Relative Minimal Polynomials

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Having looked at the lecture notes in the cold light of office space, I understand a bit more about what is meant in the notation by a relative minimal polynomial for a given vector. This is defined as the unique monic polynomial of smallest degree p such that $p(T)v = 0$. The point here is that $p(T)$ is a matrix, and we are multiplying this matrix by a given vector and seeing when we can get zero. For instance, if p is the minimal polynomial of T, then $p(T)$ will be the zero matrix and so $p(T)v = 0$ necessarily; but we may not need all of the terms.

For instance, suppose we have the diagonal matrix: $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$,

Then the characteristic polynomial is $(x-1)^3(x-2)^4$, and the minimal polynomial is $(x-1)(x-2)$ (the latter you can check quite easily, since $(A-I)(A-2I) = 0$ but $A - I \neq 0$, $A - 2I \neq 0$.

But suppose we have the vector $v = (1, 0, 0, 0, 0, 0, 0)^T$ (a column vector). Then the relative minimal polynomial here is $x - 1$, since $(A - I)v = 0$ (even though $A - I$ is not the zero matrix). Similarly, the relative minimal polynomial of $v = (0, 0, 0, 0, 0, 0, 1)^T$ is $x - 2$. It isn't always strictly smaller than the minimal polynomial, though, since for instance the relative minimal polynomial of $v = (1, 0, 0, 0, 0, 0, 1)^T$ is still $(x - 1)(x - 2)$. This idea leads on to an alternative way of finding the minimal polynomial of a matrix by repeating this process for all basis vectors and looking at the lowest common multiple. For instance here, let v_i be the vector with a 1 in the ith place and 0s everywhere else. Then the relative minimal polynomial of $v_1, v_2, .v_3$ is $(x - 1)$, the relative minimal polynomial of v_4, v_5, v_6, v_7 is $(x - 2)$, hence the minimal polynomial is $(x - 1)(x - 2)$.

The relative minimal polynomials also need not always be linear - for instance, take the JCF of a matrix which is not diagonalisable. Then you should discover that, for the standard basis on the JCF, the highest power of $x - \lambda$ you encounter in this way (for λ an eigenvalue) will be the size of the largest Jordan block, which relates very strongly to the result I was talking about right at the end of the supervision.

I hope that clears up what is meant there - if not, do let me know!