THE EQUIVARIANT COVERING HOMOTOPY PROPERTY

ANDREW RONAN

In this article, we will prove the equivariant covering homotopy property of $(\Pi;\Gamma)$-bundles and relate it to the theory of Hurewicz fibrations. Our main source of reference will be [ML86], where the theory of generalised equivariant bundles was first introduced, and the discussion here can be viewed as an elaboration on the material covered in [ML86] along with an alternative proof of the ECHP. As in [ML86], we work throughout with an exact sequence of compact Lie groups:

$$1 \to \Pi \to \Gamma \to G \to 1$$

where the maps are continuous, and we assume that all subgroups of compact Lie groups under discussion are closed. The main result of this article is the following equivariant covering homotopy property for principal $(\Pi;\Gamma)$-bundles, from which everything else follows:

Theorem 1: *If $p : E \to B$ is a numerable principal $(\Pi;\Gamma)$-bundle, then p is a Hurewicz Γ-fibration.*

We deduce that numerable $(\Pi;\Gamma)$-bundles with fibre F are Hurewicz G-fibrations in Corollary 12, via the more general result of Lemma 9, and we deduce the ECHP for $(\Pi;\Gamma)$-bundles with fibre F from Theorem 1 in Theorem 13.

We begin with some routine lemmas that we will use throughout the article:

Lemma 2: *Suppose that we have a pullback square of G-spaces, where G is a compact (Lie) group:*

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[\phi]{} & C \end{array}$$

Then if f is injective when restricted to each orbit of A, the following square is also a pullback:

$$\begin{array}{ccc} P/G & \xrightarrow{\alpha} & A/G \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f} \\ B/G & \xrightarrow[\phi]{} & C/G \end{array}$$

Proof. Let Q be the actual pullback and $\sigma : P/G \to Q$ the induced map. Any element of Q is of the form $([a],[b])$, where $[f(a)] = [\phi(b)]$. In particular, there exists a g such that $f(ga) = \phi(b)$ and so σ is surjective. Suppose that (a,b) and (x,y) in P are both sent to the same element of Q. There exists g such that $gb = y$ and so $f(ga) = f(x)$. Since f is injective on orbits, $(ga, gb) = (x,y)$. So we have a continuous bijection $\sigma : P/G \to Q$. Since we are working in the category of CGWH spaces, σ will be a homeomorphism if whenever $K \subset Q$ is compact Hausdorff, $\sigma^{-1}(K)$ is compact. It suffices to show that the preimage of K in P is compact. This is a closed subspace of the preimage under $\pi \times \pi$ of $\tilde{\alpha}(K) \times \tilde{\beta}(K) \subset A/G \times B/G$ in $A \times B$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are the relevant maps from Q, so is compact as desired. □

The next result goes back to at least [Pal60, Theorem 3.7]:

Lemma 3: *If $f : X \to G \times_H S$ is a G-map, then the canonical map $\sigma : G \times_H f^{-1}(S) \to X$ is a homeomorphism.*

Proof. Since every orbit of $G \times_H S$ contains an element of S, σ is surjective. If $g_1x_1 = g_2x_2$ with $g_i \in G$ and $x_i \in f^{-1}(S)$, then $g_1f(x_1) = g_2f(x_2)$, so $g_1 = g_2h$ for some $h \in H$ and so σ is bijective. If $K \subset X$ is compact Hausdorff then so is $GK \subset X$, and since $f^{-1}(S)$ is closed, $f^{-1}(S) \cap GK$ is compact. So $\sigma^{-1}(K)$ is a closed subspace of a compact space, hence compact. Therefore, σ is a homeomorphism. □

We will use the following familiar result about Hurewicz G-fibrations (:= hG-fibrations), whose proof is identical to the non-equivariant case, [May99, pg.51]:

Lemma 4: *Let $p : E \to B$ be a map of G-spaces and let $\{U_i\}$ be a G-numerable open cover of B. Then p is an hG-fibration iff $p : p^{-1}(U_i) \to U_i$ is an hG-fibration for every i.*

Finally, we have the following local description of completely regular G-spaces:

Lemma 5: *Let X be a completely regular G-space. Then, for any $x \in X$ with isotropy group G_x, there is a G_x-invariant subspace V_x containing x, called a slice through x, such that the canonical map $G \times_{G_x} V_x \to X$ is a homeomorphism onto an open subspace of the orbit Gx.*

Proof. See [Bre72, Theorem 5.4]. □

We now move on to the proof of the equivariant covering homotopy property. Firstly, recall the definition of a numerable principal $(\Pi; \Gamma)$-bundle:

Definition 6: *A numerable principal $(\Pi; \Gamma)$-bundle is a Π-free Γ-space, E, such that there exists an Γ-equivariant open cover $\{U_i\}$ of E with $U_i \cong \Gamma \times_{\Lambda_i} S_i$ as Γ-spaces, where $\Lambda_i \cap \Pi = \{e\}$ and S_i is a Λ_i-space. Let $B = E/\Pi$ and let $p : E \to B$ be the quotient map.*

Lemma 7: *Suppose that Π is a normal subgroup of Γ and $G \cong \Gamma/\Pi$. If Λ is a subgroup of Γ such that $\Lambda \cap \Pi = 1$, then $\Gamma/\Lambda \to G/\Lambda$ is an $h\Lambda$-fibration (that is a Hurewicz Λ-fibration with the action of Λ on the left).*

Proof. We will induct on the dimension/no. of connected components of Λ, with the usual ordering where dimension n_1 with m_1 components is $\leq$ dimension n_2 with m_2 components iff $n_1 < n_2$ or $n_1 = n_2$ and $m_1 \leq m_2$. If $\Lambda = 1$, then $\Gamma \to G$ is a non-equivariant h-fibration since Γ is locally Π-homeomorphic to $\Pi \times U$ over $U \subset G$. So suppose that $\Lambda \neq 1$. If we view Γ/Λ as a $\Pi\Lambda$-space, then it is locally homeomorphic to $\Pi\Lambda \times_H S$, where S is a slice through some $x \in \Gamma/\Lambda$. The isotropy group of x under the Γ-action is conjugate to Λ, so H, which is the isotropy group of x under the $\Pi\Lambda$-action, is subconjugate to Λ in Γ. Note also that $H \cap \Pi = 1$. By the paracompactness of G/Λ, it suffices to prove that $\Pi\Lambda \times_H S \to \Lambda \times_H S$ is a Λ-fibration. Since $H \cap \Pi = 1$, Lemma 2 implies that we have a pullback square:

$$\begin{array}{ccc} \Pi\Lambda \times_H S & \longrightarrow & \Pi\Lambda \times_H * \\ \downarrow & & \downarrow \\ \Lambda \times_H S & \longrightarrow & \Lambda \times_H * \end{array}$$

Since H is subconjugate to Λ in Γ, either H has a lower dimension/ fewer connected components than Λ, or $\Lambda \times_H * = *$, and in both cases we can conclude that $\Pi\Lambda \times_H * \to \Lambda \times_H *$ is an hH-fibration, by induction in the first case. If we have a lifting problem of $\Pi\Lambda$-spaces:

$$\begin{array}{ccc} X & \xrightarrow{f} & \Pi\Lambda \times_H * \\ \downarrow & \nearrow & \downarrow \\ X \times I & \longrightarrow & \Lambda \times_H * \end{array}$$

then Lemma 3 implies that $X \cong \Pi\Lambda \times_H f^{-1}(*)$, and so the lifting problem can be reduced to a lifting problem of H-spaces. Therefore $\Pi\Lambda \times_H * \to \Lambda \times_H *$ is an $h\Pi\Lambda$-fibration and, therefore, an $h\Lambda$-fibration, as desired. □

More generally, we can deduce the equivariant covering homotopy property for principal bundles:

Theorem 8: *If $p : E \to B$ is a numerable principal $(\Pi;\Gamma)$-bundle, then p is an $h\Gamma$-fibration.*

Proof. Using Lemma 4, numerability allows us to reduce to the case where $p : \Gamma \times_\Lambda S \to G \times_\Lambda S$ is a trivial principal $(\Pi;\Gamma)$-bundle. Since $\Lambda \cap \Pi = 1$, Lemma 2 implies that we have a pullback:

$$\begin{array}{ccc} \Gamma \times_\Lambda S & \longrightarrow & \Gamma \times_\Lambda * \\ \downarrow & & \downarrow \\ G \times_\Lambda S & \longrightarrow & G \times_\Lambda * \end{array}$$

So it suffices to show that $\Gamma/\Lambda \to G/\Lambda$ is an $h\Gamma$-fibration. As above, this follows from reducing a Γ-equivariant lifting problem:

$$\begin{array}{ccc} X & \xrightarrow{f} & \Gamma \times_\Lambda * \\ \downarrow & \nearrow & \downarrow \\ X \times I & \longrightarrow & G \times_\Lambda * \end{array}$$

to a Λ-equivariant lifting problem, using the fact that $X \cong \Gamma \times_\Lambda f^{-1}(*)$ by Lemma 2. Then we're done since Lemma 7 tells us that $\Gamma/\Lambda \to G/\Lambda$ is an $h\Lambda$-fibration. $\square$

Lemma 9: *If $p : E \to B$ is an $h\Gamma$-fibration and Π acts trivially on B, then $p/\Pi : E/\Pi \to B$ is an hG-fibration.*

Proof. Consider a lifting problem of G-spaces and G-maps:

$$\begin{array}{ccc} X & \longrightarrow & E/\Pi \\ \downarrow & \nearrow & \downarrow{\scriptstyle p/\Pi} \\ X \times I & \xrightarrow[g]{} & B \end{array}$$

Let P denote the pullback of g along p, so P is a Γ-space. Since each G-space is also a Γ-space on which Π acts trivially, each Π-orbit of $X \times I$ consists of a single point and so Lemma 2 implies that we have a pullback of G-spaces on the right hand side of the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{f} & P/\Pi & \longrightarrow & E/\Pi \\ \downarrow & \nearrow & \downarrow{\scriptstyle q/\Pi} & & \downarrow{\scriptstyle p/\Pi} \\ X \times I & \xrightarrow[1]{} & X \times I & \xrightarrow[g]{} & B \end{array}$$

Since q, being a pullback of p, is an $h\Gamma$-fibration, there is a map $\alpha : P_0 \times I \to P$ over $X \times I$, where P_0 is $q^{-1}(X \times \{0\})$ and the restriction of α to $P_0 \times \{0\}$ is the inclusion of P_0 into P. Note that f factors through the inclusion $P_0/\Pi \to P/\Pi$, which allows us to define the map k in the following diagram. It suffices to solve the lifting problem:

$$\begin{array}{ccc} X & \xrightarrow{k\times\{0\}} & P_0/\Pi \times I \\ \downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle q_0/\Pi\times 1} \\ X \times I & \xrightarrow[1]{} & X \times I \end{array}$$

For this define $H(x,t) = (k(x), t)$, so we're done. □

At this point, it is worth recalling the definitions of principal $(\Pi;\Gamma)$-bundles and $(\Pi;\Gamma)$-bundles with fibre F, where F is always a Γ-space on which Π acts effectively.

Definition 10: *A principal $(\Pi;\Gamma)$-bundle is a Π-free Γ-space, E, such that there is a Π-equivariant open cover $\{U_i\}$ of E such that $U_i \cong \Pi \times V_i$ as Π-spaces, for some V_i with trivial Π-action. Define $B := E/\Pi$ and let $p : E \to B$ denote the quotient map.*

Definition 11: *A $(\Pi;\Gamma)$-bundle with fibre F is a map of G-spaces $\sigma : D \to B$ equipped with an open cover $\{U_i\}$ of B and homeomorphisms over U_i:*

$$\begin{array}{ccc} U_i \times F & \xrightarrow{\psi_i} & \sigma^{-1}(U_i) \\ & \searrow \quad \swarrow & \\ & U_i & \end{array}$$

such that if $u \in U_i$ and $gu \in U_j$ and $\gamma \in \Gamma$, then the composite:

$$F \cong \{u\} \times F \xrightarrow{\gamma^{-1}} \{u\} \times F \xrightarrow{\psi_j^{-1} g \psi_i} \{gu\} \times F \cong F$$

is multiplication by some $\pi \in \Pi$, where g is the image of γ in G.

It makes sense to also require that $\{(U_i, \psi_i)\}$ is maximal with respect to sets of trivialisations satisfying the above properties.

Recall from [LMS86, pg. 179] or [Zou21, Theorem 2.29] that there is an equivalence of categories between the categories of principal $(\Pi;\Gamma)$-bundles and $(\Pi;\Gamma)$-bundles with fibre F, which we will refer to loosely as a 1-1 correspondence for the rest of this article. The correspondence sends a principal $(\Pi;\Gamma)$-bundle $p : E \to B$ to the $(\Pi;\Gamma)$-bundle $1 \times_\Pi * : E \times_\Pi F \to E \times_\Pi *$ with fibre F. We define a numerable $(\Pi;\Gamma)$-bundle with fibre F to be one which corresponds to a numerable principal $(\Pi;\Gamma)$-bundle. In particular, if σ is numerable, there is a numerable open cover of B over which σ is locally homeomorphic to bundles of the form $(\Gamma \times_\Lambda S) \times_\Pi F \to (\Gamma \times_\Lambda S) \times_\Pi *$. Using Theorem 8 and Lemma 9 we can deduce:

Corollary 12: *If $\sigma : D \to B$ is a numerable $(\Pi;\Gamma)$-bundle with fibre F, then σ is an hG-fibration.*

We can also use the 1-1 correspondence to deduce the equivariant covering homotopy property for $(\Pi;\Gamma)$-bundles with fibre F:

Theorem 13: *Let:*

$$\begin{array}{ccc} D_1 & \xrightarrow{f} & D_2 \\ {\scriptstyle \sigma_1}\downarrow & & \downarrow{\scriptstyle \sigma_2} \\ B_1 & \xrightarrow[g]{} & B_2 \end{array}$$

be a map of $(\Pi;\Lambda)$-bundles with fibre F, with σ_2 numerable, and let $G : B_1 \times I \to B_2$ be a G-homotopy starting at g. Then there exists a map of $(\Pi;\Gamma)$-bundles with the fibre F of the form:

$$\begin{array}{ccc} D_1 \times I & \xrightarrow{F} & D_2 \\ {\scriptstyle \sigma_1 \times 1}\downarrow & & \downarrow{\scriptstyle \sigma_2} \\ B_1 \times I & \xrightarrow[G]{} & B_2 \end{array}$$

Proof. The map of bundles given is equivalent under the 1-1 correspondence to a map of principal bundles, which is simply a Γ-equivariant map $\alpha : E_1 \to E_2$. Note that $\alpha/\Pi = g$. Since $E_2 \to B_2$ is an $h\Gamma$-fibration, there is a Γ-equivariant homotopy $A : E_1 \times I \to E_2$ such that $A/\Pi = G$. Given the correspondence, this implies the result. □

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