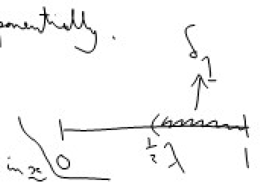
Now we prove exp attracting unique stationary probability for "weakly dependent" PCA: i.e. those for which $\|P\| = \delta < 1$.

Then P is a contraction on $\mathcal{P}$ so it has a unique sta prob $\rho \overset{?}{=} \rho^P$ ($\mathcal{P}$ complete w.r.t. our norm)
and ρ attracts all $\nu \in \mathcal{P}$ exponentially

$$D(\nu P^t, \rho) \leq \|P^t\| D(\nu, \rho) \leq \delta^t D(\nu, \rho)$$

<u>Example</u>: Stavskaya PCA has $\delta = 2(1-\lambda)$ (use interaction metric). So $\delta < 1$ for all $\lambda > \frac{1}{2}$.
We already have one sta prob $\delta_{\underline{1}}$. So it is unique & attracts everything exponentially.

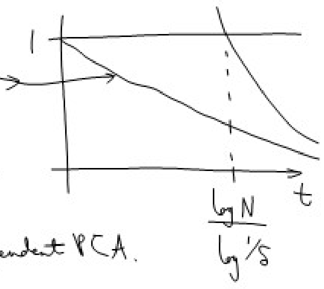
Rapid absorption for finite system:



On $\mathbb{Z}_N$ take $f(\underline{x}) = \#0\text{s in } \underline{x}$

So $\|f\|_F = N$. So $\|P^t f\|_F \leq \delta^t N$

So $P\{\text{not absorbed at time } t \text{ given initial cond}^n\ \underline{x}\} \leq \delta^t N$



<u>Robustness of exponentially attracting stationary probability</u>

Generalise from weakly dependent PCA.

<u>Thm</u>: If P_0 has sta prob ρ_0 which attracts exponentially in sense $\exists\, C > 0,\ r_0 < 1$ s.t. $D(\rho P_0^t, \rho_0) \leq C r_0^t D(\rho, \rho_0)$
then same is true for all P near P_0 ($\|P - P_0\|_Z$ small)
(with possibly different ρ_0, C, r_0)

<u>Proof</u>: Let $\varphi(r) = \sum_{t \geq 0} \frac{\|P_0^t\|}{r^t} < \infty$ for $r > r_0$

Take adapted norm on Z: $\|\mu\|_r = \sum_{t \geq 0} \frac{\|\mu P_0^t\|}{r^t}$.

It is equivalent to $\|\ \|$, because

$$\|\mu\| \leq \|\mu\|_r \leq \varphi(r) \|\mu\|$$

Then $\|\mu P_0\|_r = \sum_{t \geq 0} \frac{\|\mu P_0^{t+1}\|}{r^t} = r \sum_{t \geq 1} \frac{\|\mu P_0^t\|}{r^t}$

$$= r\left(\|\mu\|_r - \|\mu\|\right) \leq r\left(1 - \frac{1}{\varphi(r)}\right) \|\mu\|_r$$

Write $\varphi(r) = 1 + \underset{\geq 0}{\Psi(r)}$ then $r\left(1 - \frac{1}{\varphi(r)}\right) = r \frac{\Psi(r)}{\varphi(r)} \leq r$

So P_0 is a contraction in $\|\ \|_r$ for $r \in (r_0, 1]$:

$$\|P_0\|_r \leq \frac{r \Psi(r)}{\varphi(r)} < 1$$

Suppose $\varepsilon := \|P - P_0\| < \frac{1-r}{\varphi(r)}$ Then
$\|P - P_0\|_r \leq \varphi(r) \varepsilon < 1 - r$ so $\|P\|_r \leq \|P_0\|_r + \|P - P_0\|_r < 1$

So $\|P\|_r < 1$ is a contraction in r-norm

So P has a unique sta prob ρ and it attracts all of $\mathcal{P}$ like $\le \|P\|_r^t$ in r-norm

Furthermore $\|\rho-\rho_0\|_r \le \dfrac{\|\rho_0(P-P_0)\|_r}{1-\|P\|_r}$ so

$$\|\rho-\rho_0\| \le \frac{\varphi(r)\,\|\rho_0(P-P_0)\|}{1-r^{\psi(r)/\varphi(r)}-\varphi(r)\,\varepsilon}.$$

Similarly get $D(\tilde\rho P^t, \rho) \le C_1 r_1^t D(\tilde\rho, \rho)$

with some explicit C_1, r_1. $\square$

<u>Next address λ near 0 for Stavskaya PCA</u>

We'll show, following Toom, that

$$\delta_{\underline{0}} P^t \not\to \delta_{\underline{1}} \quad \text{for } \lambda < \frac{1}{54}$$

Start from $\underline{0}$ (all infected), so "worst case"

and ask for $\mathbb{P}\{x_s^t = 1\}$ $t>0$ wlog $s=0$

Trick is to consider prob distn on space-time

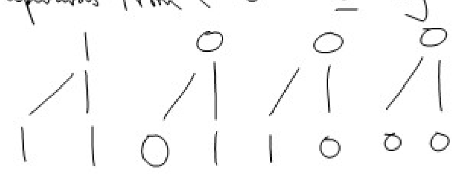
configurations $\underline{\underline{x}} \in \{0,1\}^{\mathbb{Z}\times\mathbb{Z}_+}$ space time

by putting "stoppers" in $\mathbb{R}^2$ from

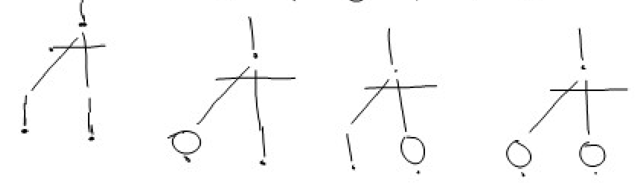
$(s-\frac{1}{2}, t-\frac{1}{4})$ to $(s+\frac{1}{2}, t-\frac{1}{4})$ with indept probs λ

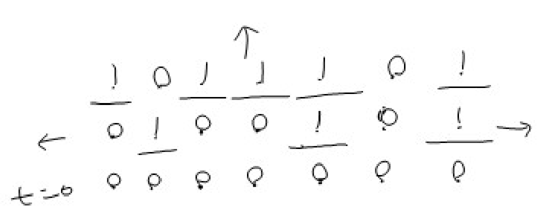
and construct configs upwards from $t=0$ $\underline{0}$ by

rules: no stopper



stopper





This generates the same prob distn as PCA with initial condn $\underline{0}$.