Beilinson at weight -1

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BSD

Let E be an elliptic curve over number field $\mathbb{Q}$ of rank r

- BSD conjecture (Beilinson style):
  1. $\mathrm{ord}_{s=1} L(E,s) = r$
  2. $L^{(r)}(E,1) \equiv \Omega_E R_E \mod \mathbb{Q}^\times$
- Ω_E is the real period.
- If $\{P_i\}$ is a basis for $E(\mathbb{Q})$, then $R_E = \det\langle P_i, P_j\rangle_E$ where $\langle -, -\rangle_E$ is the Néron-Tate height pairing.

Néron-Tate height pairing

- $\langle -, - \rangle_E : E(\mathbb{Q})^2 \to \mathbb{R}$ non-singular on $E(\mathbb{Q})^2 \otimes \mathbb{Q}$.
- $h(P) = \langle P, P \rangle_E$ is canonical height function, a quadratic form.
- h can be constructed as a sum of 'almost quadratic' local terms $h_v : E(\mathbb{Q}_v) \backslash \{\mathcal{O}\} \to \mathbb{R}$ for each place v:

$$h(P) = \sum_v h_v(P)$$

  for $P \neq \{\mathcal{O}\}$.
- Since $E(\mathbb{Q}) = \mathrm{CH}^1(E)^0$ (homologically trivial subspace) get a perfect pairing

$$\mathrm{CH}^1(E)^0_{\mathbb{Q}} \otimes \mathrm{CH}^1_{\mathbb{Q}}(E)^0 \to \mathbb{R}.$$

Relation to Beilinson's conjectures

- Let $M = h^1(E)(1)$. M is a pure motive of weight $\omega = -1$ and

$$L(E,1) = L(M,0)$$

- What does Beilinson's conjecture say in this case?
- Problems at $\omega = -1$:
  1. For $\omega = -2, -1$, $s = 0$ is not in the convergence region.
  2. Deligne conjectures that zeroes can only occur at $\omega = -1$.
  3. Deligne's conjecture: Pure motives are always critical when $\omega = -1$. But conjecture becomes vacuous in the presence of zeroes.
- BSD shows us that the order of zeroes can carry important arithmetic information.

Relation to Beilinson's conjectures

- Let X be a smooth projective variety over $\mathbb{Q}$ equidimensional of dimension N and let $M = h^{2a-1}(X)(a)$.
- For $a + b = N + 1$, Beilinson has, under some assumptions, constructed a 'geometric' height pairing

$$\langle -, - \rangle_X : \mathrm{CH}^a(X)^0_{\mathbb{Q}} \otimes \mathrm{CH}^b(X)^0_{\mathbb{Q}} \to \mathbb{R}$$

- Beilinson conjectures:
  1. $\langle -, - \rangle_X$ is non-degenerate.
  2. $\mathrm{ord}_{s=0} L(M, 0) = \dim_{\mathbb{Q}} \mathrm{CH}^n(X)^0_{\mathbb{Q}}$
  3. $L^*(M, 0) = c_+(M) \det\langle -, - \rangle_X \cdot \mathbb{Q}^*$,where L^* denotes the leading term and $c_+(M)$ is Deligne's period.

Outline

- A discussion of mixed motives and their ext groups
- Beilinson's construction of geometric height pairings
- Scholl's construction of motivic height pairings
- Relation to L-values: Scholl's unification.

Mixed motives

- Let $\mathcal{MM}_{\mathbb{Q}}$ denote the conjectural category of mixed motives over $\mathbb{Q}$.
- $\mathcal{MM}_{\mathbb{Q}}$ should be abelian and generated by the full subcategory of pure motives $\mathcal{M}_{\mathbb{Q}}$ under homological equivalence.
- $E \in \mathcal{MM}_{\mathbb{Q}}$ has realisations $(E_B, E_{dR}, \{E_\ell\}_\ell)$. E_{dR} is mixed Hodge structure: Additional increasing weight filtration: $W_\bullet E_{dR}$ such that $\mathrm{Gr}_i^W E_{dR}$ are pure of weight i. Corresponding filtration on E.
- Scholl defines 'mixed motives over $\mathbb{Z}$' to be the subcategory of $\mathcal{MM}_{\mathbb{Q}}$ whose weight filtration splits over the inertia subgroup I_v for all v, ℓ with $v \nmid \ell$. For $E \in \mathcal{MM}_{\mathbb{Z}}$

$$L(E,s) = \prod_i L(\mathrm{Gr}_i^W E, s),$$

Ext groups in $\mathcal{MM}_K$

- We write $\mathrm{Ext}^i_{\mathbb{Q}}, \mathrm{Ext}^i_{\mathbb{Z}}$ for ext in $\mathcal{MM}_{\mathbb{Q}}$, $\mathcal{MM}_{\mathbb{Z}}$. We expect these groups to vanish for $i \notin [0,1]$. If X is a smooth proper variety over $\mathbb{Q}$, $M = h^i(X)(m)$ we expect:

$$\mathrm{Ext}^0_{\mathbb{Z}}(M, \mathbb{Q}(1)) = \begin{cases} 0 & \text{if } i \neq 2n \\ \mathrm{CH}^n(X)/\mathrm{CH}^n(X)^0 \otimes \mathbb{Q} & \text{if } i = 2n \end{cases}$$

$$\mathrm{Ext}^1_{\mathcal{O}}(M, \mathbb{Q}(1)) = \begin{cases} H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}} & i \neq 2n+1 \\ \mathrm{CH}^n(X)^0 \otimes \mathbb{Q} & i = 2n+1 \end{cases}$$

  where $n = l + 1 - m$.

- $N = M^\vee(1) = h^i(X)(n)$ then equality for $\mathrm{Ext}^i_{\mathcal{O}}(\mathbb{Q}(0), N) \cong \mathrm{Ext}^i_{\mathcal{O}}(M, \mathbb{Q}(1))$.

Beilinson's height pairing

- Let X be as before.
- Suppose X admits a regular model $\mathcal{X}$ over $\mathbb{Z}$. For $a+b=N+1$, we have an intersection pairing

$$\mathrm{CH}^a(\mathcal{X})^0_{\mathbb{Q}} \times \mathrm{CH}^b(\mathcal{X})^0_{\mathbb{Q}} \to \mathbb{R},$$

  defined as a sum of local terms.
- Define $\mathrm{CH}^n(X)^{00}_{\mathbb{Q}}$ to be the image of

$$\cap_{v,\ell,v\nmid\ell}\mathrm{Ker}(\mathscr{Z}^n(\mathcal{X})_{\mathbb{Q}} \to H^{2n}(\mathcal{X}\otimes\overline{k(v)}, \mathbb{Q}_\ell(n)).$$

  in $\mathrm{CH}^n(X)^0_{\mathbb{Q}}$. Cycles ξ, δ lying in this subspace can be lifted to ξ', δ' on $\mathcal{X}$ and we define

$$\langle\xi,\delta\rangle_X = \langle\xi',\delta'\rangle_{\mathcal{X}}$$

  which does not depend on the choice of lift.
- Beilinson conjectures:

$$\mathrm{CH}^n(X)^{00}_{\mathbb{Q}} = \mathrm{CH}^n(X)^0_{\mathbb{Q}}.$$

Beilinson's height pairing

- Beilinson describes the pairing $\langle -, - \rangle_X$ in local terms, each defined cohomologically.
- We can define the terms at primes both infinite and non-infinite in a unified way using the tensor category formulation of 'geometric' and 'arithmetic' cohomology theories discussed in Alex's talk.

Beilinson's height pairing

- Given a rigid abelian tensor category $\mathcal{T}$ with coefficient ring $A = \mathrm{End}_{\mathcal{T}}(\mathbb{1})$ and 'geometric cohomology' objects $R\Gamma_c(X)$, $R\Gamma_Y(X)$ in $\mathscr{D}^b(\mathcal{T})$ for schemes of finite type X/F and closed subsets $Y \hookrightarrow X$, letting $R\Gamma(X) = R\Gamma_X(X)$.
- Pertinent examples:
  1. F is a number field or a finite extension of $\mathbb{Q}_\ell^{ur}$ and $\mathcal{T}$ is the category of finite-dimensional $\mathbb{Q}_\ell$-linear representations of G_F and $R\Gamma(X) = R\Gamma(\bar{X}_{\acute{e}t}, \mathbb{Q}_\ell)$
  2. $F = \mathbb{R}$ and $\mathcal{T}$ is the category of mixed $\mathbb{R}$-Hodge structures over F and $R\Gamma(X)$ is the 'Hodge complex'.
- Both examples admit a Tate object $A(1)$. Write denote $R\Gamma_?(X) \otimes A(n) =: R\Gamma(X, n)$. Define arithmetic cohomology groups $H^i_{\mathcal{T}}$ as:

$$R\Gamma_{\mathcal{T},?}(X, n) := R\mathrm{Hom}_?(\mathbb{1}, R\Gamma_?(X, n)) \in \mathscr{D}(A).$$

- Produces 'absolute Hodge cohomology' 'continuous étale cohomology', 'motivic cohomology' etc.

Beilinson's height pairings

- We have exact triangles

$$R\Gamma_Y(X) \to R\Gamma(X) \to R\Gamma(X - Y) \to R\Gamma_Y(X)[1]$$
$$R\Gamma_c(X - Y) \to R\Gamma_c(X) \to R\Gamma_c(Y) \to R\Gamma_c(X - Y)[1],$$

  duality pairings

$$R\Gamma_Y(X) \otimes R\Gamma(X) \to R\Gamma_Y(X), R\Gamma_c(X) \otimes R\Gamma(X) \to R\Gamma_c(X)$$

  and trace maps

$$Tr : R\Gamma_c(X) \to A(-N)[-2N]$$

  when X is smooth of dimension N.
- X smooth, $Y \subset X$ codimension d we have

$$H^i_Y(X) = 0, \ i < 2d$$

  and a cycle class map

$$\mathrm{cl}_Y : A(-d) \to H^{2d}_Y(X),$$

  which is an isomorphism for Y absolutely irreducible.

Beilinson's height pairing

- cl_Y induces an 'absolute' cycle map

$$\mathrm{cl}_{\mathcal{T},Y} : \mathscr{Z}_Y^d(X) \to H^{2d}_{\mathcal{T},Y}(X, d).$$

  This becomes an isomorphism after tensoring with A.
- We refer to the above cases where F is not a number field as the local cases, in which case we have a natural isomorphism

$$\mathrm{Ext}_{\mathcal{T}}(A(0), A(1)) \cong A.$$

Beilinson's height pairing

- Fix one of the local $\mathcal{T}$. Let ξ, δ be cycles on X_F of respective codimensions a, b with disjoint supports Y, Z. Assume that their global absolute cohomology classes vanish in $H^{2*}_{\mathcal{T}}(X, *)$. Let $\tilde{\mathrm{cl}}_{\mathcal{T}}(\delta) \in H^{2b-1}_{\mathcal{T}}(X - Z, b)$ be any lift of $\mathrm{cl}_{\mathcal{T},Z}(\delta) \in H^{2b}_{Z}(X, b)$. The local pairing $\langle \xi, \delta \rangle_{X,\mathcal{T}}$ at $\mathcal{T}$ is defined to be the image of $-\mathrm{cl}_{\mathcal{T},Y}(\xi) \otimes \tilde{\mathrm{cl}}_{\mathcal{T}}(\delta)$ under

$$\begin{array}{ccccc}
H^{2a}_{\mathcal{T},Y}(X - Z, a) \otimes H^{2b-1}_{\mathcal{T}}(X - Z, b) & \xrightarrow{\cup} & H^{2N+1}_{\mathcal{T},Y}(X - Z, N+1) & \xrightarrow{\mathrm{Tr}} & \mathrm{Ext}^1_{\mathcal{T}}(A(0), A(1)) \\
\downarrow\wr & & & & \downarrow\wr \\
H^{2a}_{\mathcal{T},Y}(X, a) \otimes H^{2b-1}_{\mathcal{T}}(X - Z, b) & & \dashrightarrow & & A
\end{array}$$

Beilinson's height pairing

- For the non-archimedean cases when $F = \mathbb{Q}_v^{ur}$ and the archimedean cases where $F = \mathbb{R}$ write

$$\langle -, - \rangle_{X,\mathcal{T}} =: \langle -, - \rangle_{X,v}.$$

  If χ and δ have disjoint supports and their rational equivalence classes are in $\mathrm{CH}^*(X)^{00}_{\mathbb{Q}}$ (assuming a regular model) then for $v \nmid \infty$ the local pairing is in $\mathbb{Q}$ and independent of ℓ. The global pairing decomposes as

$$\langle -, - \rangle_X = \sum_{v|\infty} \langle -, - \rangle_{X,v} + \sum_{v\nmid\infty} \log q_v^{-1} \langle -, - \rangle_{X,v}$$

  where q_v is what you think it is.
- This pairing generalises the Néron-Tate pairing. Its construction is unconditional for X a curve, an abelian variety and for $a = 1$.

Motivic height pairings

- Let G be a finite dimensional $G_{\mathbb{Q}}$-representation over $\mathbb{Q}$. Such a representation defines an *Artin motive*, denoted $G(0)$.
- Let $E \in \mathcal{MM}_{\mathbb{Q}}$ satisfy

$$\mathrm{Gr}_{-1}^W E = M, \mathrm{Gr}_0^W E = G_1(0), \mathrm{Gr}_1^W E = G_2(1)$$

and $\mathrm{Gr}_i^W E = 0$ otherwise for Galois reps G_1, G_2 as above. Scholl defines local pairings

$$b_{v,E} : G_1 \times G_2^\vee \to \begin{cases} \mathbb{R} & v \mid \infty \\ \mathbb{Q}_\ell & v \nmid \ell\infty \end{cases}$$

under certain hypothesis. These pairings will transform under base change: if $K/\mathbb{Q}$ is a finite extension and $e(v'/v)$ is the ramification degree of a prime v'/v then

$$b_{v',E'} = e(v'/v) b_{v,E}$$

where $E' = E \otimes K$.

Motivic height pairings: archimedean places

There is a canonical splitting

$$E_{\mathbb{R}} = V_{\mathbb{R}} \oplus M_{\mathbb{R}}$$

where $V_{\mathbb{R}}$ is an extension

$$0 \to G_2(1)_{\mathrm{R}} \to V_{\mathbb{R}} \to G_1(0)_{\mathbb{R}} \to 0.$$

This defines an element of

$$\begin{aligned}\mathrm{Ext}_{\mathcal{MH}_{\mathbb{R}}}(G_1(0)_{\mathbb{R}}, G_2(1)_{\mathbb{R}}) &= \mathrm{Hom}(G_1, G_2) \otimes \mathrm{Ext}(\mathbb{R}(0), \mathbb{R}(1)) \\ &= \mathrm{Hom}(G_1, G_2) \otimes \mathbb{R},\end{aligned}$$

i.e. a pairing $b_{\infty,E} : G_1 \times G_2^{\vee} \to \mathbb{R}$.

Motivic height pairings: Non-archimedean pairings

- We need some assumptions at non-archimedean places. Write

$$M_1 = E/W_{-2}, M_2 = W_{-1}$$

  We assume that M_i are defined over $\mathbb{Z}$. Equivalently
  For every v, ℓ with $v \nmid \ell$ that no eigenvalue of Frob_v on $M_\ell^{I_v}$ or $M_\ell(1)_{I_v}$ is a root of unity.
- Assume G_i have trivial $G_{\mathbb{Q}}$ action. A similar argument gives a pairing

$$b_{v,E} : G_1 \times G_2^\vee \to \mathbb{Q}_\ell.$$

- The pairings satisfy the base-change property. In general take a finite extension $K/\mathbb{Q}$ such that G_K acts trivially on each G_i, then define

$$b_{v,E} = \frac{1}{e(v', v)} b_{v',E'}.$$

- Scholl conjectures these pairings to be valued in $\mathbb{Q}$ and independent of ℓ.

Mixed periods and the height pairing

- Scholl defines a notion of criticality for mixed motives in a similar way as for pure motives.
- Critical mixed motives E admit periods $c_+(E)$.
- It can be shown that the motive E as above is critical if and only if the pairing $b_{\infty,E}$ is perfect.
- In this case we have

$$c_+(E) = c_+(M)\det(b_{\infty,E}).$$

Motivic height pairing: a thought experiment

- Scholl assumes following hypothesis: $\mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Q}(0), \mathbb{Q}(1)) = 0$ and $\mathrm{Ext}^1_{\mathbb{Q}}(\mathbb{Q}(0), \mathbb{Q}(1))$ is generated by a special class of '1-motives'.
- Let M be pure of weight -1 and set G, G' to be any finite dimensional subspaces

$$G \subset \mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Q}(1))$$
$$G' \subset \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}(1), M)$$

- There are motives M_i over $\mathbb{Z}$ given by

$$0 \to M \to M_1 \to G'(0) \to 0$$
$$0 \to G^{\vee}(1) \to M_2 \to M \to 0$$

Motivic height pairing: a thought experiment

- The hypothesis allows us to infer the existence of a unique object $E \in \mathcal{MM}_{\mathbb{Z}}$ with isomorphisms

$$\alpha_1 : W_{-1}E \cong M_1, \alpha_2 : E/W_{-2}E \cong M_2$$

  such that the induced isomorphisms

$$\mathrm{Gr}^W_{-1}(\alpha_i) : \mathrm{Gr}^W_{-1}E \cong M$$

  are equal for for $i = 1, 2$.
- This defines a canonical pairing

$$b_{\infty,E} : G \times G' \to \mathbb{R},$$

  compatible with restriction to smaller subspaces $H \subset G$, $H' \subset G'$. Taking the inductive limit, define a canonical motivic height pairing

$$\langle -, - \rangle_{\mathcal{M}} : \mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Q}(1)) \times \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}(1), M) \to \mathbb{R}.$$

Global motivic height pairing

Theorem

Let G_1, G_2 be finite dimensional $\mathbb{Q}$-vector spaces with trivial Galois action. Suppose we have a mixed motive $E' \in \mathcal{M}\mathcal{M}_{\mathbb{Q}}$ satisfying

$$\mathrm{Gr}_{-1}^{W} E' = M, \mathrm{Gr}_{0}^{W} E' = G_1(0), \mathrm{Gr}_{1}^{W} E' = G_2(1)$$

and $\mathrm{Gr}_{i}^{W} E' = 0$ *for* $i \notin [-2, 0]$. *Set*

$$M_1 = E'/W_{-2}E', M_2 = W_{-1}E'$$

which we assume are defined over $\mathbb{Z}$. *Assume the pairings* $b_{p,E'}$ *are* $\mathbb{Q}$*-valued and independent of* p. *Then there is a motive* E *defined over* $\mathbb{Z}$ *satisfying*

$$M_1 = E/W_{-2}E, \; M_2 = W_{-1}E$$

and

$$b_{\infty,E} = b_{\infty,E'} + \sum_{p} \log p^{-1} \cdot b_{p,E}$$

Comparison of local pairings

- Let X be a smooth projective variety over $\mathbb{Q}$ and assume it admits a regular model over $\mathbb{Z}$. For $M = h^{2a-1}(X)(a)$ Scholl constructs canonical maps

$$\alpha : \mathrm{CH}^a(X)^{00}_{\mathbb{Q}} \to \mathrm{Ext}^1(\mathbb{Q}(0), M)$$
$$\beta : \mathrm{CH}^b(X)^{00}_{\mathbb{Q}} \to \mathrm{Ext}^1(M, \mathbb{Q}(1)).$$

  These are conjecturally isomorphisms.

- Scholl proves the following theorem:

Theorem

Let $G \subset \mathrm{CH}^a(X)^{00}_{\mathbb{Q}}$, $G' \subset \mathrm{CH}^b(X)^{00}_{\mathbb{Q}}$ be finite-dimensional subspaces. Then there is a unique motive $\tilde{M}$ over $\mathbb{Z}$ satsifying the usual conditions on its grading satisfying

$$b_{\infty,\tilde{M}}(\alpha(x), \beta(y)) = \langle x, y \rangle_X$$

Special values of L-functions

Given a motive M, Scholl constructs a mixed motive E according to the following recipe:

1. Construct M_1 by taking M_1 to be the quotient in the sequence

$$0 \to \mathrm{Hom}(\mathbb{Q}(0), M) \otimes \mathbb{Q}(0) \to M \to M_1 \to 0$$

2. Construct a motive M_2:

$$0 \to M_2 \to M_1 \to \mathrm{Hom}(M_1, \mathbb{Q}(1)) \otimes \mathbb{Q}(1) \to 0.$$

3. Take the universal extension by $\mathbb{Q}(0)$ on the left and $\mathbb{Q}(1)$ on the right:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(M_2, \mathbb{Q}(1))^\vee \otimes \mathbb{Q}(1) \to M_3 \to M_2$$
$$0 \to M_3 \to E \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}(0), M_3) \otimes \mathbb{Q}(0) \to 0$$

if $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Q}(0), \mathbb{Q}(1)) = 0$ then the order in which this is done is not important and E has a three-step weight filtration with associated graded pieces $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}(0), M_3) \otimes \mathbb{Q}(0), M_2, \mathrm{Ext}^1_{\mathbb{Z}}(M_2, \mathbb{Q}(1))^\vee \otimes \mathbb{Q}(1)$.

Special values of L-functions

- Take $M = h^{2a-1}(X)(a)$. This is the only situation in which both $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}(0), M_3)$ and $\mathrm{Ext}^1_{\mathbb{Z}}(M_2, \mathbb{Q}(1))$ can be non-zero. Set ρ, ρ' to be their respective dimensions.
- The L-function of E is given by

$$L(E, s) = L(M, s)\zeta(s)^{\rho}\zeta(s+1)^{\rho'}$$

  and E is critical if and only if the associated pairing $\langle -, - \rangle$ is non-singular. We have $L^*(E, s) \equiv L^*(M, s) \mod \mathbb{Q}^\times$ and E does not vanish at $s = 0$.
- The extended Deligne conjecture suggests that for critical E

$$L^*(E, 0) = c_+(E) \cdot \mathbb{Q}^\times.$$

- The unified Beilinson conjecture is: The height pairing $\langle -, - \rangle$ is non-singular and
  1. $\mathrm{ord}_{s=0} L(M, s) = \rho$.
  2. $L^*(M, 0) = c_+(M) \det\langle -, - \rangle \cdot \mathbb{Q}^\times$.