\|L\|_Z = \|L\|_F$ using Hahn-Banach thm

Then $\boxed{\|P\|_Z \leq \delta} = \sup_{r \in S} \sum_{s \in S} k_{rs} = \|k\|_\infty$

where $k_{rs} = \sup_{\substack{x, \tilde{x} \\ \text{agreeing off } s}} D_T(p_r^x, p_r^{\tilde{x}}) / d_s(x_s, \tilde{x}_s)$

where $p_r^x(x'_r) = p_r(x'_r \mid x)$

$$D_T(\rho, \tilde{\rho}) = \sup_{f \in L \setminus C} \frac{\rho(f) - \tilde{\rho}(f)}{\|f\|_L}$$

$\|f\|_L$ = best Lipschitz cst

for $\rho, \tilde{\rho} \in \mathcal{P}(X)$, X metric. L = Lipschitz fns : $X \to \mathbb{R}$.

k_{rs} bounds the influence of state at site s now on state at site r next step.

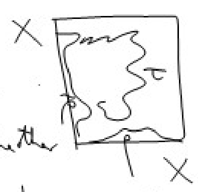
eg $k_{rs} = 0$ if r is not a nbr of s for usual PCAs

<u>Proof</u>: Use an alternative way of writing

$$D_T(\rho, \tilde{\rho}) = \inf_{\tau \in \mathcal{P}(X \times X)} \int d(x, \tilde{x})\, \tau(dx, d\tilde{x})$$

s.t. marginals on X are $\rho, \tilde{\rho}$

Such τ are called "couplings" but "joinings" is better because for me "coupling" implies some non-trivial effect of at least one on the other



D_T is called transportation metric because of origins in Monge's earth-movement questions

minimise $\int$ mass moved × distance moved

Kantorovich & Rubinstein proved equality of the 2 defns.

Take $x, \tilde{x}$ agreeing off s and take optimal joinings $\tau_r^{x,\tilde{x}}$ of P_r^{x} to $P_r^{\tilde{x}}$ $\forall r$

(if no optimal joining, take τ close to optimal & restrict $f \in F$ to depend on finitely many sites)

By marginals property of $\tau_r^{x,\tilde{x}}$, for any $f \in F$

$$(Pf)(x) - (Pf)(\tilde{x}) = \int \left(f(x') - f(\tilde{x}')\right) \prod_{r \in S} \tau_r^{x,\tilde{x}}(dx_r', d\tilde{x}_r')$$

But $f(x') - f(\tilde{x}') \leq \sum_r \Delta_r(f)\, d_r(x_r', \tilde{x}_r')$

and $\int d_r(x_r', \tilde{x}_r')\, \tau_r^{x,\tilde{x}}(dx_r', d\tilde{x}_r') \leq k_{rs}\, d_s(x_s, \tilde{x}_s)$

So $(Pf)(x) - (Pf)(\tilde{x}) \leq \sum_r \Delta_r(f)\, k_{rs}\, d_s(x_s, \tilde{x}_s)$

and $\Delta_s(Pf) \leq \sum_r \Delta_r(f)\, k_{rs}$

So now sum this result over $s \in S$

$$\|Pf\|_F = \sum_s \Delta_s(Pf) \leq \|k\|_\infty \sum_r \Delta_r(f)$$

i.e. $\|Pf\|_F \leq \|k\|_\infty \|f\|_F$. $\square$

cf. Maes Rev Mod Phys 5 (1983) 457

Example of optimal joining $\rho = (1-\lambda, \lambda)$ on $\{0,1\}$

$\tilde{\rho} = (1-\mu, \mu)$ / with $d(0,1) = 1$

$\lambda > \mu$

unique (optimal) τ

$$\begin{array}{cc|c|c|}
 & \tilde{x} & & \\
\mu \leftarrow & 1 & 0 & \mu \\
\hline
1-\mu \leftarrow & 0 & 1-\lambda & \lambda-\mu \\
\hline
 & & 0 & 1 \quad x \\
 & & \downarrow & \downarrow \\
 & & 1-\lambda & \lambda
\end{array}$$

$$D_T(\rho, \tilde{\rho}) = \lambda - \mu.$$