Maximal subgroups of type $\mathcal{S}_1$ in dimensions 16 and 17

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Progress Report - Part A

Abstract

We examine the 16- and 17-dimensional classical groups and their almost simple extensions, and aim to classify all maximal subgroups of these groups. This requires considering those groups in class $\mathcal{S}$ of Aschbacher's theorem, and classifying them using methods generalised from the work of Bray, Holt and Roney-Dougal in [1].

Contents

1 Introduction

Bray, Holt and Roney-Dougal in [1] have classified the maximal subgroups of classical groups of dimension up to 12 and their almost simple extensions, the bulk of this work being in classifying those groups in class $\mathcal{S}$ in Aschbacher's theorem. Schroeder is currently extending this work to dimensions 13, 14 and 15. In this report, we aim to classify the $\mathcal{S}_1$ candidates in dimensions 16 and 17, using similar methods.

Section 2 offers a very brief introduction to the methods used - for a full and comprehensive treatment see [1]. Section 3 provides some technical results which we will use in later sections but which also apply to any future work done in higher dimensions. Sections 4 and 5 deal with all $\mathcal{S}_1$-candidates in dimensions 17 and 16 respectively - we will consider the 17-dimensional case first as this is more straightforward and allows us to give the proofs in more detail.

1.1 Notation

We will typically follow the notational conventions from [1]. The notation for groups and extensions follows the conventions in the ATLAS; see [2] for more information. We will also be considering the outer automorphism groups of various classical groups $\bar{\Omega}$. A presentation for the outer automorphism group will be introduced at the start of each section where it is required, and we will follow the convention in [1] for this. We will often abuse notation and use the same symbol for the conjugacy class of $\mathrm{Aut}\bar{\Omega}$ and a specific element of that conjugacy class.

Also following [1], we will use $\mathrm{SL}_n^{\pm}(q)$ to refer to either the special linear group $(+)$ or the special unitary group $(-)$.

Often we are considering the representation of a group G, as a subgroup of some classical group Ω, and we will be considering automorphisms both of G and of Ω. To avoid confusion, we will refer to elements of $\mathrm{Aut}(G)$ as *group automorphisms* and elements of $\mathrm{Aut}(\Omega)$ as *matrix automorphisms*.

We also have different types of stabiliser to consider; the two we will use most frequently are the stabiliser of a set of weakly equivalent representations acted on by group automorphisms, and the stabiliser of Ω-classes (where the projection of Ω is a classical simple group) of isomorphism classes of a group acted on by matrix automorphisms. We will refer to the former as *representation stabilisers* and the latter as *class stabilisers*.

1.2 Calculations

A number of proofs use computations in MAGMA. Some computations are straightforward and for these no code is provided to check these claims; other computations reference files which can be found at

`http://www2.warwick.ac.uk/fac/sci/maths/people/staff/rogers/talks/`.

2 General Theory

In this section we provide a very brief overview of the methods we will use in future chapters. All definitions and results in this section are from [1] unless otherwise stated, and the reader is advised to refer to [1] for a more comprehensive summary of the relevant theory.

Definition 2.1. Let G be an almost simple group with $\Omega < G < \mathrm{Aut}\Omega$, where $\Omega = \mathrm{SL}_n(q), \mathrm{SU}_n(q), \mathrm{Sp}_n(q)$ or $\Omega_n^\epsilon(q)$. We say that $H < G$ is in *class* $\mathcal{S}$ if all of the following are true:

1. H/Z is almost simple, where Z is the subgroup consisting of the scalar matrices in H
2. H does not contain Ω
3. H^∞ acts absolutely irreducibly
4. $(H^\infty)^g$ is not defined over a proper subfield of $\mathbb{F}_{q^u}$ for any $g \in \mathrm{GL}_n(q^u)$ and any $u \geq 1$.
5. H^∞ preserves a non-zero unitary/quadratic form if and only if Ω is unitary/orthogonal, H^∞ preserves a non-zero symplectic form and no non-zero quadratic form if and only if Ω is symplectic, and H^∞ preserves no non-zero classical form if and only if Ω is linear.

H is in class $\mathcal{S}_1$ if H^∞ is not isomorphic to a group of Lie type in characteristic p (where $q = p^e$), and $\mathcal{S}_2$ otherwise.

Hence class $\mathcal{S}_1$ deals with the so-called *cross-characteristic* case, and these can be found by examining the character tables and Brauer character tables of quasisimple groups. A list of candidates in dimensions up to 250 can be found in [3], and in the relevant sections we include the appropriate representations along with some additional useful information.

We then consider each candidate group in turn, establishing which classical groups they are subgroups of, and whether any of the outer automorphisms of the candidate groups are realisable in the relevant classical group. Establishing the shape of the classical group can typically be derived from the algebraic irrationalities and the Schur indicator of the relevant representation, and computations regarding the outer automorphism is usually done either by considering the character table of the relevant extension from [2] or [5], or by computer calculations in Magma.

These computations suffice to establish maximal subgroups of the classical groups in 16 and 17 dimensions; to answer the question for almost simple groups requires some additional work. We also need to consider the number of conjugacy classes of subgroups of Ω isomorphic to the candidate group, and the stabiliser of one of these classes under the action of matrix automorphisms (i.e. automorphisms of Ω), as well as which group automorphisms the matrix automorphisms induce. There are a number of computational results in Section 4.6 of [1] which are used to establish these results, and copies of the computations referenced in the rest of this document can be found at

`www2.warwick.ac.uk/fac/sci/maths/people/staff/rogers/talks/`.

This process gives us a list of candidates. However, it is possible that there are containments between groups, by which we mean an abstract containment of $\mathcal{S}_1$ candidates $H_1 < H_2$, which also extends to a containment $\rho_1(H_1) < \rho_2(H_2)$ where ρ_i is the relevant (16- or 17-dimensional) representation of the group H_i. Given such an abstract containment, we can usually establish whether we have a containment by looking at the character values of the restriction of the representation $\rho_2(H_2)$ to H_1 and comparing this with the character values of $\rho_1(H_1)$.

3 General results

Here we prove some results which are applicable in the 17-dimensional case, but can also be applied in higher dimensions.

3.1 Alternating group in dividing characteristic

We quote without proof a technical lemma from [1] (Proposition 1.6.11, p28)

Lemma 3.1. *Let q and n be odd and $g \in SO_n(q)$. Let F be the symmetric bilinear form preserved by $SO_n(q)$, $A := I_n - g$, $k = \text{rank}A$, and B be a $k \times n$ matrix whose rows form a basis of the complement of the nullspace of A. Then the spinor norm of g is 1 if $\det(BAFB^T)$ is a (nonzero) square mod q, and -1 otherwise. In other words the spinor norm corresponds to the Legendre symbol $\left(\frac{\det(BAFB^T)}{q}\right)$*

Theorem 3.2. *For $n > 6$ odd and $p \mid n$ where p is prime, S_n is a subgroup of $\Omega_{n-2}(p)$ if $p = 1$ or $3 \bmod 8$ and a subgroup of $SO_{n-2}(p)$ but not $\Omega_{n-2}(p)$ if $p = 5$ or $7 \bmod 8$.*

Proof. The $(n-2)$-dimensional representation of S_n over the field $\mathbb{F} = \mathbb{F}_p$ for $p \mid n$ is constructed as follows:

Begin with the standard degree n permutation representation of S_n; in other words, for $\sigma \in \mathrm{S}_n$, define the matrix $\rho(\sigma) = (a_{ij})$, where $a_{ij} = \begin{cases} 1 & \text{if } i^\sigma = j \\ 0 & \text{otherwise.} \end{cases}$

This gives us an n-dimensional module M with basis $e_1, ..., e_n$, such that $e_i^\sigma = e_{i^\sigma}$. This has a 1-dimensional submodule $K = \langle e_1 + ... + e_n \rangle$, giving rise to the deleted permutation module M/K with dimension $n-1$ and basis $e_2 + K, ..., e_n + K$. The corresponding matrix for σ is attained by replacing the 1^σ-th row with a row of -1s, and then deleting the first row and column of the matrix.

Next, define $f_i := e_i - e_n + K$. It is routine to show that the module $N := \langle f_1, ..., f_{n-1} \rangle$ of M/K is irreducible, and if $p \nmid n$ the f_i are linearly independent and thus N is irreducible. However, if $\text{char}\,\mathbb{F} \mid |G|$ then $f_2, .., f_{n-1}$ are still linearly independent, but $f_1 = -(f_2 + ... + f_{n-1})$, giving an $(n-2)$-dimensional irreducible representation in this case.

This representation preserves the quadratic form $F = (f_{ij}) = \begin{cases} 1 & \text{if } i = j \\ \frac{p+1}{2} & \text{if } i \neq j \end{cases}$

We know already that the even elements of S_n have spinor norm 0; hence to determine whether $S_n = A_n.2$ is contained in $\Omega_{n-2}(p)$, it suffices to find the spinor norm of an odd element of S_n. We choose the element $(1,2) \in S_n$. In the construction above, the matrix of the $(n-2)$-dimensional representation of $(1,2)$ has determinant -1, so we multiply the matrix by -1 to give us an element of $\mathrm{SO}_{n-2}(p)$; namely the matrix (g_{ij}) given by $g_{ij} = \begin{cases} 1 & \text{if } i = 1 \\ -1 & \text{if } i = j \neq 1 \\ 0 & \text{otherwise} \end{cases}$

In the context of Lemma 3.1 we have $A = I_{n-2} - g = \begin{pmatrix} 0 & -1 & -1 & \dots & -1 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}$. which clearly has rank $n-3$, and a basis of the complement of the nullspace can be taken to be all the standard basis vectors bar the first, yielding the $n-3 \times n-2$ matrix;

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Performing the computation $BAFB^T$ yields the $n-3 \times n-3$ matrix
$$\begin{pmatrix} 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots & 1 \\ 1 & 1 & 2 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 2 \end{pmatrix}$$
which by standard linear algebra has determinant $n-2$. Hence the question reduces to computing the Legendre symbol

$$\left(\frac{n-2}{p}\right) = \left(\frac{-2}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p = 1,3 \bmod 8 \\ -1 & \text{if } p = 5,7 \bmod 8 \end{cases}$$

by elementary results.

□

Corollary 3.3. *Let $n > 6$ be odd and $p \mid n$ be prime. Let $\Omega = \Omega_{n-2}(p)$ and $G = A_n$ be a $\mathcal{S}_1$ subgroup of Ω.*

- *If $p = 1,3 \bmod 8$, then $S := N_\Omega(G) = G.2$, there are two conjugacy classes of subgroups of Ω isomorphic to G, and S has trivial class stabiliser.*
- *If $p = 5,7 \bmod 8$, then $S := N_\Omega(G) = G$, there is a unique conjugacy class of subgroups of Ω isomorphic to G and S has class stabiliser $\langle\delta\rangle$.*

In both cases we have a unique $\mathrm{Aut}\Omega$ *class of groups G.*

Proof. The result on normalisers is direct from Theorem 3.2. Let c denote the number of Ω-classes of G; from [1, p.168, Lemma 4.4.3(ii)] we have that $c = |C : N_C(G)\Omega|$ where in this case $C = \mathrm{Aut}\Omega = \mathrm{Inn}\Omega\langle\delta\rangle$. Thus we have

$$\begin{aligned} c = 1 &\iff C = N_C(G)\Omega \iff \delta \in N_C(G) \iff G^\delta = G \\ &\iff \exists g \in \mathrm{SO}_{n-2}(p) \setminus \Omega_{n-2}(p) \text{ such that } G^g = G \\ &\iff \exists g \in \mathrm{SO}_{n-2}(p) \setminus \Omega_{n-2}(p) \text{ such that } g \in G.2 \\ &\iff G.2 \not\leq \Omega_{n-2}(p) \end{aligned}$$

Since δ always permutes the classes, the result follows.

□

3.2 Automorphisms of orthogonal groups in odd degree

Throughout this section we will be considering the automorphism group of $\Omega = \Omega_n^\circ(q)$ for n odd. We have that $\mathrm{Out}\Omega = \langle \phi, \delta | \phi^e = \delta^2 = [\delta, \phi] = 1\rangle \cong C_e \times C_2$ where $q = p^e$. Some calculations in these cases are aided by knowing in which cosets of $\mathrm{Inn}\Omega$ elements in $\mathrm{Aut}\Omega$ of a given order lie.

The following lemma in [1, p.239, Lemma 4.9.40] performs this task for involutions.

Lemma 3.4. *Let g be an element of order 2 in $\mathrm{Aut}(\mathrm{O}_n^\circ(q))$, with $q = p^e$. Then*

$$g \in \mathrm{O}_n^\circ(q)\langle\delta\rangle \cup \mathrm{O}_n^\circ(q)\langle\phi\rangle$$

In dimension 17 we will need a similar result for elements of order 4:

Lemma 3.5. *Let g be an element of order 4 in $\mathrm{Aut}(\mathrm{O}_n^\circ(q))$, with $q = p^e$. Then*

$$g \in \mathrm{O}_n^\circ(q)\langle\delta\rangle \cup \mathrm{O}_n^\circ(q)\langle\phi^{\frac{e}{4}}\rangle \cup \mathrm{O}_n^\circ(q)\langle\phi^{\frac{e}{2}}\delta\rangle$$

Proof. We use the isomorphism $\mathrm{Aut}(O_n^\circ(q)) = O_n^\circ(q)\langle\phi,\delta\rangle \cong \mathrm{SO}_n^\circ(q)\langle\phi\rangle$, since $\mathrm{SO}_n^\circ(q) = \mathrm{O}_n^\circ(q)\langle\delta\rangle$, and take $g = A\sigma$ for $A \in \mathrm{SO}_n^\circ(q)$ and $\sigma \in \langle\phi\rangle$. Let $q = p^e$, so ϕ has order e. We will consider various orders of σ:

- $|\sigma| = 1$. Then $g = A \in \mathrm{SO}_n^\circ(q) = \mathrm{PSO}_n^\circ(q)$.
- $|\sigma| = 2$. Then $\sigma = \phi^{\frac{e}{2}}$ and so $g \in O_n^\circ(q)\langle\delta\phi^{\frac{e}{2}}\rangle$ or $O_n^\circ(q)\langle\phi^{\frac{e}{2}}\rangle$.
- $|\sigma| = 4$. Then in particular we must have that q is a fourth power, say $q = c^4$. We have that $I_n = g^4 = (A\sigma)^4 = A\sigma^4\sigma^{-3}A\sigma^3\sigma^{-2}A\sigma^2\sigma^{-1}A\sigma = AA^{\sigma^3}A^{\sigma^2}A^{\sigma}$. We have that the spinor norm of A is 1 iff μ is a square in $\mathbb{F}_q$, where μ is defined in [1][Definition 1.6.10,p28], the exact definition is not needed for this proof. Similarly the spinor norm of A^{ϕ^i} is 1 iff μ^{e^i} is a square in $\mathbb{F}_q$. Then $AA^{\sigma^3}A^{\sigma^2}A^{\sigma} = I_n$ which is an element of $\mathrm{SO}_n^\circ(c)$ with spinor norm 1. Hence μ^r is a square in $\mathbb{F}_c^\times$ where $r = 1 + c + c^2 + c^3$. This is the case iff μ is a square in $\mathbb{F}_q^\times$, so A has spinor norm 1 and $g \in \mathrm{O}_n^\circ(q)\langle\phi\rangle$.

□

Remark 3.6. Although there are more cases in Lemma 3.5 than there are in Lemma 3.4, computer simulations show that all three cases can occur.

4 $\mathcal{S}_1$ candidates in dimension 17

4.1 Candidates

Table 1: $\mathcal{S}_1$ candidates in dimension 17

Group	PmDivs	Out	Dim	Ind	# ρ	Stab	Charc	Ch Ring
$L_2(17)$	2,3,**17**	2	17	+	1	2	0	—
$L_2(16)$	**2**,3,5,17	4	17	+	1	4	$0, 5, 17$	—
$L_2(16)$	**2**,3,5,17	4	17	+	2	2	$0, 3, 17$	b_5
$L_2(16)$	**2**,3,5,17	4	17	+	4	1	$0, 17$	b_5, y_{15}
A_{18}	2,3,5,7,11,13,17	2	17	+	1	2	$0, 5, 7, 11, 13, 17$	—
A_{19}	2,3,5,7,11,13,17,19	2	17	+	1	2	19	—

Remark 4.1. We supply a brief explanation of the content of Table 1, along with details of how such information was found:

- 'Group' is the isomorphism class of the group discussed, in Atlas notation. The candidate groups can be found in [3] and [4].
- 'PmDivs' consists of the prime divisors of the order of the group. A number in bold indicates the defining characteristic of the group, if this exists - note that groups in fields of these characteristics are of type $\mathcal{S}_2$. This is an easy computation.
- 'Out' is the shape of the outer automorphism group, again in Atlas notation (although in this table all the outer automorphism groups considered are cyclic). In all cases this is found via [2], [9] or (in the case of the alternating groups) standard results, for example [10, Theorem 2.3].
- 'Dim' denotes the dimension of the representation considered. Here we are considering precisely those candidates which are 17-dimensional.
- 'Ind' denotes the Schur indicator of the representation.
  - + denotes Schur indicator 1 and indicates that the representation preserves an orthogonal form; thus the representation is a subgroup of $\Omega_n^{\epsilon}(q)$ for $\epsilon \in \{+, -, \circ\}$.
  - $-$ - denotes Schur indicator -1 and indicates that the representation preserves a symplectic form; thus the representation is a subgroup of $\mathrm{Sp}_n(q)$.
  - $\circ$ denotes Schur indicator 0 and indicates that the representation either preserves a unitary form (so is a subgroup of $\mathrm{SU}_n(q)$) or preserves no form (so is a subgroup of $\mathrm{SL}_n(q)$ and no other classical group). This can also be found in [3] and [4].
- '$\#\rho$' is the number of *weakly equivalent* representations in the character table; in other words, the number of representations that can be obtained by applying group automorphisms, algebraic conjugacy and module duality to the given representation. For instance in Table 1, we have three weak equivalence classes for $L_2(16)$. Looking at the character table in characteristic 0, we see 7 17-dimensional characters. One is entirely real-valued and hence closed under all three operations, giving one row in our table with a single representation. Two representations involve only the irrationality $b_5 = \frac{-1+\sqrt{5}}{2}$, and we can obtain one from the other by algebraic conjugacy, giving a row containing two representations. The final four representations involve the irrationalities b_5 and $y_{15} = e^{\frac{2\pi i}{15}} + e^{\frac{-2\pi i}{15}}$, and the outer automorphism of order 4 permutes these cyclically, giving us a weak equivalence class containing four representations.

- 'Charc' is the characteristics in which we have such a representation - this again comes from [3] and [4].

- 'Ch Ring' denotes the character ring of the representation; this is the ring generated by the character values, and we list in this column any irrational generators. These can be easily seen from the character tables, which can usually be found in [2], [9] or GAP. This was not possible for A_{18} and A_{19} in dimension 17; however these characters are easy to find by a result due to Wagner ([7] and [8]), which tells us that the relevant Brauer character χ of A_n (for $n > 8$) over $\mathbb{F}_p$ is given by $\chi(g) = \begin{cases} |\mathrm{Fix}(g)| - 1 & \text{if } p \nmid n \\ |\mathrm{Fix}(g)| - 2 & \text{if } p \mid n. \end{cases}$ In particular these characters all have character ring $\mathbb{Z}$.

4.2 Results

We now determine the $\mathcal{S}_1$-maximal subgroups of $\Omega_{17}^{\circ}(q)$ and its almost simple extensions. From Table 1 the candidates are $L_2(17)$ ($p \neq 2, 3, 17$), $L_2(16)$ (three times: one with $p \neq 2, 3$, one with $p \neq 2, 5$ and one with $p \neq 2, 3, 5$), A_{18} ($p \neq 2, 3$) and A_{19} ($p = 19$). We will consider these in reverse order. A reminder that MAGMA files referenced can be found at

`http://www2.warwick.ac.uk/fac/sci/maths/people/staff/rogers/talks/`.

Remark 4.2. The outer automorphism group of $\Omega_{17}^{\circ}(p^e)$ is given by the presentation

$$\langle \delta, \phi | \delta^2 = \phi^e = [\delta, \phi] = 1 \rangle \cong C_2 \times C_e.$$

Here δ, the *diagonal automorphism* is induced by conjugation by an element of $\mathrm{SO}_{17}^{\circ}(q) \setminus \Omega_{17}^{\circ}(q)$, and ϕ is the *field automorphism* given by applying the map $x \mapsto x^p$ to each entry in the matrix.

Proposition 4.3. *Let $G = A_{19} < \Omega = \Omega_{17}^{\circ}(q)$, $q = p^e$ and $S = N_{\Omega}(G)$. Then $q = p = 19$ and $S = G.2$ with two Ω-classes of subgroups isomorphic to G with trivial class stabilizer. $G.2$ is $\mathcal{S}_1$-maximal, there is a unique* AutΩ*-class of groups G, and for no other q is there a $\mathcal{S}_1$-subgroup of $\Omega_{17}^{\circ}(q)$ isomorphic to G.*

Proof. Direct from Corollary 3.3. □

Proposition 4.4. *Let $G = A_{18} < \Omega = \Omega_{17}^{\circ}(q)$ with $q = p^e$ and $S = N_{\Omega}(G)$. Then $q = p \neq 2, 3$, $S = G.2$ with two Ω-classes of subgroups isomorphic to G with trivial class stabiliser, and there is a unique* AutΩ *class of groups G. If $q = 19$ then S is not $\mathcal{S}_1$-maximal; otherwise, S is $\mathcal{S}_1$-maximal. For no other q are there $\mathcal{S}_1$-maximal subgroups of $\Omega_{17}^{\circ}(q)$ isomorphic to A_{18}.*

Proof. The condition on q follows directly from Table 1. Since this representation occurs as a deleted permutation module, we can apply Lemma 4.9.39 from [1][p.238], which says that $G.2 < \Omega_{17}^{\circ}(q)$ iff 9 is a square mod p, which is clearly always true. Hence $N_{\Omega}(G) = G.2$, with trivial class stabiliser.

The only potential containment involving G is $A_{18} < A_{19}$. This only occurs when $q = 19$; in this case, by Proposition 4.3 and from the relevant character tables, we have the containment $A_{18}.2 < A_{19}.2$, so G is not $\mathcal{S}_1$-maximal in this case. Lagrange gives no other possible containments, so if $q \neq 19$ then $G.2$ is $\mathcal{S}_1$-maximal. □

Proposition 4.5. *There are no $\mathcal{S}_1$ maximal subgroups involving $L_2(17)$.*

Proof. The 17-dimensional representation of $L_2(17)$ appears as a deleted permutation module of the action of $L_2(17)$ on the 18 lines in $\mathbb{F}_17^2$. Hence we have $L_2(17) < A_{18}$. Also, $L_2(17).2 = \mathrm{PGL}(2, 17)$ which occurs naturally as a submodule of $S_{18} = A_{18}.2$ so we have an abstract containment $L_2(17).2 < A_{18}.2$.

From Lemma 4.9.39 from [1][p.238], we have that, since the 17-dimensional representation of $L_2(17)$ consists of action matrices of a deleted permutation module, then in all characteristics not dividing 18 (which is no restriction for us since we are not considering $p = 2, 3$), $G.2 < \Omega$ iff 9 is a square mod p. Since this always happens, we have that $N_\Omega(G) = G.2$ in every characteristic we are considering. From Proposition 4.2 we have that $A_{18}.2 < \Omega$, and so it simply remains to establish what the restriction of this character is to $L_2(17).2$.

We used MAGMA to find the character table of $L_2(17).2$ - this has two 1-dimensional characters (the trivial character $\mathbb{1}$ and the sign character $\mathbb{S}$, 2 real-valued 16 dimensional characters (which we denote by χ^+ and χ^- depending on the sign of the non-zero character value on elements of order 2), and 2 17-dimensional characters (denoted similarly by τ^+ and τ^-). The restriction of the character of $A_{18}.2$ to $L_2(17).2$ must be real valued, meaning that it is either one of the 17-dimensional characters, meaning we have an inclusion, or it is some linear combination of the smaller-dimensional real-valued characters.

$L_2(17).2$ contains elements of order 2 which are the product of 8 2-cycles, which therefore have character value 1 and means that the restriction gives either τ^+ or $\mathbb{S} + \chi^+$. We also have elements in $L_2(17)$ of order 9 with cycle shape $(9, 9)$, meaning these have character value -1 which rules out the possibility of $\mathbb{S} + \chi^+$. Hence the restriction is τ^+, which is irreducible and hence means that we have a containment $L_2(17).2 < A_{18}.2$. □

Remark 4.6. We can do a similar computation with $L_2(16)$ and A_{18}. In this case, the possibilities for the restriction of the 17-dimensional character of A_{18} to $L_2(16)$ are either $\mathbb{1} + \chi$ for χ the real-valued 16-dimensional character of $L_2(16)$, or τ for τ the real-valued 17-dimensional character. Considering elements of order 3 gives us that the restriction is $\mathbb{1} + \chi$, so we do not have a containment here. An identical argument also rules out the possibility of $L_2(16) < A_{19}$ in characteristic 19.

Remark 4.7. There are three different representations of $L_2(16)$ to consider. We will denote these by $L_2(16)_1, L_2(16)_2$ and $L_2(16)_3$, based on the order they appear in Table 1.

Proposition 4.8. *Let $G = L_2(16)_1 < \Omega^\circ_{17}(q)$, $q = p^e$ and $S = N_\Omega(G)$. Then provided $p \neq 2, 3$ we have $q = p$ and the following results:*

- *If $p = \pm 1 \bmod 8$, then $S = G.4$. We have two Ω-classes of subgroups with trivial class stabiliser.*
- *If $p = \pm 3 \bmod 8$, then $S = G.2$. We have a single Ω-class of subgroups with class stabiliser $\langle \delta \rangle$.*

There is a single $\mathrm{Aut}\Omega$-class of subgroups G, S is $\mathcal{S}_1$-maximal and the only situation where we have subgroups of $\Omega^\circ_{17}(q)$ isomorphic to G are described in Proposition 4.9 and Proposition 4.11.

Proof. Since we can see from Table 1 that the character ring contains no irrationalities, we always have $G < O^\circ_{17}(p)$ for valid primes p (i.e. $p \neq 2, 3$). Calculations contained in the computer file `l216d171calc` perform the spinor norm calculation before any p-modular reductions for an element which induces the 4 automorphism on G. This element, and the form it preserves, were found by using the inbuild MAGMA commands for some finite fields and attempting from these to recover the original matrices. The calculations include checks that the group constructed is $G.4$ and that it preserves the given form.

We then construct matrices following Lemma 3.1 of [1] to compute the spinor norm. The computer calculations show that this determinant is $\frac{2^9}{3^{12}} = 2(2^4 3^{-6})^2$. Hence whether or not this matrix has square determinant depends on whether 2 is a square modulo p, which by well-known resuts is precisely the conditions supplied above. If the element g inducing the 4 automorphism G has spinor norm 1, then g^2 inducing the 2 automorphism of G must have spinor norm 0.

Standard computations confirm the number of Ω-classes. When there are two classes, δ interchanges them, and so the results on the class stabiliser follow as well. □

Proposition 4.9. *Let $G = L_2(16)_2 < \Omega^\circ_{17}(q)$, with $q = p^e$, and let $S = N_\Omega(G)$. Then provided $p \neq 2, 5$, we have the following results:*

(i) If $p = 2, 3 \bmod 5$ then $q = p^2$ and $S = G.2$. There are two Ω-classes of subgroups of Ω isomorphic to S, but we do not currently have a proof of what the class stabilisers are.

(ii) If $p = 1, 4 \mod 5$ *and* $p = 1, 7 \mod 8$ *then* $q = p$ *and* $S = G.2$. *There are two* Ω*-classes of subgroups of* Ω *isomorphic to* S*, and they have trivial class stabilisers.*

(iii) If $p = 1, 4 \mod 5$ *and* $p = 3, 5 \mod 8$ *then* $q = p$ *and* $S = G$. *There is a single* Ω*-class of subgroups of* Ω *isomorphic to* G*, with class stabiliser* $\langle \delta \rangle$.

There is a single AutΩ*-class of subgroups* G*,* S *is* $\mathcal{S}_1$*-maximal and the only situation where we have subgroups of* $\Omega_{17}^{\circ}(q)$ *isomorphic to* G *are described in Proposition 4.8 and Proposition 4.11.*

Proof. The form of the field depends on the existence of the quadratic irrationality b_5, which lies in $\mathbb{F}_p$ if $p = 1, 4 \mod 5$ and $\mathbb{F}_{p^2}$ if $p = 2, 3 \mod 5$.

We perform similar calculations to Proposition 4.8 in `l216d172calc`, although this time the characteristic 0 group we are dealing with has a quadratic irrationality b_5, which we denote here by w and which satisfies $w^2 = w + 1$.

The computation tells us that the matrix relating to the 2 automorphism of G has spinor norm 0 iff

$$\frac{2^{19}5^5(1+w)}{13^{10}}$$

is a square in the field we are considering. Since $1 + w = w^2$ is clearly a square, this reduces to determining when 10 is a square in the field.

If $q = p^2$ then 10 is contained in the prime field and so is always a square; hence here we can always realise the 2 automorphism. If $q = p$, then in particular $p = \pm 1 \mod 5$; thus, from the Legendre symbol for 5 we can see that 5 is always a square, and the Legendre symbol for 2 gives the required congruences.

For the class stabiliser calculations, note that if $G.2 < \Omega$, then $N_C(G) < Z\Omega$, where Z is the group of scalar matrices. Then $N_C(G) = Z\Omega$ and so, by Lemma 4.4.3 in [1] the number of Ω-classes is $|C : N_C(G)\Omega| = |C : \Omega| = 2$. Likewise, if $G.2 \not< \Omega$, then $N_C(G)\Omega = C$ and so we have a single Ω-class in this case.

In case (ii) we have two classes permuted by δ and no further outer automorphisms exist, so the class stabiliser is trivial.

In case (iii) we have one class; hence the class stabiliser is $\langle \delta \rangle$, the whole outer automorphism group. □

Remark 4.10. Computer calculations give us the conjecture that in Proposition 4.9(i), the class stabiliser should be $\langle \phi \rangle$ when $p = \pm 1 \mod 8$ and $\langle \phi\delta \rangle$ when $p = \pm 3 \mod 8$

Proposition 4.11. *Let* $G = L_2(16)_3 < \Omega_{17}^{\circ}(q)$*, with* $q = p^e$. *Then provided* $p \neq 2, 3, 5$*, we have* $N_\Omega(G) = G$*, two* Ω*-classes of subgroups isomorphic to* G*, and the following results:*

(i) If $p = \pm 1 \mod 15$ *then* $q = p$*, with trivial class stabiliser.*

(ii) If $p = \pm 4 \mod 15$ *then* $q = p^2$*, with class stabiliser* $\langle \phi \rangle$.

(iii) If $p = \pm 2, \pm 7 \mod 15$ *then* $q = p^4$ *with class stabiliser* $\langle \phi \rangle$.

There is a single AutΩ*-class of subgroups* G*,* G *is* $\mathcal{S}_1$*-maximal and the only situation where we have subgroups of* $\Omega_{17}^{\circ}(q)$ *isomorphic to* G *are described in Proposition 4.8 and Proposition 4.9.*

Proof. $L_2(16)_3$ involves the irrationalities b_5 and $y_{15} = \theta + \theta^{-1}$, where θ is a primitive 15th root of unity. Lemma 4.2.1 in [1] tells us that, provided $p \nmid 15$ (which is the case here since we are considering representations not in characteristic 2, 3 or 5), then $y_{15} \in \mathbb{F}_q$ if and only if $q = \pm 1 \bmod 15$, where

$q = p^e$. Thus straightforward calculations give us that y_{15} exists over $\mathbb{F}_q$ for $q = p^e$ for e as small as possible, where

$$e = \begin{cases} 1 & \text{if } p = 1, 14 \bmod 15 \\ 2 & \text{if } p = 4, 11 \bmod 15 \\ 4 & \text{if } p = 2, 7, 8, 13 \bmod 15 \end{cases}$$

It turns out that, whenever y_{15} exists in a field, so does b_5; hence this suffices to determine everything we need to know.

In all cases, we have 2 Ω-classes, by arguments similar to above, and δ interchanges these two classes.

For (i), since $\mathrm{Out}\Omega = \langle\delta\rangle$, the above is enough to confirm that the class stabiliser is trivial.

For (ii), since $G.2 \setminus G$ contains involutions, we can apply Lemma 3.4 to conclude that the class stabiliser here is $\langle\phi\rangle$.

For (iii), let α be an element of the class stabiliser. We have 2 classes and 8 outer automorphisms, so the stabiliser must have 4 elements; so the stabiliser is either C_4 or $C_2 \times C_2$. If it were $C_2 \times C_2$, then we would have $|\alpha| = 2$ for every choice of α. Then by Lemma 3.4 we would have that $\alpha \in \langle\delta\rangle$ or $\alpha \in \langle\phi\rangle$. This would force the stabiliser to be $\langle\delta, \phi^2\rangle$, which is not possible as δ does not stabilise either class. Hence we have that the stabiliser has shape C_4; this leaves us with the option of either $\langle\phi\rangle$ or $\langle\phi\delta\rangle$. Since $G.4 \setminus G$ contains elements of order 4, we can apply Lemma 3.5 which tells us that the stabiliser cannot contain $\phi\delta$, so we must have the class stabiliser is $\langle\phi\rangle$ as claimed. □

4.3 Summary

We summarise the results of the previous section, using the same convention used in [1] in Section 4.10.

Theorem 4.12. *Let G and Ω be as in the convention in [1] with $\Omega = \Omega_{17}^{\circ}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_1$-maximal subgroups of G are described in the list below:*

Proof. See Section 4.2 □

(1) $S = A_{19}.2 < \Omega_{17}^{\circ}(19)$ with trivial class stabiliser.

(2) $S = A_{18}.2 < \Omega_{17}^{\circ}(p)$ with $p \neq 2, 3, 19$, with trivial class stabiliser.

(3) $S = L_2(16).4$ with $p = \pm 1 \mod 8$, or $S = L_2(16).2$ with $p \neq 3$ and $p = \pm 3 \mod 8$. The group $S < \Omega_{17}^{\circ}(p)$ for all such p. The class stabiliser is trivial when $S = L_2(16).4$ and $\langle\delta\rangle$ when $S = L_2(16).2$.

(4) Results relating to $L_2(16)_2$ which are not yet complete.

(5) $S = L_2(16)$ with $p \neq 2, 3, 5$. We have $S < \Omega_{17}^{\circ}(p)$ if $p = \pm 1 \mod 15$, $S < \Omega_{17}^{\circ}(p^2)$ if $p = \pm 4 \mod 15$, and $S < \Omega_{17}^{\circ}(p^4)$ if $p = \pm 2, \pm 7 \mod 15$. The class stabiliser is trivial if $p = \pm 1 \mod 15$ and $\langle\phi\rangle$ otherwise.

5 $\mathcal{S}_1$ candidates in dimension 16

5.1 Candidates

Table 2 contains the candidates in dimension 16; these were found using the same method used to construct Table 1. The table is ordered first by Schur indicator and then by the order of the group from smallest to largest.

Table 2: $\mathcal{S}_1$ candidates in dimension 16

Group	PmDivs	Out	Dim	Ind	# ρ	Stab	Charc	Ch Ring
$2.L_2(17)$	2,3,**17**	2	16	-	1	2	0	—
$2.L_2(17)$	2,3,**17**	2	16	-	3	2	0	y_9
$2.A_7$	2,3,5,7	2	16	-	1	2	7	—
$2.A_8$	2,3,5,7	2	16	-	1	2	7	—
A_{18}	2,3,5,7,11,13,17	2	16	-	1	2	2	—
$L_2(17)$	2,3,**17**	2	16	+	1	2	$0, 2, 3$	—
$L_2(17)$	2,3,**17**	2	16	+	3	2	$0, 2$	y_9
$L_2(16)$	**2**,3,5,17	4	16	+	1	4	$0, 3, 5$	—
$L_3(3)$	2,**3**,13	2	16	+	1	2	13	—
M_{11}	2,3,5,11	1	16	+	1	1	11	—
2.Sz(8)	2,5,7,13	1	16	+	1	1	13	y_7
M_{12}	2,3,5,11	2	16	+	1	2	11	—
A_{10}	2,3,5,7	2	16	+	1	2	2	—
$2.A_{10}$	2,3,5,7	2	16	+	1	2	$0, 3, 7$	—
$2.A_{11}$	2,3,5,7,11	2	16	+	1	2	11	—
A_{17}	2,3,5,7,11,13,17	2	16	+	1	2	$0, 2, 3, 5, 7, 11, 13$	—
A_{18}	2,3,5,7,11,13,17	2	16	+	1	2	3	—
$L_3(3)$	2,**3**,13	2	16	$\circ$	4	1	$0, 2$	d_{13}
M_{11}	2,3,5,11	1	16	$\circ$	2	1	$0, 2, 5$	b_{11}
$2.L_2(31)$	2,3,5,**31**	2	16	$\circ$	2	1	$0, 3, 5$	b_{31}
$4_2.L_3(4)$	**2**,3,5,7	2^2	16	$\circ$	2	2_2	3	i, b_7
M_{12}	2,3,5,11	2	16	$\circ$	2	1	$0, 2, 5$	b_{11}
$4.M_{22}$	2,3,5,7,11	2	16	$\circ$	2	1	7	i, b_{11}
A_{11}	2,3,5,7,11	2	16	$\circ$	2	1	2	b_{11}
$2.A_{11}$	2,3,5,7,11	2	16	$\circ$	2	1	$0, 3, 5, 7$	b_{11}
A_{12}	2,3,5,7,11	2	16	$\circ$	2	1	2	z_3, b_{11}, b_{35}
$2.A_{12}$	2,3,5,7,11	2	16	$\circ$	2	1	3	i_2, i_5, b_{11}, b_{35}

5.2 Results

A reminder that MAGMA files referenced can be found at
`http://www2.warwick.ac.uk/fac/sci/maths/people/staff/rogers/talks/`.

5.2.1 Symplectic case

We now determine the $\mathcal{S}_1$-maximal subgroups of $\mathrm{Sp}_{16}(q)$ and its almost simple extensions. From Table 2 the candidates are $2.L_2(17)$ (twice, both with $p \neq 2, 3, 17$), $2.A_7$ ($p = 7$), $2.A_8$ ($p = 7$) and A_{18} ($p = 2$). We will denote the twp different weak equivalence classes of $2.L_2(17)$ by $2.L_2(17)_1$ and $2.L_2(17)_2$ respectively, depending on the order they appear in the table. We will consider these in reverse order.

Remark 5.1. The outer automorphism group of $\mathrm{Sp}_{16}(p^e)$ is given by the presentation

$$\langle \delta, \phi | \delta^{(q-1,2)} = \phi^e = [\delta, \phi] = 1 \rangle \cong C_{(q-1,2)} \times C_e$$

Here δ is the *diagonal automorphism* induced (when $p \neq 2$) by conjugation by the element $\delta = (\omega, ..., \omega, 1, ..., 1)$ with eight ω's and eight 1's, where ω is a generator of the multiplicative group $\mathbb{F}_{p^e}^*$. δ scales the preserved form by a factor of ω and hence lies in $\mathrm{CSp}_{16}(p^e) \setminus \mathrm{Sp}_{16}(p^e)$. ϕ is the *field automorphism* given by applying the map $x \mapsto x^p$ to each entry in the matrix.

Proposition 5.2. *Let $G = A_{18} < \Omega = \mathrm{Sp}_{16}(q)$, $q = p^e$ and $S = N_\Omega(G)$. Then $q = p = 2$, $S = G.2$ and we have trivial class stabiliser. The group S is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Out}\bar{\Omega}$ class of groups S, and for no other q is there a subgroup of $\mathrm{Sp}_{16}(q)$ isomorphic to A_{18}.*

Proof. Table 2 shows that we only have a 16-dimensional character in characteristic 2, and since the character ring is trivial we have that $q = p$ here. Since A_{18} in characteristic 2 is not in [5], a short computer calculation in `a18d16f2calc` shows that $G.2$ preserves a symplectic form, and hence $G.2 < \mathrm{Sp}_{16}(2)$. Since we have a single representation, and since the outer automorphism group of $\mathrm{Sp}_{16}(2)$ is trivial, the rest of the results follow. Lagrange rules out all possible containments. □

Proposition 5.3. *Let $\Omega = \mathrm{Sp}_{16}(q)$ and $q = p^e$.*

(i) $2.A_7$ is not $\mathcal{S}_1$-maximal in Ω for any q.

(ii) Let $G = 2.A_8$. Then $G < \Omega$ for $q = p = 7$ only, $N_{\bar{\Omega}}(G) = G$ and there is a unique conjugacy class of subgroups of $\bar{\Omega}$ isomorphic to G, with class stabiliser $\langle \delta \rangle$. The group G is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Out}\bar{\Omega}$ class of groups G, and for no other q is there a subgroup of $\mathrm{Sp}_{16}(q)$ isomorphic to $2.A_8$.

Proof. Both $2.A_7$ and $2.A_8$ exist in dimension 16 only as a single representation in characteristic 7, with trivial character ring, and the extensions $2.A_7.2$ and $2.A_8.2$ both require the irrationalities r_5 and r_6, neither of which exist in $\mathbb{F}_7$. Hence we have that both are only defined when $q = p = 7$, with $2.A_7.2, 2.A_8.2 \not\leq \mathrm{Sp}_{16}(7)$. For $G = 2.A_8$, this tells us that $N_{\bar{\Omega}}(G) = G$, meaning we have a unique $\bar{\Omega}$ class of subgroups of $S_{16}(q)$ isomorphic to G, with class stabiliser $\langle \delta \rangle$.

We clearly have $2.A_7 < 2.A_8$ as abstract groups, and the character values found in [5] show that the restriction of the representation of $2.A_8$ to $2.A_7$ is the representation we are considering here, so we have a containment of $\mathcal{S}_1$-maximal subgroups. □

Proposition 5.4. *We do not have an abstract containment $2.L_2(17) < A_{18}$.*

Proof. By looking at the maximal subgroups of $2.L_2(17)$ we see that the largest subgroup is of index 18. Thus this subgroup must contain the central element of order 2 in $2.L_2(17)$, giving us a permutation representation on 18 points of $L_2(17)$ but not $2.L_2(17)$. Hence the result follows. □

Proposition 5.5. *Let $G = 2.L_2(17)_1 < \Omega = \mathrm{Sp}_{16}(q)$ with $q = p^e$, and $S = N_{\bar{\Omega}}(G)$. Then:*

(i) If $p = \pm 1 \mod 12$ then $q = p$, $S = G.2$ and there are two $\bar{\Omega}$-classes of subgroups of Ω isomorphic to S, with trivial class stabiliser

(ii) If $p = \pm 5 \mod 12$ and $p \neq 17$ then $q = p$, $S = G$ and there is a single $\bar{\Omega}$-class of subgroups of Ω isomorphic to G, with class stabiliser $\langle \delta \rangle$.

In both cases the group S is $\mathcal{S}_1$-maximal, there is a unique $\mathrm{Aut}\bar{\Omega}$ class of such groups G, and the only other cases where we may have a containment of G in $\mathrm{Sp}_{16}(q)$ are those covered in Proposition 5.6.

Proof. Table 2 gives us that $q = p \neq 2, 3, 17$. From [2] we see that there is a single such representation, and for this representation that elements of $G.2 \setminus G$ are isometries, but their character values involve r_3, which lies in $\mathbb{F}_p$ if and only if $p = \pm 1 \mod 12$. Multiplying elements of $G.2 \setminus G$ by a suitable scalar would give us the isoclinic group $G.2^-$, but this will not consist of isometries of the form.

Then standard results give the number of $\bar{\Omega}$ classes and the class stabilisers. The only possible containment involving S is dealt with in Proposition 5.4. □

Proposition 5.6. *Let $G = 2.L_2(17)_2 < \mathrm{Sp}_{16}(q) = \Omega$ and let $S = N_\Omega(G)$. Then provided $p \neq 2, 3, 17$ we have the following:*

- *If $p = 1, 35 \mod 36$ then $q = p$ and $S = G.2$. There are three $\mathrm{Out}\bar{\Omega}$-classes of subgroups, each dividing into 2 conjugacy classes of subgroups of $\bar{\Omega}$ isomorphic to G with trivial class stabiliser.*
- *If $p = 11, 13, 23, 25 \mod 36$ then $q = p^3$ and $S = G.2$. There is a single $\mathrm{Out}\bar{\Omega}$-class of subgroups of Ω, splitting into 6 conjugacy classes of subgroups of $\bar{\Omega}$ isomorphic to G, with trivial class stabiliser.*
- *If $p = 17, 19, \mod 36$ then $q = p$ and $S = G$. There are three $\mathrm{Out}\bar{\Omega}$ classes of subgroups, and 3 conjugacy classes of subgroups of $\bar{\Omega}$ isomorphic to G, each with class stabiliser $\langle\delta\rangle$.*
- *If $p = 5, 7, 29, 31 \mod 36$ then $q = p^3$ and $S = G$. There is a single $\mathrm{Out}\bar{\Omega}$ class of subgroups, and 3 conjugacy class of subgroups of $\bar{\Omega}$ isomorphic to G, each with class stabiliser $\langle\delta\rangle$.*

The group S is $\mathcal{S}_1$-maximal and the only other cases where we may have a containment of G in $\mathrm{Sp}_{16}(q)$ are those covered in Proposition 5.5.

Proof. From Table 2 we have that the character ring involves the irrationality y_9, which is defined over $\mathbb{F}_p$ if $p = 1, 3, 8 \mod 9$ and $\mathbb{F}_{p^3}$ otherwise. From [2] we can see that the representation of $G.2$ consists of isometries (since it has Schur indicator -) but the character values on elements of $G.2 \setminus G$ involve y_{36}, which, given that $p \neq 2, 3$, exists over $\mathbb{F}_q$ iff $q = \pm 1 \mod 36$; hence we obtain the given conditions on p.

The three representations are fixed by δ and permuted by ϕ if it is non-trivial. Depending on whether the 2 automorphism is realisable over Ω or not, each of these representations divides into 1 or 2 conjugacy classes. If there are two classes then δ permutes them, and so the results on class stabilisers follow. The only possible containment involving S is dealt with in Proposition 5.4. □

5.2.2 Orthogonal case

We now determine the $\mathcal{S}_1$-maximal subgroups of $\Omega^{\pm}_{16}(q)$ and its almost simple extensions. From Table 2 the candidates are $L_2(17)$ (twice; one with $p \neq 17$ and one with $p \neq 3, 17$), $L_2(16)$ ($p \neq 17$), $L_3(3)$ ($p = 13$), M_{11} ($p = 11$), $2.\mathrm{Sz}(8)$ ($p = 13$), M_{12} ($p = 11$), A_{10} ($p = 2$), $2.A_{10}$ ($p \neq 2, 5$), $2.A_{11}$ ($p = 11$), A_{17} ($p \neq 17$) and A_{18} ($p = 3$). We denote the two copies of $L_2(17)$ by $L_2(17)_1$ and $L_2(17)_2$ depending on the order they appear in Table 2. We will consider these groups in reverse order.

Remark 5.7. Note that in the orthogonal + case here, we have that for $q = p^e$, $\mathrm{OutO}^+_{16}(q) = \langle \delta', \gamma, \delta, \phi | \delta'^2 = \gamma^2 = \delta^2 = 1, (\gamma\delta)^2 = \delta', \phi^2 = [\delta, \phi] = [\gamma, \phi] = 1 \rangle \cong D_8 \times C_e$. In [1] the generators are often given as conjugation by a specific matrix, but for the purposes of determining the class stabilisers here it suffices to note that the automorphism δ' is induced by conjugation by a matrix in $\mathrm{SO}^+_{16}(q) \setminus \mathrm{O}^+_{16}(q)$, γ is induced by an element of $\mathrm{GO}^+_{16}(q) \setminus \mathrm{SO}^+_{16}(q)$, δ by an element of $\mathrm{CGO}^+_{16}(q) \setminus \mathrm{GO}^+_{16}(q)$ and ϕ (where non-trivial) by an element of $\mathrm{C\Gamma O}^+_{16}(q) \setminus \mathrm{CG}^+_{16}(q)$. These split into $5e$ conjugacy classes, and we have that the only conjugacy class that lies in $\mathrm{SO}^+_{16}(q) \setminus \mathrm{O}^+_{16}(q)$ has representative δ', and the only conjugacy class that lies in $\mathrm{GO}^+_{16}(q) \setminus \mathrm{SO}^+_{16}(q)$ has representative γ; thus if a group automorphism lies in either of these sets then we know the class stabiliser without further computation. If the group automorphism lies in $\mathrm{CGO}^+_{16}(q) \setminus \mathrm{GO}^+_{16}(q)$ then we have to decide between the conjugacy classes with representatives δ and $\gamma\delta$. Since $\det\gamma \neq 1$ we can distinguish between these two classes by considering the determinant of the matrix inducing the group automorphism.

In these computations the orthogonal $-$ case is always defined over $q = p$, where we have $\mathrm{Out}\mathrm{O}^-_{16}(p) = \langle\gamma,\delta|\gamma^2 = \delta^2 = [\gamma,\delta] = 1\rangle \cong C_2 \times C_2$, with γ induced by conjugation by an element of $\mathrm{GO}^-_{16}(p)\setminus\mathrm{SO}^-_{16}(p)$ and δ by an element of $\mathrm{CGO}^+_{16}(q) \setminus \mathrm{GO}^+_{16}(q)$. Hence we essentially have a slight simplification of the case in the previous paragraph, where the class stabiliser is γ if the group automorphism lies in $\mathrm{GO}^-_{16}(p) \setminus \mathrm{SO}^-_{16}(p)$, and otherwise we have to decide between γ and $\gamma\delta$ as the class stabiliser, which again we can establish by considering the determinant of the inducing matrix.

Proposition 5.8. *Let $G = A_{18} < \Omega = \Omega^{\pm}_{16}(q)$. Then $q = p = 3$ and $G < \Omega^-_{16}(3)$. There are two Ω-classes of subgroups isomorphic to G, with class stabiliser $\langle\gamma\rangle$. G is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Aut}\bar{\Omega}$ class of groups G and for no other q is there a $\mathcal{S}_1$-subgroup of $\Omega^{\pm}_{16}(q)$ isomorphic to A_{18}.*

Proof. From Table 2 we see that this representation in dimension 16 is only defined in characteristic 3. The character ring is trivial and so, with a brief computer calculation found in `a18d16calc` we can see that $G < \Omega^-_{16}(3)$. We can apply Remark 4.1 to $G.2 = \mathrm{Sym}(18)$ to see that the conjugacy class of $(1,2)$ lies inside $G.2\setminus G$, consists of elements of order 2 and has character value $|\mathrm{Fix}((1,2))|-2 = 14$; this must therefore have eigenvalues consisting of 15 1s and one -1, and so this has determinant -1. Thus $G.2 \not< \mathrm{SO}^-_{16}(3)$. From `a18d16calc` we see that $G.2$ preserves the same form, so $G.2 \setminus G \subset \mathrm{GO}^-_{16}(3) \setminus \mathrm{SO}^-_{16}(3)$. Hence the class stabiliser is $\langle\gamma\rangle$, since the other elements in the outer automorphism group of Ω are δ and $\gamma\delta$, neither of which lie in $\mathrm{GO}^-_{16}(3)$. Thus there are two classes of subgroups of Ω isomorphic to G.

There are no possible containments involving A_{18} so this is $\mathcal{S}_1$-maximal. □

Proposition 5.9. *Let $G = A_{17} < \Omega = \Omega^{\pm}_{16}(q)$, $q = p^e$ and $S = N_{\bar{\Omega}}(G)$. Then $q = p \neq 17$. If $p = \pm1, \pm2, \pm4, \pm8 \mod 17$ then $G < \Omega^+_{16}(p)$ and there are four Ω-classes of subgroups isomorphic to G, whereas if $p = \pm3, \pm5, \pm6, \pm7 \mod 17$ then $G < \Omega^-_{16}(p)$ and there are two Ω-classes of subgroups isomorphic to G. In both cases $S = G$ and we have class stabiliser $\langle\gamma\rangle$. If $p \neq 3$ then the group G is $\mathcal{S}_1$-maximal; if $p = 3$ then G is not $\mathcal{S}_1$-maximal. There is a single $\mathrm{Aut}\bar{\Omega}$ class of groups G and for no other q is there a $\mathcal{S}_1$-subgroup of $\Omega^{\pm}_{16}(q)$ isomorphic to A_{17}.*

Proof. Calculations in `a17d16calc` find a representation of A_{17} in $\mathrm{GL}(16,\mathbb{Z})$ which preserves a form with determinant 17 times a square; hence the type of form preserved by G depends on whether 17 is a square modulo p, which leads us to a mod 17 dependence on p. Table 2 gives us that $q = p \neq 17$. The calculation also shows that $G.2 \setminus G \subset \mathrm{GO}^{\pm}_{16}(q) \setminus \mathrm{SO}^{\pm}_{16}(q)$; hence by Remark 5.7 the class stabiliser is $\langle\gamma\rangle$.

There is an abstract containment $A_{17} < A_{18}$, with the former a point stabiliser of the latter. From the above and Proposition 5.8 we see that characteristic 3 is the only characteristic where both these groups exist in dimension 16, and both preserve an orthogonal minus form. From Remark 4.1 it is clear that the characters will be the same, and so we have a containment here. There are no other possible containments. □

Proposition 5.10. *Let $\Omega = \Omega^{\pm}_{16}(q)$ and let $G = 2{}^{\cdot}A_{11}$ be a $\mathcal{S}_1$-subgroup of $\bar{\Omega}$. Then $\Omega = \Omega^+_{16}(11)$ with $S := N_{\bar{\Omega}}(G) = G.2$, and eight Ω-classes of subgroups isomorphic to G, with trivial class stabiliser. The group S is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Aut}\bar{\Omega}$-class of groups S, and for no other q is there subgroups of $\Omega^{\pm}_{16}(q)$ isomorphic to G.*

Proof. The condition on q is direct from Table 2; the rest of the claims, excluding maximality, follow from a computer calculation in `2a11d16f11calc`.

For maximality, the only potential abstract containment is $2.A_{11} < A_{17}$. If $2.A_{11}$ could be expressed as a subgroup of A_{17} then we would have a corresponding 17-dimensional representation of $2.A_{11}$ given by the number of fixed points for a representative of each conjugacy class. However the only representation of $2.A_{11}$ of degree at most 17 is the 16-dimensional representation we are considering, so the only possibility for a character of such a representation is this 16-dimensional character plus the trivial character. However this character takes negative values, for instance on class 3A, so we cannot have a containment here. Hence S is $\mathcal{S}_1$-maximal. □

Proposition 5.11. *Let $G = 2{}^{\cdot}A_{10} < \Omega = \Omega^{\pm}_{16}(q)$ and $S = N_{\Omega}(G)$. Then $\Omega = \Omega^{+}_{16}(q)$, and $q = p$. Also:*

- *If $p = \pm 1 \mod 10$ then $S = G.2$ with eight Ω-classes isomorphic to G and trivial class stabiliser.*
- *If $p = \pm 3 \mod 10$ then $S = G$ with four Ω-classes isomorphic to G and class stabiliser $\langle \delta \rangle$.*

There is a single $\mathrm{Out}\bar{\Omega}$-class of subgroups isomorphic to S. If $p = 11$ then S is not $\mathcal{S}_1$-maximal, whilst if $p \neq 11$ then S is $\mathcal{S}_1$-maximal. For no other q is there a $\mathcal{S}_1$-subgroup of $\Omega^{\pm}_{16}(q)$.

Proof. A computer calculation in `2a10d16calc` shows that the form preserved is always of + type. From the character tables in [2] and [5] we see that the 16-dimensional irreducible character of $G.2 \setminus G$ takes the value 0 on all conjugacy classes except one, which requires the irrationality r_5; and elements of $G.2 \setminus G$ are isometries of the form. In particular, there is an element of order 2 in $G.2 \setminus G$ with character value 0, requiring 8 1s and 8 -1s as eigenvalues; hence this extension has determinant 1, so when r_5 exists we have $G.2 < \mathrm{SO}^{+}_{16}(q)$; this happens precisely when $p = \pm 1 \mod 10$. Multiplying elements of $G.2 \setminus G$ by a scalar element of order 4 would give us the isoclinic group $G.2^{-}$, but this does not preserve the form; thus when $p = \pm 3 \mod 10$, $G.2 \setminus G \subset \mathrm{CGO}^{+}_{16}(p) \setminus \mathrm{GO}^{+}_{16}(p)$.

The calculations in `2a10d16calc` also construct a matrix $L \in \mathrm{GL}_{10}(\mathbb{Q})$ that normalises G and induces the unique nontrivial outer automorphism. L has square determinant and transforms the form F to $5F$. L also has spinor norm 1. Hence, if r_5 exists over $\mathbb{F}_p$ then L can be multiplied by a scalar matrix to give us that $G.2 \setminus G \subset \Omega$. Otherwise, L lies in $\mathrm{CGO}^{+}_{16}(p) \setminus \mathrm{GO}^{+}_{16}(p)$, and by comparing the determinant of L and the action of L on F to those of matrices inducing outer automorphisms of G, we see by Remark 5.7 that in the case $G = S$ we have class stabiliser $\langle \delta \rangle$.

As there is a single representation, there is a single $\mathrm{Out}\bar{\Omega}$ class of subgroups.

Lagrange gives us possible abstract containments of $2.A_{10}$ in $2.A_{11}$, A_{17} and A_{18}. We can rule out the latter two cases by examining the maximal subgroups of $2.A_{10}$ - the only subgroup of index less than 18 has index 10 and contains the central element of $2.A_{10}$; hence $2.A_{10}$ acts on the 10 conjugacy classes of this subgroup as A_{10}. Thus the smallest possible permutation representation for $2.A_{10}$ would be on 20 points (and in fact for that permutation representation to exist we would require this maximal subgroup to have a centreless subgroup of index 2, which it does not).

There are, however, abstract containments $2.A_{10} < 2.A_{11}$ and $2.A_{10}.2 < 2.A_{11}.2$. In dimension 16 this occurs only in characteristic 11. Since $2.A_{10}$ has no faithful irreducible representations of degree smaller than 16, and all groups we are considering are contained in $\Omega^{+}_{16}(11)$, we have a containment here. □

Proposition 5.12. *Let $G = A_{10} < \Omega = \Omega^{\pm}_{16}(q)$, and $S = N_{\bar{\Omega}}(G)$. Then $q = p = 2$, $G < \Omega^{+}_{16}(2)$ and $S = G.2$. There are two Ω-classes of subgroups isomorphic to G, with trivial class stabiliser. G is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Aut}\bar{\Omega}$ class of subgroups isomorphic to S and for no other q is there a $\mathcal{S}_1$-subgroup of $\Omega^{\pm}_{16}(q)$ isomorphic to G.*

Proof. That G only exists in dimension 16 in characteristic 2 is direct from Table 2. The rest of the claims (except those regarding maximality) follow from computer calculations in `a10d16f2`.

We have an abstract containment $A_{10} < A_{17}$, but considering the characteristic 2 representations we see that the character values on, for instance, elements of order 3 do not correspond. (In fact the restriction of the 16-dimensional character of A_{17} to A_{10} is the sum of the 8-dimensional character of A_{10} with 8 copies of the trivial character). There are no other possible containments. □

Proposition 5.13. *Let $\Omega = \Omega^{\pm}_{16}(q)$ and let $G = M_{12}$ be a $\mathcal{S}_1$-subgroup of $\bar{\Omega}$. Then $\Omega = \Omega^{+}_{16}(11)$ with $N_{\bar{\Omega}}(G) = G$ and four Ω-classes of subgroups isomorphic to G, with class stabiliser $\langle \gamma \rangle$. The group G is*

$\mathcal{S}_1$*-maximal, there is a single* $\mathrm{Aut}\bar{\Omega}$*-class of groups* G*, and for no other* q *is there subgroups of* $\Omega^{\pm}_{16}(q)$ *isomorphic to* G*.*

Proof. A straightforward computer calculation using Table 2 and the group in [9] shows that $\Omega = \Omega^+_{16}(11)$. From [5] we see that $G.2 \setminus G$ has character ring contained in $\mathbb{Z}[r_3, r_5]$. Both r_3 and r_5 exist in $\mathbb{F}_{11}$, so we have $G.2 < \mathrm{GO}^+_{16}(11)$. Also from the character table, we see that elements of $G.2 \setminus G$ have determinant -1, and hence $G.2 \setminus G \in \mathrm{GO}^+_{16}(11) \setminus \mathrm{SO}^+_{16}(11)$. Thus we have by Remark 5.7 that the class stabiliser is $\langle\gamma\rangle$. There is a single representation, and hence a single $\mathrm{Aut}\bar{\Omega}$ class.

There is an abstract containment $M_{12} < A_{17}$ in characteristic 11. However, the 11-dimensional characters of M_{12} in characteristic 11 are clearly deleted permutation modules from the standard definition of M_{12} on 12 points, and the 16-dimensional character of A_{17} is also a deleted permutation module. Hence it is clear that the restriction of the 16-dimensional character of A_{17} to M_{12} is given by one of the 11-dimensional characters plus 5 copies of the trivial character, and there is no containment here. □

Proposition 5.14. *Let* $\Omega = \Omega^{\pm}_{16}(q)$ *and let* $G = 2^{\cdot}\mathrm{Sz}(8)$ *be a* $\mathcal{S}_1$*-subgroup of* $\bar{\Omega}$*. Then* $\Omega = \Omega^+_{16}(13)$ *with* $N_{\bar{\Omega}}(G) = G$ *and four* Ω*-classes of subgroups isomorphic to* G*, with trivial class stabiliser. The group* G *is* $\mathcal{S}_1$*-maximal, there is a single* $\mathrm{Aut}\bar{\Omega}$*-class of groups* G*, and for no other* q *is there subgroups of* $\Omega^{\pm}_{16}(q)$ *isomorphic to* G*.*

Proof. All claims are straightforward from Table 2, since G has a trivial outer automorphism group and we only have a single representation to consider. The result on the characteristic comes from the table, and the fact that y_7 exists over $\mathbb{F}_{13}$.

Lagrange leaves us with A_{17} as the only possible group which may contain 2.Sz(8); however, an analysis of the maximal subgroups of G shows that there are no subgroups of index less than 65, so we certainly cannot have a degree 17 permutation representation of G. Hence G is maximal. □

Proposition 5.15. *There are no maximal subgroups involving the group* M_{11}*.*

Proof. From Table 2 we see that the 16-dimensional irreducible representation of M_{11} is only defined in characteristic 11. There is an abstract containment of M_{11} in M_{12} (M_{11} is a point stabiliser of M_{12} as a permutation on 12 points). From the character tables in [5], the 16-dimensional representation of M_{12} has character value 0 on elements of order 8, which is also the case for the 16-dimensional representation of M_{11}, but not for any other 16-dimensional representation consisting of linear combinations of smaller-degree characters. Hence we have a containment here, and since the representation stabiliser of M_{11} is trivial, we are done. □

Proposition 5.16. *Let* $\Omega = \Omega^{\pm}_{16}(q)$ *and let* $G = L_3(3)$ *be a* $\mathcal{S}_1$*-subgroup of* $\bar{\Omega}$*. Then* $\Omega = \Omega^+_{16}(13)$ *with* $N_{\bar{\Omega}}(G) = G$ *and four* Ω*-classes of subgroups isomorphic to* G*, with class stabiliser* $\langle\gamma\rangle$*. The group* G *is* $\mathcal{S}_1$*-maximal, there is a single* $\mathrm{Aut}\bar{\Omega}$*-class of groups* G*, and for no other* q *is there subgroups of* $\Omega^{\pm}_{16}(q)$ *isomorphic to* G*.*

Proof. From Table 2 we get the result on the characteristic of Ω, from the matrix in [9] a straightforward Magma calculation shows that the orthogonal form preserved is of + type, and from [5] we see that the character ring of $G.2$ is $\mathbb{Z}[r_3]$. r_3 exists in $\mathbb{F}_{13}$ so that $G.2 < \mathrm{GO}^+_{16}(13)$. From the character table we see that elements of $G.2 \setminus G$ have determinant -1, so that $G.2 \setminus G \subset \mathrm{GO}^+_{16}(13) \setminus \mathrm{SO}^+_{16}(13)$. Hence by Remark 5.7 we have that the class stabiliser is $\langle\gamma\rangle$. There a single representation, and hence a single $\mathrm{Aut}\bar{\Omega}$ class.

The only possible containment we have is $L_3(3) < A_{17}$ in characteristic 13. However $L_3(3)$ has a natural definition as a subgroup of A_{13}, and embedding this in the natural way into A_{17} we see that, whilst the 16-dimensional character of $L_3(3)$ takes value 0 on elements of order 2, the restriction of the 16-dimensional character of A_{17} has value 8. In fact, similarly to the example in Proposition 5.13 this character is the 11-dimensional character plus 5 copies of the trivial character. Hence there is no containment here. □

Proposition 5.17. *There are no $\mathcal{S}_1$-maximal subgroups involving $G = L_2(16) < \Omega = \Omega_{16}^{\pm}(q)$.*

Proof. Computer calculations in `l216d16calc` find a representation of $L_2(16)$ in $\mathrm{GL}(16, \mathbb{Z})$ which preserves a form with determinant 17 times a square; hence the type of form preserved by G depends on whether 17 is a square modulo p, which leads us to a mod 17 dependence on p. Table 2 gives us that $q = p \neq 17$. The calculation also shows that $G.4 < \Omega$ regardless of the value of p, giving us a trivial class stabiliser. We have an abstract containment $L_2(16) < A_{17}$ and A_{18} (and these are the only possible potential containments), and in fact $L_2(16).4 = \mathrm{P\Gamma L}(2, 16)$ has a representation on 17 points. Computing this character table in Magma shows that there are no faithful representations of degree smaller than 16, so we have a containment $L_2(16).4 < A_{17}$. Since the representation of A_{17} is also defined when $p \neq 17$, the result follows. □

Proposition 5.18. *Let $\Omega = \Omega_{16}^{\pm}(q)$ and let $G = L_2(17)_1$ be a $\mathcal{S}_1$-subgroup of $\bar{\Omega}$. Then $q = p \neq 17$. If $p = \pm1, \pm2, \pm4, \pm8 \mod 17$ then $G < \Omega_{16}^{+}(p)$ with four Ω-classes of subgroups isomorphic to G, whereas if $p = \pm3, \pm5, \pm6, \pm7 \mod 17$ then $G < \Omega_{16}^{-}(p)$ with two Ω-classes of subgroups isomorphic to G. In both cases $N_{\bar{\Omega}}(G) = G$ and we have class stabiliser $\langle\gamma\rangle$. If $p \neq 3$ then the group G is $\mathcal{S}_1$-maximal; if $p = 3$ then G is not $\mathcal{S}_1$-maximal. There is a single $\mathrm{Aut}\Omega$ class of subgroups isomorphic to G and the only other situation where there are subgroups of $\Omega_{16}^{\pm}(q)$ isomorphic to G is described in Proposition 5.19.*

Proof. Calculations in `l2171d16calc` find a representation of $L_2(17)_1$ in $\mathrm{GL}(16, \mathbb{Z})$ which preserves a form with determinant 17 times a square; hence the type of form preserved by G depends on whether 17 is a square modulo p, which leads us to a mod 17 dependence on p. Table 2 gives us that $q = p \neq 17$. The calculation also shows that $G.2 \setminus G \subset \mathrm{GO}_{16}^{\pm}(q) \setminus \mathrm{SO}_{16}^{\pm}(q)$; hencehence by Remark 5.7 the class stabiliser is $\langle\gamma\rangle$.

Lagrange rules out all possible containments except for $L_2(17) < A_{17}$ or A_{18}. We do not have $L_2(17) < A_{17}$ as $L_2(17)$ has no permutation representation on fewer than 18 points; hence the only possible containment is $L_2(17)_1 < A_{18}$, which can only occur in characteristic 3. Looking at [5] and the standard definition of $L_2(17)$ as a permutation group on 18 points, we see that the value of the 16-dimensional character $L_2(17)_1$ on an element g (with order coprime to 3) corresponds with $|\mathrm{Fix}(g)| - 2$, the character value of the 16-dimensional representation of A_{18} in characteristic 3, so we have a containment here. A similar check with $\mathrm{PGL}(2, 17) = L_2(17).2$ shows that we also have a containment $L_2(17)_1.2 < A_{18}.2$, and hence $L_2(17)_1$ is not maximal when $p = 3$. □

Proposition 5.19. *Let $\Omega = \Omega_{16}^{\pm}(q)$ and let $G = L_2(17)_2$ be a $\mathcal{S}_1$-subgroup of $\bar{\Omega}$. Then $N_{\bar{\Omega}}(G) = G$ and we have class stabiliser $\langle\gamma\rangle$. If $p = \pm1 \mod 9$ then $q = p$ and there are three $\mathrm{Aut}\bar{\Omega}$ classes of subgroups isomorphic to G, whereas if $p = \pm2, \pm4 \mod 9$ then $q = p^3$ and there is a single $\mathrm{Aut}\bar{\Omega}$ class of subgroups isomorphic to G. If $p = \pm1, \pm2, \pm4, \pm8 \mod 17$ then $G < \Omega_{16}^{+}(q)$ and we have twelve Ω-classes of subgroups isomorphic to G, whereas if $p = \pm3, \pm5, \pm6, \pm7 \mod 17$ then $G < \Omega_{16}^{-}(q)$ and we have six Ω-classes of subgroups isomorphic to G. G is $\mathcal{S}_1$-maximal and the only other situation where there are subgroups of $\Omega_{16}^{\pm}(q)$ isomorphic to G is described in Proposition 5.18.*

Proof. Table 2 show that the character ring of $L_2(17)_2$ involves the irrationality y_9, which gives us the shape of the field depending on $p \mod 9$. The field automorphisms permute the three representations if they exist, whilst all other outer automorphisms of $\bar{\Omega}$ fix all three representations, so the result on $\mathrm{Aut}\bar{\Omega}$ follows.

Calculations in `l2172d16calc` find a representation of $L_2(17)_2$ in $\mathrm{GL}(16, \mathbb{Z})$ which preserves a form with determinant 17 times a square; hence the type of form preserved by G depends on whether 17 is a square modulo p, which leads us to a mod 17 dependence on p. The calculation also shows that $G.2 \setminus G \subset \mathrm{GO}_{16}^{\pm}(q) \setminus \mathrm{SO}_{16}^{\pm}(q)$; hence by Remark 5.7 the class stabiliser is $\langle\gamma\rangle$.

The existence of the irrationality y_9 in the character ring for $L_2(17)_2$ but for no other + type representations ensures that $L_2(17)_2$ is $\mathcal{S}_1$-maximal. □

5.2.3 Linear and unitary cases

We now determine the $\mathcal{S}_1$-maximal subgroups of $\mathrm{SL}_{16}^{\pm}(q)$ and its almost simple extensions. (Recall that this notation is a compact way of referring to both linear and unitary groups.) From Table 2 the candidates are $L_3(3)$ $(p \neq 3, 13)$, M_{11} $(p \neq 11)$, $2.L_2(31)$ $(p \neq 2)$, $4_2.L_3(4)$ $(p = 3)$, M_{12} $(p \neq 3, 11)$, $4.M_{22}$ $(p = 7)$, A_{11} $(p = 2)$, $2.A_{11}$ $(p \neq 2, 11)$, A_{12} $(p = 2)$ and $2.A_{12}$ $(p = 3)$. We will consider these in reverse order.

We will need the following irrationalities not described in [1]:

Remark 5.20.

- b_{35} has minimal polynomial $X^2 - X + 9$, is not real, and we have that p-modular reductions have degree $\begin{cases} 1 & \text{if } p = 0,1,3,4,5,7,9,11,12,13,16,17,27,29,33 \mod 35 \\ 2 & \text{if } p = 2,6,8,18,19,22,23,24,26,31,32,34 \mod 35 \end{cases}$, which comes from the Legendre symbol $\left(\dfrac{35}{p}\right)$.
- b_{31} has minimal polynomial $X^2 + X + 8$, is not real, and we have that p-modular reductions have degree $\begin{cases} 1 & \text{if } p = 0,1,2,4,5,7,8,9,10,14,16,18,19,20,25,28 \mod 31 \\ 2 & \text{if } p = 3,6,11,12,13,15,17,21,22,23,24,26,27,29,30 \mod 31 \end{cases}$, which comes from the Legendre symbol $\left(\dfrac{31}{p}\right)$.
- d_{13} has minimal polynomial $X^4 + X^3 + 2X^2 - 4X + 3$, is not real, and we have that p-modular reductions have degree $\begin{cases} 1 & \text{if } p = 0,1,3,9 \mod 13 \\ 2 & \text{if } p = 4,10,12 \mod 13 \\ 4 & \text{if } p = 2,5,6,7,8,11 \mod 13 \end{cases}$. This is because we have $\mathbb{Q} \subset \mathbb{Q}(d_{13}) \subset \mathbb{Q}(r_{13})$ where r_{13} denotes a primitive 13th root of unity, whose existence depends on p modulo 13. Hence we also have a modulo 13 dependence on the existence of d_{13} and case-by-case analysis produces the result. In fact, existence of d_{13} over $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$ is dependent on whether p is a fourth power or a square respectively modulo 13.

Remark 5.21. The outer automorphism group of $\mathrm{SL}_{16}^{+}(p^e) = \mathrm{SL}_{16}(p^e)$ is given by the presentation

$$\langle \delta, \gamma, \phi | \delta^{(p^e-1,16)} = \gamma^2 = \phi^e = [\gamma, \phi] = 1, \delta^\gamma = \delta^{-1}, \delta^\phi = \delta^p \rangle$$

Here δ is the *diagonal automorphism* induced by conjugation by the element $\delta = (\omega, 1, ..., 1) \in \mathrm{GL}_{16}(p^e) \setminus \mathrm{SL}_{16}(p^e)$ where ω is a generator of the multiplicative group $\mathbb{F}_{p^e}^*$. ϕ is the *field automorphism* given by applying the map $x \mapsto x^p$ to each entry in the matrix. γ is the *duality automorphism* $g \mapsto g^{-T}$ for matrices g.

Remark 5.22. The outer automorphism group of $\mathrm{SL}_{16}^{-}(p^e) = \mathrm{SU}_{16}(p^e)$, defined over the field $\mathbb{F}_{p^{2e}}$ is given by the presentation

$$\langle \delta, \gamma, \phi | \delta^{(p^e+1,16)} = \gamma^2 = 1, \phi^e = \gamma, \delta^\gamma = \delta^{-1}, \delta^\phi = \delta^p \rangle$$

Here δ is the *diagonal automorphism* induced by conjugation by the element $\delta = (\omega^{p^e-1}, 1, ..., 1) \in \mathrm{GL}_{16}(p^e) \setminus \mathrm{SL}_{16}(p^e)$ where ω is a generator of the multiplicative group $\mathbb{F}_{p^{2e}}^*$. ϕ is the *field automorphism* given by applying the map $x \mapsto x^p$ to each entry in the matrix. γ is the *duality automorphism* $g \mapsto g^{-T}$ for matrices g.

To decide whether the representation preserves a unitary form or not, we can use the following lemma:

Lemma 5.23 (Corollary 4.4.2, [1]). *Let G have an n-dimensional representation defined over the field $\mathbb{F} = \mathbb{F}_{q^2}$, with Schur indicator $\circ$ and character ring over $\mathbb{Z}$ generated by quadratic irrationalities $a_1, ..., a_r$, with p-modular reductions to $\mathbb{F}$ given by $\bar{a_1}, ..., \bar{a_r}$. Then $G < \mathrm{SU}_n(q)$ if and only if the $bara_i$ which are realisable over $\mathbb{F}_q$ are precisely those for which the corresponding $a_i \in \mathbb{R}$.*

Proposition 5.24. *Let $G = 2.A_{12}$. Then $G < \mathrm{SL}_{16}(3)$. $N_{\bar{\Omega}}(G) = G$, and we have two Ω-classes of subgroups isomorphic to G, with class stabiliser $\langle\gamma\rangle$. There is a single $\mathrm{Aut}\bar{\Omega}$-class of subgroups G, G is $\mathcal{S}_1$-maximal and for no other q is G a subgroup of $\mathrm{SL}^{\pm}_{16}(q)$.*

Proof. The character ring of $2.A_{12}$ in type $\circ$ depends on the existence of the four irrationalities i_2, i_5, b_{11}, b_{35}. All four of these irrationalities are not real but exist over $\mathbb{F}_3$; hence we have that $2.A_{12} < \mathrm{SL}_{16}(3)$. The two representations are interchanged by the outer automorphism of G and also by the duality automorphism γ; hence we must have the group automorphism is induced by a conjugate of γ. In particular this means that it cannot lie in Ω and so the result on the normaliser follows. There are $2 = (3-1, 16)$ Ω-classes of subgroups isomorphic to G, and computer calculations, found in `2a12d16f3calc`, show that the class stabiliser is $\langle\gamma\rangle$. There are no possible containments. □

Proposition 5.25. *There are no $\mathcal{S}_1$-maximal subgroups involving A_{11} in dimension 16.*

Let $G = A_{12}$. Then $G < \Omega = \mathrm{SU}_{16}(2)$. $N_{\bar{\Omega}}(G) = G$, we have a unique $\bar{\Omega}$-class of subgroups isomorphic to G and G has class stabiliser $\langle\gamma\rangle$. There is a single $\mathrm{Aut}\bar{\Omega}$ class of groups G, G is $\mathcal{S}_1$-maximal and for no other q is there a $\mathcal{S}_1$-subgroup of $\mathrm{SL}^{\pm}_{16}(q)$ isomorphic to G.

Proof. From the table, we see that A_{11} only occurs in characteristic 2, and since $b_{11} \notin \mathbb{R}$ and $b_{11} \notin \mathbb{F}_2$, we have that $A_{11} < \mathrm{SU}_{16}(2)$. A_{12} is also only defined in characteristic 2, and since z_3 and b_{35} are both non-real and not realisable over $\mathbb{F}_2$, we have that $A_{12} < \mathrm{SU}_{16}(2)$ also. There is clearly an abstract containment $A_{11} < A_{12}$, and the representation will either be one of the 16-dimensional ones, or the sum of the 10 dimensional one and 6 copies of the trivial character. However the character values on elements of order 11 must involve the irrationality b_{11} whereas the latter character is integer-valued; hence we must have a containment of $\mathcal{S}_1$-subgroups here. Since the representation stabiliser is trivial, it follows that there are no maximal subgroups involving A_{11}.

Now let $G = A_{12}$. Results on the characteristic and type of form preserved are in the previous paragraph. Table 2 tells us that the representation has trivial stabiliser in $\mathrm{Out}G$ so we get the result on the normaliser. Since q is even and n is a power of 2, δ acts trivially, and since $q = p$ ϕ is also trivial; hence Ω has a single outer automorphism γ which must stabilise the unique $\bar{\Omega}$ class here. □

Proposition 5.26. *Let $G = 2.A_{11} < \Omega = \mathrm{SL}^{\pm}_{16}(q)$ with $q = p^e$ and $S = N_{\bar{\Omega}}(G)$. Then $S = G$ and:*

- *If $p = 1, 3, 4, 5, 9 \mod 11$ and $p \neq 2$ then $q = p$ and $G < \mathrm{SL}_{16}(q)$.*
- *If $p = 2, 6, 7, 8, 10 \mod 11$ then $q = p$ and $G < \mathrm{SU}_{16}(q)$.*

In either case the class stabiliser is $\langle\gamma\rangle$.

When $p = 3$, G is not $\mathcal{S}_1$-maximal; otherwise G is $\mathcal{S}_1$-maximal. There are d $\bar{\Omega}$-classes of subgroups isomorphic to G, where $d = (q-1, 16)$ in the linear case and $(q+1, 16)$ in the unitary case. There is a single $\mathrm{Aut}\bar{\Omega}$-class of subgroups isomorphic to G. For no other q are there $\mathcal{S}_1$-subgroups of $\mathrm{SL}^{\pm}_{16}(q)$ isomorphic to $2.A_{11}$.

Proof. The congruences on p follow from Table 2.

In the file `2a11d16calc` we perform calculations in $\mathrm{SL}_{16}(K)$ where $K := \mathbb{Q}(b_{11})$, which show that the outer automorphism of $2.A_{11}$ is induced by γ composed with a diagonal automorphism of $\mathrm{SL}_{16}(K)$ with determinant a square, telling us that in all cases the class stabiliser is γ.

The only possibility for containments is in characteristic 3 with $2.A_{11} < 2.A_{12}$. Clearly there is an abstract containment of groups, and looking at the character values we see that on elements of order 11, the 16-dimensional representations of both $2.A_{11}$ and $2.A_{12}$ take character values involving the irrationality b_{11}, whereas all representations of $2.A_{11}$ of smaller degree have integer-valued characters. Hence we have a containment in characteristic 3. □

Proposition 5.27. *Let $G = 4.M_{22}$. Then $G < \Omega = \mathrm{SU}_{16}(7)$. $N_{\bar{\Omega}}(G) = G$, we have eight $\bar{\Omega}$-classes of subgroups isomorphic to G, and G has class stabiliser $\langle\gamma\rangle$. G is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Aut}\bar{\Omega}$ class of subgroups G, and for no other q is G a subgroup of $\mathrm{SL}^{\pm}_{16}(q)$.*

Proof. The character ring involves the irrationalities i and b_{11}, neither of which are real, and neither of which exist in $\mathbb{F}_7$; hence we have that $4.M_{22}$ preserves a unitary form. Standard computer calculations found in `4m22d16f7calc` gives the result on the class stabiliser, we have 8=(16,7+1) $\bar{\Omega}$-classes, and the rest of the results are standard. □

Proposition 5.28. *There are no $\mathcal{S}_1$-maximal subgroups involving M_{11}. Let $G = M_{12} < \Omega = \mathrm{SL}^{\pm}_{16}(q)$ with $q = p^e$ and $S = N_{\bar{\Omega}}(G)$. Then $S = G$ and:*

- *If $p = 1, 3, 4, 5, 9 \mod 11$ and $p \neq 3$ then $q = p$ and $G < \mathrm{SL}_{16}(q)$. If $p = 1 \mod 6$ then the class stabiliser is $\langle\gamma\rangle$, whilst if $p = 5 \mod 6$ then the class stabiliser is $\langle\gamma\delta\rangle$.*
- *If $p = 2, 6, 7, 8, 10 \mod 11$ then $q = p$ and $G < \mathrm{SU}_{16}(q)$. If $p = 1 \mod 6$ then the class stabiliser is $\langle\gamma\delta\rangle$, whilst if $p = 5 \mod 6$ then the class stabiliser is $\langle\gamma\rangle$.*

If $p = 2$ then G is not $\mathcal{S}_1$-maximal; otherwise G is $\mathcal{S}_1$-maximal, there are d $\bar{\Omega}$-classes of subgroups isomorphic to G, where $d = (q-1, 16)$ in the linear case and $(q+1, 16)$ in the unitary case, and there is a single $\mathrm{Aut}\bar{\Omega}$-class of such groups G. For no other q are there $\mathcal{S}_1$-subgroups M_{12} of $\mathrm{SL}^{\pm}_{16}(q)$.

Proof. From Table 2 we can see that M_{11} and M_{12} are both defined in the same characteristic, involve the same character ring and have trivial class stabiliser. There is a well-known abstract containment of M_{11} in M_{12}, and looking at the character tables in [2] and [5] shows that the 16-dimensional representation of M_{12} has character values involving the irrationality b_{11} on elements of order 11, whilst the character table for M_{11} shows that the 16-dimensional representation also has character values involving b_{11}, and all smaller-dimensional representations have integer-valued characters. Hence the restriction of the 16-dimensional representation of M_{12} to M_{11} gives the 16-dimensional representation of M_{11} and hence we have no $\mathcal{S}_1$-maximal subgroups involving M_{11}. (In fact there are two classes of subgroups of M_{12} isomorphic to M_{11} which are interchanged by the outer automorphism of M_{12}).

The congruences for M_{12} follow from 2 and the result on S is clear since M_{12} has no stabilising automorphisms.

In the file `m12d16calc` we perform calculations in $\mathrm{SL}_{16}(K)$ where $K := \mathbb{Q}(b_{11})$, which show that the outer automorphism of M_{12} is induced by γ composed with a diagonal automorphism of $\mathrm{SL}_{16}(K)$ with determinant a square multiplied by -3. In the linear case, -3 is a square modulo p iff $p = 1 \mod 6$; if this is the case then the class stabiliser is $\langle\gamma\rangle$, otherwise it is $\langle\gamma\delta\rangle$. In the unitary case we use Proposition 4.6.6 from [1], which tells us that the class stabilisers are the other way round, due to the sign of -3. (We also have 41 occurring as an exceptional prime, which the file deals with separately, but this also behaves as expected). In all cases δ permutes the d classes.

We next consider containments. Lagrange rules out a number of possibilities, and we cannot have $M_{12} < A_{11}$ as M_{12} has no permutation representation on fewer than 12 points. An analysis of the maximal subgroups of $2.A_{12}$ shows that the only possibility of a containment here is if $M_{12} < 2.A_{11}$; but this would imply (using the second isomorphism theorem and quotienting by the centre) that $M_{12} < A_{11}$ so this is not possible. Similar analysis of maximal subgroups of $4.M_{22}$ rules out a containment there as well, leaving A_{12} in characteristic 2 as the only possibility for a containment.

We certainly have an abstract containment $M_{12} < A_{12}$ since M_{12} has a permutation representation on 12 points. The 16-dimensional representation of A_{12} takes values involving b_{11} on elements of order 11, as does the 16-dimensional representation of M_{12} whilst all smaller-dimensional representations of M_{12} are integer-valued on such elements; hence this is a containment. □

Proposition 5.29. *Let $G = 4_2.L_3(4)$. Then $G < \Omega = \mathrm{SU}_{16}(3)$. $S := N_{\bar{\Omega}}(G) = G.2_2$, we have four $\bar{\Omega}$-classes of subgroups isomorphic to S, and S has class stabiliser $\langle\gamma\rangle$. There is a single $\mathrm{Aut}\bar{\Omega}$ class of subgroups G, S is $\mathcal{S}_1$-maximal and for no other q is S a subgroup of $\mathrm{SL}^{\pm}_{16}(q)$.*

Proof. From the table we obtain that the only possibility is $4_2.L_3(4) < \mathrm{SU}_{16}(3)$, since both i and b_7 are not real and not realisable over $\mathbb{F}_3$. From [5] we see that the character ring of $4_2.L_3(4).2_2 \setminus 4_2.L_3(4)$

contains no additional irrationalities, so that $4_2.L_3(4).2_2 < \mathrm{GU}_{16}(3)$. From the character tables we can also see that the element in class 2C has trace 4, and must square to an element of trace 16. Hence it must have 1 (10 times) and -1 (6 times) as its eigenvalues, and hence it has determinant 1. Thus $4_2.L_3(4).2_2 < \mathrm{SU}_{16}(3)$, and so $N_{\bar{\Omega}} = G.2_2$. Computer calculations found in `42l34d16f3calc` confirm that γ stabilises the class, and hence induces the unique outer automorphism of S, while δ permutes the classes.

Lagrange limits the possibility of abstract containments to $2.A_{12}$, and an analysis of the maximal subgroups of $2.A_{12}$ confirms that none of them can contain $4_2.L_3(4).2_2$, so there are no containments in this case. □

Proposition 5.30. *Let $G = 2{}^{\cdot}L_2(31) < \Omega = \mathrm{SL}^{\pm}_{16}(q)$ with $q = p^e$ and $S = N_{\bar{\Omega}}(G)$. Then $S = G$ and:*

- *If $p = 1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28 \mod 31$ and $p \neq 2$ then $q = p$ and $G < \mathrm{SL}_{16}(q)$.*
- *If $p = 3, 6, 11, 12, 13, 15, 17, 21, 22, 23, 24, 26, 27, 29, 30 \mod 31$ then $q = p$ and $G < \mathrm{SU}_{16}(q)$.*

In all cases there are d $\bar{\Omega}$-classes of subgroups isomorphic to G, where $d = (q-1, 16)$ in the linear case and $(q+1, 16)$ in the unitary case. If $p = \pm 1 \mod 8$ then the class stabiliser is $\langle\gamma\rangle$; if $p = \pm 3 \mod 8$ then the class stabiliser is $\langle\gamma\delta\rangle$. G is $\mathcal{S}_1$-maximal in all cases, there is a single $\mathrm{Aut}\bar{\Omega}$-class of such groups G, and for no other q are there $\mathcal{S}_1$-subgroups $2{}^{\cdot}L_2(31)$ of $\mathrm{SL}^{\pm}_{16}(q)$.

Proof. The congruences on p for Ω follow directly from Table 2.

In the file `2l231d16calc` we perform calculations in $\mathrm{SL}_{16}(K)$ where $K := \mathbb{Q}(b_{31})$, which show that the outer automorphism of $2{}^{\cdot}L_2(31)$ is induced by γ composed with a diagonal automorphism of $\mathrm{SL}_{16}(K)$ with determinant twice a square. In the linear case, 2 is a square modulo p iff $p = \pm 1 \mod 8$; hence if this is the case then the class stabiliser is $\langle\gamma\rangle$, otherwise it is $\langle\gamma\delta\rangle$. (Separate calculations are required for $p = 5, 193$). A similar calculation applies in the unitary case to give the same result. In all cases δ permutes the d classes.

$2{}^{\cdot}L_2(31)$ is the only candidate with order divisible by 31 so this is $\mathcal{S}_1$-maximal. □

Proposition 5.31. *Let $G = L_3(3) < \Omega = \mathrm{SL}^{\pm}_{16}(q)$ with $q = p^e$ and $S = N_{\bar{\Omega}}(G)$. Then $S = G$ and:*

- *If $p = 1, 3, 9 \mod 13$ and $p \neq 3$ then $q = p$ and $G < \mathrm{SL}_{16}(q)$. If $p = 1 \mod 6$ then the class stabiliser is $\langle\gamma\rangle$; if $p = 5 \mod 6$ then the class stabiliser is $\langle\gamma\delta\rangle$. There are two $\mathrm{Aut}\bar{\Omega}$-classes of such groups G.*
- *If $p = 4, 10, 12 \mod 13$ then $q = p$ and $G < \mathrm{SU}_{16}(q)$. If $p = 1 \mod 6$ then the class stabiliser is $\langle\gamma\delta\rangle$; if $p = 5 \mod 6$ then the class stabiliser is $\langle\gamma\rangle$. There are two $\mathrm{Aut}\bar{\Omega}$-classes of such groups G.*
- *If $p = 2, 5, 6, 7, 8, 11 \mod 13$ then $q = p^2$ and $G < \mathrm{SU}_{16}(q)$. If $p = 1 \mod 6$ then the class stabiliser is $\langle\gamma\delta\rangle$; if $p = 5 \mod 6$ then the class stabiliser is $\langle\gamma\rangle$. There is a single $\mathrm{Aut}\bar{\Omega}$-class of such groups G.*

In all cases there are $2d$ $\bar{\Omega}$-classes of subgroups isomorphic to G, where $d = (q-1, 16)$ in the linear case and $(q+1, 16)$ in the unitary case. G is $\mathcal{S}_1$-maximal in all cases, and for no other q are there $\mathcal{S}_1$-subgroups $L_3(3)$ of $\mathrm{SL}^{\pm}_{16}(q)$.

Proof. The congruences on p for Ω follow directly from Table 2.

We have a single nontrivial outer automorphism of G, but 4 representations, upon which this automorphism acts as a (2,2)-cycle. Hence, we have two orbits of the action of the conformal group $\mathrm{CGL}^{\pm}_{16}(q)$ on the representations of G. Standard computations confirm that each of these orbits splits into d Ω-classes, with $d = |\delta|$, giving us $2d$ Ω-classes in total.

In the file `l33d16calc` we perform calculations in $\mathrm{SL}_{16}(K)$ where $K := \mathbb{Q}(d_{13})$, which show that the outer automorphism of $L_3(3)$ is induced by γ composed with a diagonal automorphism of $\mathrm{SL}_{16}(K)$

with determinant a square multiplied by -3. In the linear case, -3 is a square modulo p iff $p = 1 \mod 6$; if this is the case then the class stabiliser is $\langle\gamma\rangle$, otherwise it is $\langle\gamma\delta\rangle$. In the unitary case we use Proposition 4.6.6 from [1], which tells us that the class stabilisers are the other way round, based on the sign of -3. In all cases δ permutes the d classes.

In the case where $q = p^2$, we have ϕ which acts as the p-power map on character values. However from [2] we see that the nontrivial outer automorphism of G acts on character values as the p^2-power map on character values. Hence we have a single Aut$\bar{\Omega}$-class here, and so again the computation in the previous paragraph suffices to confirm that the class stabiliser is $\langle\gamma\rangle$ here also.

$L_3(3)$ is the only candidate with order divisible by 13 so this is $\mathcal{S}_1$-maximal. □

5.3 Summary

Theorem 5.32. *Let G and Ω be as in the convention in [1] with $\Omega = \mathrm{Sp}_{16}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_1$-maximal subgroups of G are described in the list below:*

Proof. See discussion in Section 5.2.1. □

1. $S = A_{18}.2 < \mathrm{Sp}_{16}(2)$ with trivial class stabiliser
2. $S = 2.A_8 < \mathrm{Sp}_{16}(7)$ with class stabiliser $\langle\delta\rangle$.
3. $S = 2.L_2(17) < \mathrm{Sp}_{16}(p)$ with class stabiliser $\langle\delta\rangle$ if $p = \pm 5 \mod 12$ and $p \neq 17$, or $S = 2.L_2(17).2$ with trivial class stabiliser if $p = \pm 1 \mod 12$,
4. $S = 2.L_2(17).2$ with trivial class stabiliser if $p = 1, 11, 13, 23, 25, 35 \mod 36$, or $S = 2.L_2(17)$ class stabiliser $\langle\delta\rangle$ otherwise, provided $p \neq 2, 3, 17$. If $p = \pm 1 \mod 9$ then $S < \mathrm{Sp}_{16}(p)$; otherwise $S < \mathrm{Sp}_{16}(p^3)$.

Theorem 5.33. *Let G and Ω be as in the convention in [1] with $\Omega = \Omega^+_{16}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_1$-maximal subgroups of G are described in the list below:*

Proof. See discussion in Section 5.2.2. □

1. $S = A_{17}$ with $q = p = \pm 1, \pm 2, \pm 4, \pm 8 \mod 17$ and class stabiliser $\langle\gamma\rangle$.
2. $S = 2.A_{11}.2 < \Omega^+_{16}(11)$ with trivial class stabiliser.
3. $S = 2.A_{10}$ with class stabiliser $\langle\delta\rangle$ if $p = \pm 3 \mod 10$, or $S = 2.A_{10}.2$ with trivial class stabiliser if $p = \pm 1 \mod 10$, $p \neq 11$.
4. $S = A_{10}.2 < \Omega^+_{16}(2)$ with trivial class stabiliser.
5. $S = M_{12} < \Omega^+_{16}(11)$ with class stabiliser $\langle\gamma\rangle$.
6. $S = 2.\mathrm{Sz}(8) < \Omega^+_{16}(13)$ with trivial class stabiliser.
7. $S = L_3(3) < \Omega^+_{16}(13)$ with class stabiliser $\langle\gamma\rangle$.
8. $S = L_2(17)$ with $q = p = \pm 1, \pm 2, \pm 4, \pm 8 \mod 17$ and class stabiliser $\langle\gamma\rangle$.
9. $S = L_2(17)$ with $p = \pm 1, \pm 2, \pm 4, \pm 8 \mod 17$ and class stabiliser $\langle\gamma\rangle$. If $p = \pm 1 \mod 9$ then $S < \Omega^+(p)$, and if $p = \pm 2, \pm 4 \mod 9$ then $S < \Omega^+(p^3)$.

Theorem 5.34. *Let G and Ω be as in the convention in [1] with $\Omega = \Omega^-_{16}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_1$-maximal subgroups of G are described in the list below:*

Proof. See discussion in Section 5.2.2. □

1. $S = A_{18} < \Omega^-_{16}(3)$ with class stabiliser $\langle\gamma\rangle$.
2. $S = A_{17} < \Omega^-_{16}(p)$ with $p = \pm3, \pm5, \pm6, \pm7 \mod 17$, $p \neq 3$ and class stabiliser $\langle\gamma\rangle$.
3. $S = L_2(17) < \Omega^-_{16}(p)$ with $p = \pm3, \pm5, \pm6, \pm7 \mod 17$, $p \neq 3$ and class stabiliser $\langle\gamma\rangle$.
4. $S = L_2(17)$ with $p = \pm3, \pm5, \pm6, \pm7 \mod 17$ and class stabiliser $\langle\gamma\rangle$. If $p = \pm1 \mod 9$ then $S < \Omega^-(p)$, and if $p = \pm2, \pm4 \mod 9$ then $S < \Omega^-(p^3)$.

Theorem 5.35. *Let G and Ω be as in the convention in [1] with $\Omega = \mathrm{SL}_{16}(q)$ or $\mathrm{SU}_{16}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_1$-maximal subgroups of G are described in the list below:*

Proof. See discussion in Section 5.2.3. □

1. $S = 2.A_{12} < \mathrm{SL}_{16}(3)$ with class stabiliser $\langle\gamma\rangle$.
2. $S = A_{12} < \mathrm{SU}_{16}(2)$ with class stabiliser $\langle\gamma\rangle$.
3. $S = 2.A_{11}$ with $p \neq 2, 3, 11$. If $p = 1, 3, 4, 5, 9 \mod 11$ then $S < \mathrm{SU}_{16}(p)$ and if $p = 2, 6, 7, 8, 10 \mod 11$ then $S < \mathrm{SL}_{16}(p)$. The class stabiliser is $\langle\gamma\rangle$.
4. $S = 4.M_{22} < \mathrm{SU}_{16}(7)$ with class stabiliser $\langle\gamma\rangle$.
5. $S = M_{12}$ with $p \neq 2, 3, 11$. If $p = 2, 6, 7, 8, 10 \mod 11$ then $S < \mathrm{SU}_{16}(p)$ with class stabiliser $\langle\gamma\rangle$ if $p = 5 \mod 6$ and $\langle\gamma\delta\rangle$ if $p = 1 \mod 6$. If $p = 1, 3, 4, 5, 9 \mod 11$ then $S < \mathrm{SL}_{16}(p)$, with class stabiliser $\langle\gamma\rangle$ if $p = 1 \mod 5$ or $\langle\gamma\delta\rangle$ if $p = 5 \mod 6$.
6. $S = 4_2.L_3(4).2_2 < \mathrm{SU}_{16}(3)$ with class stabiliser $\langle\gamma\rangle$.
7. $S = 2.L_2(31)$ with $p \neq 2, 31$. If p is not a square $\mod 31$ then $S < \mathrm{SU}_{16}(p)$, whilst if p is a square $\mod 31$ then $S < \mathrm{SL}_{16}(p)$. We have class stabiliser $\langle\gamma\rangle$ if $p = \pm1 \mod 8$ and $\langle\gamma\delta\rangle$ if $p = \pm3 \mod 8$.
8. $S = L_3(3)$ with $p \neq 3, 13$. If $p = 1, 3, 9 \mod 13$ then $S < \mathrm{SL}_{16}(p)$. If $p = 4, 10, 12 \mod 13$ then $S < \mathrm{SU}_{16}(p)$. If $p = 2, 5, 6, 7, 8, 11 \mod 13$ then $S < \mathrm{SU}_{16}(p^2)$. If S preserves a unitary form then the class stabiliser is $\langle\gamma\rangle$ if $p = 5 \mod 6$ and $\langle\gamma\delta\rangle$ if $p = 1 \mod 6$, whilst if S does not preserve a unitary form then the class stabiliser is $\langle\gamma\rangle$ if $p = 1 \mod 6$ and $\langle\gamma\delta\rangle$ if $p = 5 \mod 6$.

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