

# Jordan Canonical Form example

Daniel Rogers (D.Rogers@warwick.ac.uk)

Here is an example from the sheet I meant to go through in the supervision but ran out of time. Hopefully this will put together everything I was gunning for in the supervision itself.

We're looking at Sheet 1, q7)ii), with the matrix  $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & -3 \end{pmatrix}$ ,

We find the eigenvalues of this matrix using our favourite method (mine is via having the solutions, yours hopefully is not that), and we find that they are  $-2, -2, 2$ . Hence we have characteristic polynomial  $(x + 2)^2(x - 2)$ .

At this point, we have two options for what the JCF might be - it could be  $J_{-2,1} + J_{-2,1} + J_{2,1}$ , giving the (diagonal) JCF  $B_1 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ , or it could be  $J_{-2,2} + J_{2,1}$ , giving us the JCF

$$B_2 \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In order to decide which it is, we're going to find the minimal polynomial. Remember that the degree of the term relating to each eigenvalue determines the size of the largest Jordan block, so we know that the minimal polynomial must be either  $(x - 2)(x + 2)$  (which would give us JCF  $B_1$ ), or  $(x - 2)(x + 2)^2$  (which would give us JCF  $B_2$ ).

We do the mathematics and discover that whilst  $A - 2I \neq 0$ ,  $A + 2I \neq 0$  (neither of these should shock you) and  $(A - 2I)(A + 2I) \neq 0$  (this requires computation), we see that  $(A - 2I)(A + 2I)^2 = 0$ . Indeed, it must do since this is the characteristic polynomial of  $A$  and a result in the lectures tells us that this must be the case - the new information here is that the characteristic polynomial and the minimal polynomial coincide. Hence the JCF must be  $B_2$ , the non-diagonal one.

The next question is to find a matrix  $P$  such that  $P^{-1}AP = B_2$ . We consider each eigenvalue in turn:

For the eigenvalue 2, this is easier. Here, we just need to find an eigenvector  $v_3$ ; i.e. a vector such that  $(A - 2I)v_3 = 0$ . This is standard, and you should find  $v_3 = (0, 1, -1)^T$  (up to multiplication by a scalar).

For the eigenvalue -2, if the matrix were diagonalisable we would be able to find two linearly independent eigenvectors and be done. Since we can't do this, we need to find generalised eigenvectors instead. In other words, here we need a vector  $v_2$  such that  $(A + 2I)v_2 \neq 0$  but  $(A + 2I)^2v_2 = 0$ .

$$A + 2I = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & -1 \\ -6 & -5 & -1 \end{pmatrix} \text{ (if you want to be sure you haven't made a mistake, make sure this$$

$$\text{matrix has rank 2), and } (A + 2I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 16 & 16 & 0 \\ -16 & -16 & 0 \end{pmatrix}.$$

So for instance we can take  $v_2 = (0, 0, 1)^T$  (check this satisfies both conditions above). We then take  $v_1 = (A + 2I)v_2 = (1, -1, -1)^T$ .

Now we have all the vectors we need to construct  $P$ , exactly as we would had we found eigenvectors

instead. Namely, we let  $P = \langle v_1 | v_2 | v_3 \rangle$ ; i.e.  $P = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix}$

Then you can check by hand that  $P^{-1}AP = B_2$ , and we are done.

Notice that the order we write the  $v_i$  in is important (this doesn't matter when you're just writing a diagonal matrix). If we had  $P = \langle v_2 | v_1 | v_3 \rangle$ , then we would end up with a matrix with eigenvalues along the diagonal and 1s and 0s on the subdiagonal. If we wrote  $P = \langle v_1 | v_3 | v_2 \rangle$ , then you'll end up with something which is not even close to a Jordan Normal form.  $P = \langle v_3 | v_1 | v_2 \rangle$  would be fine and if you do the computation hopefully you get the matrix you'd expect. This matrix is also a JCF but hopefully you'll agree that it makes no sense to consider these as two different JCFs.

I hope this helps and makes the construction of a JCF a bit more clear and explicit. Apologies for not getting to this in the supervision. If you have any questions about this or about constructing JCFs in general, please don't hesitate to ask.