Localizations and completions of nilpotent G-spaces

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Abstract

We develop the theory of nilpotent G-spaces and their localizations, for G a compact Lie group, via reduction to the non-equivariant case using Bousfield localization. One point of interest in the equivariant setting is that we can choose to localize or complete at different sets of primes at different fixed point spaces - and the theory works out just as well provided that you invert more primes at $K \leq G$ than at $H \leq G$, whenever K is subconjugate to H in G. We also develop the theory in an unbased context, allowing us to extend the theory to G-spaces which are not G-connected.

Introduction

The purpose of this paper is to develop the theory of localizations and completions of nilpotent G-spaces at sets of primes, where G is a compact Lie group. The main reference for the equivariant theory is [1, Ch. II], which itself is a summary of the older papers [2] and [3], where it was explained how the foundations of the theory could be developed using the same arguments as in the non-equivariant setting, with some additional complications when G is compact Lie rather than just finite. Our approach is slightly different, in that we use the theory of Bousfield localization to deduce the foundations of the theory from the non-equivariant case. This approach leads to fewer difficulties in the compact Lie case, and allows us to use a more general definition of a nilpotent G-space than in [1], see Definition 2.1.1. For example, we prove that a nilpotent G-space X is p-complete iff all homotopy groups of the form $\pi_i(X^H)$ are p-complete. This fact was proved in [3, Theorem 2], but only under the assumption that, for fixed i, the nilpotency classes of $\pi_i(X^H)$, as H varies, have a common bound.

Another contribution of this paper is that we allow the set of primes we are localizing or completing at to vary over the orbit category of G, and we show that this provides no extra difficulties provided that you 'invert more primes' at $K \leq G$ than at $H \leq G$, whenever K is subconjugate to H in G - we call this property the *poset condition*. For example, we could localize at p at one subgroup and complete at p at another, where, loosely speaking, completing at p 'inverts more primes' than localizing at p. One might ask, why consider these localizations? In the non-equivariant setting, Bousfield proved in [4, Theorem 1.1] that all localizations at connective homology theories are equivalent to localizations with respect to either $H(-;\mathbb{Z}_T)$ or $H(-;\oplus_{p\in T}\mathbb{F}_p)$ for some set of primes T. Therefore, in this paper we are considering localizations at pointwise connective homology theories, where pointwise means we choose a connective homology theory for every closed subgroup H of G, and the localizations which satisfy the poset condition are precisely those with the property that a G-space is local iff it is pointwise local.

We develop the theory in both a based and unbased context - with different parts of the theory working better in each setting. For example, we derive fracture theorems for nilpotent G-spaces (Theorems 2.3.1 and 2.3.6), relate nilpotent G-spaces to equivariant Postnikov towers, and show that our homological approach to the theory is equivalent to the classical cohomological appraoch of [2] and [3], all in the based context. We use the unbased theory to extend our results on nilpotent G-spaces to G-spaces whose fixed point spaces are disjoint unions of nilpotent spaces. This is especially pertinent in the equivariant setting, since there are many examples of G-spaces which are non-equivariantly connected, but which have disconnected fixed point spaces, or no possible choice of a G-fixed basepoint at all.

Notations and Prerequisites

We will work with the model categories of G-spaces and based G-spaces, where G is a compact Lie group, basepoints are G-fixed, and the model structures are the Quillen or q-model structures. All subgroups of G are assumed to be closed. Unless otherwise stated, we build G-CW complexes out of the maps $(\frac{G}{H})_+ \wedge S^n_+ \to (\frac{G}{H})_+ \wedge D^n_+$ in the based context, rather than using based maps out of $(\frac{G}{H})_+ \wedge S^n$. The notation $[A, B]$ denotes homotopy classes of maps, which may be based/unbased/equivariant depending on the context.

This paper should be accessible to any reader who is familiar with the non-equivariant theory of nilpotent spaces and their localizations, as well as the basics of equivariant homotopy theory.

1 Localization systems

1.1 Bousfield localization at the T-equivalences

In this section, we define localization systems, $\mathbf{T}$, as well as the notion of a $\mathbf{T}$-equivalence between based G-spaces. We develop the basic properties of the $\mathbf{T}$-equivalences, and then use the Bousfield cardinality argument to show that there exists a model structure on the category of based G-spaces, where a map is a weak equivalence iff it is a $\mathbf{T}$-equivalence. We develop the basic properties of this model structure, including Theorem 1.1.13 below, which is the key to deducing the equivariant theory of nilpotent G-spaces from the non-equivariant theory.

Let $\mathbf{P}$ denote the poset of subsets of the set of prime numbers partially ordered by inclusion, and let $\mathcal{O}$ denote the orbit category of a compact Lie group G.

Definition 1.1.1: *A localization system is a functor* $\mathbf{T} : \mathcal{O}^{op} \to \mathbf{P}^{op} \times \mathbf{1}$*, where we denote by* $\mathbf{1}$ *the category with objects 0 and 1 and a single arrow from 0 to 1.*

We think of $\mathbf{T}([{}^{G}/_{H}])$ as a set of primes with coefficient, where the coefficient is either 0 or 1. If we drop the bold font on the $\mathbf{T}$, then $T([{}^{G}\!/\!{}_{H}])$ denotes only the underlying set of primes of $\mathbf{T}([{}^{G}\!/\!{}_{H}])$. Recall that a map of spaces is called a $\mathbb{Z}_T$-equivalence if it induces an isomorphism on homology with coefficients in $\mathbb{Z}_T$. Similarly, a map is called an $\mathbb{F}_T$-equivalence if it induces an isomorphism on homology with coefficients in $\mathbb{F}_p$, for every $p \in T$. When the basepoints are nondegenerate, it is equivalent to define these equivalences using the respective reduced homology theories instead.

Definition 1.1.2: *Let $\mathbf{T}$ be a set of primes with coefficient and $f : X \to Y$ a map of spaces. If the coefficient is 0, then we call f a $\mathbf{T}$-equivalence if it is a $\mathbb{Z}_T$-equivalence. If the coefficient is 1, then we call f a $\mathbf{T}$-equivalence if it is an $\mathbb{F}_T$-equivalence.*

Intuitively, a coefficient of 0 means we are localizing at T, and a coefficient of 1 means we are completing at T. In a similar vein, we have:

Definition 1.1.3: *Let $\mathbf{T}$ be a set of primes with coefficient and let X be a space. If the coefficient is 0, we say that X is $\mathbf{T}$-local if it is T-local after forgetting the coefficient. If the coefficient is* 1*, we say that X is $\mathbf{T}$-local if it is T-complete after forgetting the coefficient.*

We can now make the following definition:

Definition 1.1.4: *Let $\mathbf{T}$ be a localization system and $f : X \to Y$ be a map of based G-spaces. We say that f is a $\mathbf{T}$-equivalence if for all $H \leq G$, $f^H : X^H \to Y^H$ is a $\mathbf{T}([{}^{G}\!/\!{}_{H}])$-equivalence.*

We will need the following minimal list of properties of the $\mathbf{T}$-equivalences, where a property is *pointwise* if it holds for all fixed point spaces:

Lemma 1.1.5: *i) The class of $\mathbf{T}$-equivalences is closed under retracts, satisfies 2-out-of-3, and every weak equivalence is a $\mathbf{T}$-equivalence,*
ii) the pushout of a $\mathbf{T}$-equivalence that is a pointwise h-cofibration is a $\mathbf{T}$-equivalence,
iii) the colimit of a transfinite sequence of $\mathbf{T}$-equivalences which are closed inclusions is a $\mathbf{T}$-equivalence.

Proof. i) is easy. For ii), since taking fixed points preserves pushouts along closed inclusions, we can work pointwise and replace the spaces with nondegenerately based ones. The result then follows from consideration of cofibre sequences. For iii), taking fixed points preserves transfinite colimits of closed inclusions, and so the result follows from the fact that homology preserves these colimits. □

We can use the Bousfield-Smith cardinality argument on the $\mathbf{T}$-equivalences. The argument is essentially the same as the classical case of localizing spaces with respect to homology theories, which is treated in [5, Section 19.3]. The key lemma is as follows:

Lemma 1.1.6: *There exists a cardinal κ with the following property: if $i : A \to B$ is the inclusion of a subcomplex into cell complex B which is also a $\mathbf{T}$-equivalence, then, for any cell e of B, there is a subcomplex C of size $< \kappa$ containing e such that $A \cap C \to C$ is a $\mathbf{T}$-equivalence.*

Proof. Choose a regular cardinal, $\kappa > \aleph_0$, with the following properties:

i) every cell of any cell complex is contained within a subcomplex of size $< \kappa$,

ii) if Z is a cell complex of size $< \kappa$, then $\bigoplus H_*(Z^H; \mathbf{T}([G/H]))$ has cardinality $< \kappa$,

iii) if W is any cell complex, then:

$$H_*(W; \mathbf{T}) = colim_{<\kappa} H_*(Z; \mathbf{T})$$

where $H_*(W; \mathbf{T})$ denotes the system of abelian groups with values $H_*(W^H; \mathbf{T}([G/H]))$, and the colimit is over all subcomplexes of size $< \kappa$. To start the proof, choose a subcomplex C_0 of B of size $< \kappa$ which contains e, so we have a map $C_0 \cap A \to C_0$. By some $c \in H_*(C_0; \mathbf{T})$, we mean an element of $H_n(C_0; \mathbf{T})([G/H])$ for some H and n. For each $c \in H_*(C_0; \mathbf{T})$, its image in $H_*(B; \mathbf{T})$ is the image of an element, a, in the homology of a $< \kappa$ dimensional subcomplex of A, D. Moreover, there is a $< \kappa$ dimensional subcomplex E of B, containing C_0 and D, such that the images of a and c in $H_*(E; \mathbf{T})$ are equal. Define C_0^1 by adding such a subcomplex E to C_0 for every $c \in H_*(C_0; \mathbf{T})$ - the conditions i) - iii) above ensure that C_0^1 has size $< \kappa$. So every element of $H_*(C_0^1; \mathbf{T})$ which is in the image of $H_*(C_0; \mathbf{T})$ is also in the image of $H_*(C_0^1 \cap A; \mathbf{T})$. Now if $k \in H_*(C_0 \cap A; \mathbf{T})$ is sent to 0 in $H_*(C_0; \mathbf{T})$, it is also sent to 0 in $H_*(B; \mathbf{T})$ and $H_*(A; \mathbf{T})$, so there is a $< \kappa$ dimensional subcomplex of A, L, containing $C_0 \cap A$, such that the image of k in $H_*(L; \mathbf{T})$ is 0. Define C_1 by adding such a subcomplex L to C_0^1, for every $k \in H_*(C_0 \cap A; \mathbf{T})$ which is sent to 0 in $H_*(C_0; \mathbf{T})$. It follows that if $k \in H_*(C_0 \cap A; \mathbf{T})$ is sent to 0 in $H_*(C_0; \mathbf{T})$, it is also sent to 0 in $H_*(C_1 \cap A; \mathbf{T})$. Moreover, every element of $H_*(C_1; \mathbf{T})$ which is in the image of $H_*(C_0; \mathbf{T})$ is also in the image of $H_*(C_1 \cap A; \mathbf{T})$. We can repeat this process to form $e \in C_0 \subset C_1 \subset C_2 \subset ...$, and we let C be the union of the C_i which still has size $< \kappa$. Since homology preserves these sequential colimits, it follows that $H_*(C \cap A; \mathbf{T}) \to H_*(C; \mathbf{T})$ is an isomorphism, as desired. □

We now deduce the standard consequences of Lemma 1.1.6. Firstly, using transfinite induction, we have:

Corollary 1.1.7: *A map has the RLP with respect to all inclusions of cell complexes which are* $\mathbf{T}$*-equivalences iff it has the RLP with respect to all inclusion of cell complexes of dimension* $< \kappa$ *which are* $\mathbf{T}$*-equivalences.*

Proof. See [6, Proposition 4.5.6]. □

Any map with the RLP with respect to inclusions of cell complexes that are $\mathbf{T}$-equivalences is a q-fibration, since the generating acyclic cofibrations $(\frac{G}{H})_+ \wedge (D^n)_+ \to (\frac{G}{H})_+ \wedge (D^n \times I)_+$ are inclusions of subcomplexes. Therefore, using left properness we have:

Lemma 1.1.8: *A map has the RLP with respect to all q-cofibrations which are* $\mathbf{T}$*-equivalences iff it has the RLP property with respect to all inclusions of cell complexes that are* $\mathbf{T}$*-equivalences.*

Proof. See [6, Proposition 13.2.1]. □

If we call such a map a $\mathbf{T}$-fibration, then we see that a $\mathbf{T}$-fibration that is a $\mathbf{T}$-equivalence is a q-acyclic q-fibration by the retract argument. Using this and the small object argument we can now conclude:

Theorem 1.1.9: *There is a left proper model structure on the category of based G-spaces where the weak equivalences are the $\mathbf{T}$-equivalences, the cofibrations are the q-cofibrations and the fibrations are the $\mathbf{T}$-fibrations.*

This model structure is monoidal:

Lemma 1.1.10: *If $i : A \to B$ and $j : C \to D$ are cofibrations, then $i \square j : A \wedge D \cup B \wedge C \to B \wedge D$ is a cofibration which is a $\mathbf{T}$-equivalence if either i or j is a $\mathbf{T}$-equivalence.*

Proof. The fact that $i \square j$ is a cofibration is classical and is a consequence of the fact that $\frac{G}{H} \times \frac{G}{K}$ is G-homeomorphic to a G-CW complex. Similarly, since $(\frac{G}{H})^K$ is homeomorphic to a CW-complex, by [7, Corollary 7.2] and [8, Ch. VI, Corollary 2.5], we have that a cofibration is a pointwise cofibration. Therefore, for the remaining statement concerning $\mathbf{T}$-equivalences we can assume that G is the trivial group. Note also that the cofibre of $i \square j$ is homotopy equivalent to $\frac{B}{A} \wedge \frac{D}{C}$. Suppose that j is a $\mathbf{T}$-equivalence and p is a $\mathbf{T}$-fibration. Then $i \square j$ has the left lifting property with respect to p iff i has the left lifting property with respect to $p^{\square j}$. Therefore, it suffices to show that $i \square j$ is a $\mathbf{T}$-equivalence in the case where $i : (\frac{G}{H})_+ \wedge (S^{n-1})_+ \to (\frac{G}{H})_+ \wedge (D^n)_+$ and j is an inclusion of a subcomplex which is a $\mathbf{T}$-equivalence. Since we are assuming that G is trivial, the cofibre of $i \square j$ is homotopy equivalent to $S^n \wedge \frac{D}{C}$, which has vanishing reduced homology with the required coefficients as desired. $\square$

We have the following characterisation of the fibrant objects:

Lemma 1.1.11: *A based G-space Z is $\mathbf{T}$-local (that is fibrant in the model structure of Theorem 1.1.9) iff for all $\mathbf{T}$-equivalences $f : A \to B$ between cofibrant objects, the map $f^* : [B, Z] \to [A, Z]$ is a bijection.*

Proof. If for all $\mathbf{T}$-equivalences $f : A \to B$ between cofibrant objects, the map $[B, Z] \to [A, Z]$ is a bijection, then it is easy to show that $Z \to *$ has the right lifting property with respect to any inclusion of cell complexes that is a $\mathbf{T}$-equivalence, using the fact that inclusions of cell complexes are h-cofibrations, and so Z is $\mathbf{T}$-local by Lemma 1.1.8. On the other hand, if Z is $\mathbf{T}$-local, we can assume that f is a cofibration. Considering lifts of $Z \to *$ with respect to f shows that f^* is surjective, and considering lifts with respect to $f \square i$, where i is the inclusion $\{0, 1\}_+ \to I_+$, shows that f^* is injective. $\square$

The next lemma is the key to deducing our results on nilpotent G-spaces from the non-equivariant theory. It is also the first lemma to make use of the fact that $\mathbf{T}$ is a localization system rather than just an arbitrary choice of abelian group at each subgroup of G.

Lemma 1.1.12: *If a based G-space Z is $\mathbf{T}$-local, then Z^H is $\mathbf{T}([{}^{G}\!/\!{}_{H}])$-local for every $H \leq G$.*

Proof. Let $f : A \to B$ be a $\mathbf{T}([{}^{G}/_{H}])$-equivalence between cofibrant spaces. Let $g = 1 \wedge f : ({}^{G}/_{H})_+ \wedge A \to ({}^{G}/_{H})_+ \wedge B$. We have $({}^{G}/_{H})^K_+ = \mathcal{O}([{}^{G}/_{K}], [{}^{G}/_{H}])_+$, and it follows that g is a $\mathbf{T}$-equivalence by Lemma 1.1.10, the fact that $\mathbf{T}$ is a localization system and the following observations:

i) if $S \subset T$, then a $\mathbb{Z}_T$-equivalence is a $\mathbb{Z}_S$-equivalence,

ii) if $S \subset T$, then an $\mathbb{F}_T$-equivalence is an $\mathbb{F}_S$-equivalence,

iii) a $\mathbb{Z}_T$-equivalence is an $\mathbb{F}_T$-equivalence.

It follows that $g^* : [({}^{G}/_{H})_+ \wedge B, Z] \to [({}^{G}/_{H})_+ \wedge A, Z]$ is a bijection. This is equivalent to $[B, Z^H] \to [A, Z^H]$ being a bijection, and it follows that Z^H is $\mathbf{T}([{}^{G}/_{H}])$-local. □

Using Lemma 1.1.12, we can deduce:

Theorem 1.1.13: *i) A based G-space Z is $\mathbf{T}$-local iff Z^H is $\mathbf{T}([{}^{G}/_{H}])$-local for every $H \leq G$,*

ii) A map of based G-spaces $X \to Y$ is a $\mathbf{T}$-localization iff $X^H \to Y^H$ is a $\mathbf{T}([{}^{G}/_{H}])$-localization for every $H \leq G$.

Proof. i) For the direction we haven't already proved, let Z be a based G-space such that Z^H is $\mathbf{T}([{}^{G}/_{H}])$-local for every $H \leq G$. Consider a $\mathbf{T}$-localization $Z \to W$. Then, each map $Z^H \to W^H$ is a $\mathbf{T}([{}^{G}/_{H}])$-equivalence between $\mathbf{T}([{}^{G}/_{H}])$-local objects, and so a weak equivalence as desired.

ii) This follows from i). □

To end this subsection, we quickly give a counterexample to indicate what can happen if $\mathbf{T}$ is not a localization system. Let $G = C_2$, and define a $\mathbf{T}$-equivalence to be a map of based G-spaces, $f : X \to Y$, such that $H_*(f^e; \mathbb{Z}[p^{-1}])$ and $H_*(f^G; \mathbb{Z}[p^{-1}, q^{-1}])$ are isomorphisms, where p and q are distinct primes. If $\mathbf{T}$-local G-spaces were always pointwise local, then the analogue of Theorem 1.1.13 would also have to hold. Consider the map $K(\underline{\mathbb{Z}[p^{-1}]}, 1) \to K(\underline{\mathbb{Z}[p^{-1}, q^{-1}]}, 1)$. Since $K(\underline{\mathbb{Z}[p^{-1}, q^{-1}]}, 1)$ would be $\mathbf{T}$-local, the factorisation of the map through $K(\underline{\mathbb{Z}[p^{-1}]}, 1)_{\mathbf{T}}$ would result in a commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}[p^{-1}, q^{-1}] & \longrightarrow & \mathbb{Z}[p^{-1}] \\ {\scriptstyle 1}\downarrow & & \downarrow \\ \mathbb{Z}[p^{-1}, q^{-1}] & \xrightarrow[1]{} & \mathbb{Z}[p^{-1}, q^{-1}] \end{array}$$

Since the top map has to be zero, this is a contradiction. Therefore, a $\mathbf{T}$-local space is not necessarily pointwise local.

1.2 Unbased T-localizations

The theory described in Section 1.1 goes through essentially unchanged in the unbased context. We have:

Theorem 1.2.1: *Let* $\mathbf{T}$ *be a localization system. There is a left proper, monoidal model structure on the category of* G*-spaces where the weak equivalences are the* $\mathbf{T}$*-equivalences, the cofibrations are the q-cofibrations and the fibrations are the* $\mathbf{T}$*-fibrations (which are defined as in Lemma 1.1.8). A* G*-space* Z *is* $\mathbf{T}$*-local (that is fibrant in this model structure) iff for all* $\mathbf{T}$*-equivalences* $f : A \to B$ *between cofibrant objects, the map* $f^* : [B, Z] \to [A, Z]$ *is a bijection.*

Proof. The existence of the left proper model structure follows from the Bousfield cardinality argument as in Section 1.1. If $i : A \to B$ is a cofibration and $f : X \to Y$ is a cofibration which is a $\mathbf{T}$-equivalence, then $A \times Y \cup B \times X \to B \times Y$ is a cofibration as in Lemma 1.1.10 and it will be a $\mathbf{T}$-equivalence if $(A \times Y \cup B \times X)_+ \to (B \times Y)_+$ is a $\mathbf{T}$-equivalence. The latter map can be identified with $i_+ \Box f_+$, which is a $\mathbf{T}$-equivalence by Lemma 1.1.10. The characterisation of the fibrant objects now follows as in Lemma 1.1.11. □

Since $\mathbf{T}$ is a localization system, the arguments of Lemma 1.1.12 and Theorem 1.1.13 show:

Theorem 1.2.2: *i) A* G*-space* Z *is* $\mathbf{T}$*-local iff* Z^H *is* $\mathbf{T}([{}^{G}/_{H}])$*-local for every* $H \leq G$*,*
ii) A map of G*-spaces* $X \to Y$ *is a* $\mathbf{T}$*-localization iff* $X^H \to Y^H$ *is a* $\mathbf{T}([{}^{G}/_{H}])$*-localization for every* $H \leq G$*.*

At this point, it is helpful to compare based and unbased localizations in the non-equivariant setting. We have:

Lemma 1.2.3: *Let* Z *be an unbased space. Then:*
i) Z *is* $\mathbf{T}$*-local iff* $f^* : [B, Z] \to [A, Z]$ *is a bijection for all* $\mathbf{T}$*-equivalences,* $f : A \to B$*, between connected cofibrant spaces,*
ii) if $Z = \sqcup_{i \in I} Z_i$*, then* Z *is* $\mathbf{T}$*-local iff* Z_i *is* $\mathbf{T}$*-local for every* i*. In particular, a map of spaces which induces a bijection on connected components is a* $\mathbf{T}$*-localization iff each component is a* $\mathbf{T}$*-localization.*

Proof. If $f : A \to B$ is a $\mathbf{T}$-equivalence between cofibrant spaces, then f induces a bijection between the connected components of A and B, so f is a disjoint union of $\mathbf{T}$-equivalences $A_i \to B_i$, for i in the set of connected components of A. Now, $[\sqcup A_i, Z] = \prod_i [A_i, Z]$, and i) follows. For ii), if A is connected we have $[A, \sqcup_i Z_i] = \sqcup_i [A, Z_i]$, and so ii) follows from i). □

Lemma 1.2.4: *Let* $f : X \to Y$ *be a map of unbased spaces, with* X *non-empty. Then the following are equivalent:*

i) f *is an unbased* $\mathbf{T}$*-localization,*

ii) f *is a based* $\mathbf{T}$*-localization for some* $x \in X$*,*

iii) f *is a based* $\mathbf{T}$*-localization for all* $x \in X$*,*

iv) f_+ *is a based* $\mathbf{T}$*-localization, with respect to the adjoined basepoint* $+$*.*

Proof. The key point is that if Z is a $\mathbf{T}$-local based space, then it is also $\mathbf{T}$-local as an unbased space. This is a consequence of the fact that unbased homotopy classes $[A, Z]$ are equivalent to based homotopy classes $[A_+, Z]$, and the observation that if $A \to B$ is a $\mathbf{T}$-equivalence between cofibrant unbased spaces, then $A_+ \to B_+$ is a $\mathbf{T}$-equivalence between cofibrant based spaces. Now, $iii) \implies ii)$ is trivial, and $ii) \implies i)$ follows from the above. For $i) \implies iii)$, let $x \in X$. Since $\mathbf{T}$-localizations are preserved by composing with weak equivalences, we can assume that X is a CW-complex and f is a cofibration. Let $f_{\mathbf{T}} : X \to X_{\mathbf{T}}$ be a based $\mathbf{T}$-localization. Then $f_{\mathbf{T}}$ is also an unbased $\mathbf{T}$-localization, since $ii) \implies i)$. Therefore, there is a weak equivalence, g, such that $gf = f_{\mathbf{T}}$, and so f is also a based $\mathbf{T}$-localization, as desired. The fact that $iv) \implies i)$ follows from $ii) \implies i)$ and Lemma 1.2.3ii), and $i) \implies iv)$ follows from Lemma 1.2.3ii) and $i) \implies iii)$. □

Returning to the equivariant setting, we have the following consequence:

Theorem 1.2.5: *i) if $f : X \to Y$ is a based $\mathbf{T}$-localization, then it is also an unbased $\mathbf{T}$-localization,*
ii) if $f : X \to Y$ is a map of unbased G-spaces, then f is a $\mathbf{T}$-localization iff f_+ is a based $\mathbf{T}$-localization. Moreover, if X^G is non-empty, then f is a $\mathbf{T}$-localization iff f is a based $\mathbf{T}$-localization with respect to any G-fixed basepoint iff f is a based $\mathbf{T}$-localization with respect to all G-fixed basepoints.

1.3 An algebraic analogue

Before moving on to the theory of nilpotent G-spaces, we record the following result, which can be viewed as an algebraic analogue of the above theory. Recall that coefficient systems are functors $h\mathcal{O}^{op} \to \mathbf{Ab}$, and there are free coefficient systems defined by:

Definition 1.3.1: *The free coefficient system associated to the object $[G/H]$ is defined by $\mathbf{F}_{[G/H]}([G/K]) = \bigoplus_{h\mathcal{O}([G/K],[G/H])} \mathbb{Z}$ along with the evident definition on morphisms.*

The free coefficient systems have the property that $Hom_{[h\mathcal{O}^{op},\mathbf{Ab}]}(A \otimes \mathbf{F}_{[G/H]}, \mathbf{L}) \cong Hom_{\mathbf{Ab}}(A, \mathbf{L}([G/H]))$, where A is any abelian group.

Lemma 1.3.2: *Let $\mathbf{T}$ be a localization system and let $\mathbf{A}$ and $\mathbf{B}$ be coefficient systems such that:*

i) if the coefficient of $\mathbf{T}([G/H])$ is 0, then $\mathbf{A}([G/H]) \otimes \mathbb{Z}_{T([G/H])} = 0$ and $\mathbf{B}([G/H])$ is $T([G/H])$-local,
ii) if the coefficient of $\mathbf{T}([G/H])$ is 1, then $\mathbf{A}([G/H])$ is a $\mathbb{Z}[T([G/H])^{-1}]$-module and $\mathbf{B}([G/H])$ is $T([G/H])$-complete.

Then $Ext^i_{[h\mathcal{O}^{op},\mathbf{Ab}]}(\mathbf{A}, \mathbf{B}) = 0$ for all $i \geq 0$.

Proof. We first claim that if $\mathbf{T}([G/H])$ has coefficient 0, and n is a product of primes not in $T([G/H])$, then $Ext^i_{[h\mathcal{O}^{op},\mathbf{Ab}]}(\mathbf{F}_{[G/H]} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbf{B}) = 0$ for all $i \geq 0$. The category $[h\mathcal{O}^{op}, \mathbf{Ab}]$ has enough injectives, [9, Exercise 2.3.7], so we can calculate this by taking an injective resolution $\{\mathbf{Q}_i\}$ of $\mathbf{B}$. Such a resolution is an objectwise injective resolution of $\mathbf{B}([{}^{G}/_{H}])$, and $Hom_{[h\mathcal{O}^{op},\mathbf{Ab}]}(\mathbf{F}_{[G/H]} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbf{Q}_i) = Hom_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbf{Q}_i([{}^{G}/_{H}]))$, so taking homology calculates $Ext^i(\mathbb{Z}/n\mathbb{Z}, \mathbf{B}([{}^{G}/_{H}]))$, which vanishes by the non-equivariant case. Similarly, if $\mathbf{T}([{}^{G}/_{H}])$ has coefficient 1, then $Ext^i_{[h\mathcal{O}^{op},\mathbf{Ab}]}(\mathbf{F}_{[G/H]} \otimes \mathbb{Z}[\mathbf{T}([{}^{G}/_{H}])^{-1}], \mathbf{B}) = 0$ by [5, 10.1.22].

We will use this to define a $Hom_{[h\mathcal{O}^{op},\mathbf{Ab}]}(-, \mathbf{B})$-acyclic resolution, $\{\mathbf{P}_i\}$, of $\mathbf{A}$. If $\mathbf{T}([{}^{G}/_{H}])$ has coefficient 0, there is a coproduct, $\mathbf{K}_{[G/H]}$, of functors of the form $\mathbf{F}_{[G/H]} \otimes \mathbb{Z}/n\mathbb{Z}$, with n being a product of primes not in $T([{}^{G}/_{H}])$, such that there is a natural transformation $\mathbf{K}_{[G/H]} \to \mathbf{A}$ which is a surjection at $[{}^{G}/_{H}]$. If $\mathbf{T}([{}^{G}/_{H}])$ has coefficient 0, then there is a coproduct, $\mathbf{K}_{[G/H]}$, of functors of the form $\mathbf{F}_{[G/H]} \otimes \mathbb{Z}[\mathbf{T}([{}^{G}/_{H}])^{-1}]$, such that there is a natural transformation $\mathbf{K}_{[G/H]} \to \mathbf{A}$ which is a surjection at $[{}^{G}/_{H}]$. We define $\mathbf{P}_0 := \bigoplus_{[G/H]} \mathbf{K}_{[G/H]}$, so we have a surjection $\mathbf{P}_0 \to \mathbf{A}$, and $\mathbf{P}_0$ is $Hom_{[h\mathcal{O}^{op},\mathbf{Ab}]}(-, \mathbf{B})$-acyclic by the previous paragraph.

The key point now is that the functor $\mathbf{P}_0$ satisfies the conditions in i) and ii) that $\mathbf{A}$ does, and this follows from the fact that $\mathbf{T}$ is a localization system. In more detail, $\mathbf{F}_{[G/H]}([{}^{G}/_{K}])$ is only non-zero when there is a map $[{}^{G}/_{K}] \to [{}^{G}/_{H}]$ in $\mathcal{O}$, and then we have the following observations:

i) if $S \subset T$, then a torsion group with no T-torsion is also a torsion group with no S-torsion,
ii) if $S \subset T$, then a $\mathbb{Z}[T^{-1}]$-module is a $\mathbb{Z}[S^{-1}]$-module,
iii) a torsion group with no T-torsion is a $\mathbb{Z}[T^{-1}]$-module.

Therefore, we can inductively construct a $Hom_{[h\mathcal{O}^{op},\mathbf{Ab}]}(-, \mathbf{B})$-acyclic resolution $\{\mathbf{P}_i\}$ of $\mathbf{A}$, since the kernel of $\mathbf{P}_0 \to \mathbf{A}$ also satisfies i) and ii) in the statement of the lemma. Using the first paragraph of the proof, we can use this acyclic resolution to compute $Ext^i_{[h\mathcal{O}^{op},\mathbf{Ab}]}(\mathbf{A}, \mathbf{B}) = 0$ for all $i \geq 0$, as desired. □

2 Nilpotent G-Spaces

2.1 The main theorems

We now move on to the theory of nilpotent G-spaces and we begin with the definition of a nilpotent G-space. This differs from the definition given in [1, Ch. II] in that we do not require a common bound on the nilpotency classes at each fixed point space. To understand this, we will show in Subsection 2.4 that any nilpotent G-space can be approximated by a weak Postnikov tower, but if we assume a common bound on the nilpotency classes, then a nilpotent G-space can be approximated by a (strict) Postnikov tower, a distinction which becomes important when using co-HELP to deduce theorems about nilpotent spaces, as in [5, Section 3.3].

Definition 2.1.1: *A based G-space X is said to be nilpotent if X^H is a connected nilpotent space for all subgroups H of G.*

In the unbased context, we have the following definition:

Definition 2.1.2: *An unbased G-space X is said to be componentwise nilpotent if for every subgroup H of*

G, every component of X^H is a nilpotent space.

In general, if we speak about componentwise nilpotent G-spaces we are working in an unbased context, and if we speak about nilpotent G-spaces we are working in a based context.

By reduction to fixed point spaces, we can immediately deduce one of the most important properties of localizations of componentwise nilpotent G-spaces:

Theorem 2.1.3: *Let $\mathbf{T}$ be a localization system where all the coefficients are 0. Let $f : X \to Y$ be a map from a componentwise nilpotent G-space X to a $\mathbf{T}$-local unbased G-space Y, such that for every $H \leq G$, f^H induces a bijection on connected components. Then, the following are equivalent:*

i) f is a $\mathbf{T}$-localization,
ii) for all $H \leq G$, $* \geq 1$, and $b \in X^H$, $f^H_* : \pi_*(X^H, b) \to \pi_*(Y^H, f^H(b))$ is a $\mathbf{T}([G/H])$-localization of nilpotent groups,
iii) for all $H \leq G$ and $* \geq 1$, $f^H_* : H_*(X^H) \to H_*(Y^H)$ is a direct sum of $\mathbf{T}([G/H])$-localizations, where the sum ranges over the connected components of X^H.

Proof. This follows from [5, Theorem 6.1.2], as well as Lemma 1.2.3ii). □

Recall that if T is a set of primes and A is an abelian group, then $\mathbb{E}_T A$ and $\mathbb{H}_T A$ denote the zeroth and first derived functors of T-completion applied to A, respectively. These functors can be extended to take nilpotent groups as input by using the homotopy groups of completions of Eilenberg-Maclane spaces. In the current context, we use the above definition of $\mathbb{E}_{\mathbf{T}} G$ and $\mathbb{H}_{\mathbf{T}} G$ for sets of primes with coefficient 1. If, instead, $\mathbf{T}$ is a set of primes with coefficient 0, and G is a nilpotent group, we define $\mathbb{E}_{\mathbf{T}} G = G_T$ and $\mathbb{H}_{\mathbf{T}} G = 0$. This corresponds to using the homotopy groups of localizations of Eilenberg-Maclane spaces. A system of nilpotent groups, $\underline{G}$, is a continuous functor from $\mathcal{O}^{op}$ to the category of nilpotent groups, and we call such a system $\mathbf{T}$-local if it is pointwise $\mathbf{T}([G/H])$-local. The $\mathbf{T}$-localization $K(\underline{G}, 1) \to K(\underline{G}, 1)_{\mathbf{T}}$ specifies a homomorphism $\underline{G} \to \mathbb{E}_{\mathbf{T}}(\underline{G})$ and the, up to homotopy, universal property of $\mathbf{T}$-localization implies the following universal property:

Lemma 2.1.4: *Let $\underline{G}$ and $\underline{H}$ be systems of nilpotent groups, with $\underline{H}$ $\mathbf{T}$-local. Then any homomorphism $f : \underline{G} \to \underline{H}$ factors uniquely through the $\mathbf{T}$-localization $\underline{G} \to \mathbb{E}_{\mathbf{T}} \underline{G}$.*

Proof. This follows from the fact that $[K(\underline{G}, 1)_{\mathbf{T}}, K(\underline{H}, 1)] \cong [K(\underline{G}, 1), K(\underline{H}, 1)]$. □

If X is a nilpotent G-space, then this universal property defines a map from $\mathbb{E}_{\mathbf{T}} \underline{\pi}_i(X) \to \underline{\pi}_i(X_{\mathbf{T}})$ and we have the following theorem:

Theorem 2.1.5: *If X is a nilpotent G-space, then there is a natural short exact sequence:*

$$1 \to \mathbb{E}_{\mathbf{T}} \underline{\pi}_i(X) \to \underline{\pi}_i(X_{\mathbf{T}}) \to \mathbb{H}_{\mathbf{T}} \underline{\pi}_{i-1}(X) \to 1$$

If $f : X \to Y$ is a map between componentwise nilpotent G-spaces such that each f^H induces a bijection on connected components, and $\mathbb{H}_{\mathbf{T}([G/H])}(\pi_i(X^H, x)) = 0$ for all $H \leq G, i \geq 1$ and $x \in X^H$, then the following are equivalent:

i) f is a $\mathbf{T}$-localization,

ii) for all $i \geq 1$, $H \leq G$ and $x \in X^H$, $\pi_i(X^H, x) \to \pi_i(Y^H, f^H(x))$ is a $\mathbf{T}([G/H])$-localization.

For example, the hypothesis holds if, for all H, $X^H_{\mathbf{T}([G/H])}$ is a disjoint union of $f\mathbb{Z}_{T([G/H])}$-nilpotent spaces.

Proof. This follows from [5, Theorem 11.1.2, Proposition 10.1.23], as well as Lemma 1.2.3. □

Non-equivariantly, the fact that $Ext(\mathbb{H}_T B, \mathbb{E}_T A) = 0$, [5, Corollary 10.4.9], implies that the short exact sequence of Theorem 2.1.5 splits, however, equivariantly the sequence does not necessarily split as the following example shows. Take $G = C_2$. Then, consideration of Elmendorf's theorem, [1, Ch. V, Theorem 3.2], shows that to find a counterexample to the splitting, we can use the following counterexample to the naturality of the splitting in the non-equivariant case. For this, we let $X = K(\frac{\mathbb{Z}[p^{-1}]}{\mathbb{Z}}, 1)$, so that $\hat{X}_p = K(\hat{\mathbb{Z}}_p, 2)$, and a map $X \to K(\hat{\mathbb{Z}}_p, 2)$ is equivalent to a homomorphism $\hat{\mathbb{Z}}_p \to \hat{\mathbb{Z}}_p$. Then, any non-zero homomorphism, such as the identity, suffices to show that the splitting cannot be natural.

2.2 T-localization and fibre squares

From now on, we work in a based context. In this section, we discuss how $\mathbf{T}$-localization interacts with fibre sequences and homotopy pullbacks. Let $N(f, g)$ denote the double mapping path space of f and g. We have:

Theorem 2.2.1: *Let $f : X \to Z$ and $g : Y \to Z$ be maps of nilpotent G-spaces such that $N(f, g)$ is G-connected. If we have a commutative diagram:*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Z & \xleftarrow{g} & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \xrightarrow[f']{} & Z' & \xleftarrow[g']{} & Y' \end{array}$$

such that the vertical maps are $\mathbf{T}$-localizations, then the induced map $N(f, g) \to N(f', g')$ is a $\mathbf{T}$-localization.

Proof. This follows from [10, Proposition 3.1.2]. □

It follows that any functorial $\mathbf{T}$-localization, such as one obtained via the small object argument applied to the model structure of Theorem 1.1.9, preserves homotopy fibre squares of nilpotent G-spaces. The special case where $Y = Y' = *$ results in the connected fibre lemma.

2.3 Fracture Theorems

In this section, we move on to discuss fracture squares associated to nilpotent G-spaces. In the non-equivariant setting, we have fracture squares relating to localization and completion, [5, Theorem 8.1.3, Theorem 13.1.4], and we would like to generalise these results to the equivariant setting, perhaps localizing and completing at different sets of primes at each fixed point space. For example, the following two squares are homotopy fibre squares associated to a T-local connected nilpotent space X, where T is a set of primes containing 7:

$$\begin{array}{ccc} X & \longrightarrow & \hat{X}_T \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & (\hat{X}_T)_0 \end{array} \tag{1}$$

$$\begin{array}{ccc} X & \longrightarrow & (\prod_{p\in T\setminus\{7\}} X_p) \times \hat{X}_7 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & (\prod_{p\in T\setminus\{7\}} X_p)_0 \times (\hat{X}_7)_0 \end{array} \tag{2}$$

In 2, we complete at 7 to illustrate to point that there are an abundance of fracture squares that we can ask for, especially in the equivariant case. With this in mind, the following theorem subsumes all of the examples that we are aware of:

Theorem 2.3.1: *Consider a commutative square of nilpotent G-spaces:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ A & \xrightarrow[g]{} & B \end{array}$$

such that, for each subgroup H of G, there are sets of primes T_H, S_H satisfying:

i) X^H, Y^H are T_H-local and A^H, B^H are S_H-local,

ii) f^H is an $\mathbb{F}_{T_H}$-equivalence and g^H is an $\mathbb{F}_{S_H}$-equivalence,

iii) ϕ^H, ψ^H are $\mathbb{Q}$-equivalences.

Then the square is a homotopy fibre square.

Proof. Since taking fixed points detects homotopy fibre squares, we can reduce the theorem to a pointwise statement with sets of primes T and S. In this case, the theorem follows from (1) and successive applications of the pasting lemma for homotopy pullbacks, [6, Proposition 13.3.15]. □

When **T** is a constant localization system, we can derive, as in [5, Theorem 13.1.1], a fracture square for homotopy classes $[K, X]$, under certain finiteness hypotheses on K and X. In particular, K will always be a finite *based* G-CW complex, by which we mean that K is built by starting with a G-fixed basepoint and attaching finitely many cells along based maps out of G-spaces of the form $(\frac{G}{H})_+ \wedge S^n$, with $n \geq 0$. In order to give what we feel is the cleanest exposition of our main result, and the corresponding counterexample when $\mathbf{T}$ is not a constant localization system, we begin by recalling some preliminaries on homotopy pullbacks.

Definition 2.3.2: *Let $f : K \to X$ be a map of G-spaces. We define $[K \wedge (I^n)_+, f]$ to be the set of homotopy classes of maps $K \wedge (I^n)_+ \to X$ relative to the boundary $K \wedge (\partial I^n)_+$, where at each point on the boundary ∂I^n, the induced map is equal to f.*

Lemma 2.3.3: *Let $A \to^i B \to Ci$ be a cofibre sequence of G-spaces, and $f : Ci \to X$ a map of G-spaces. Then there is a natural long exact sequence of groups:*

$$... \to [B \wedge (I^2)_+, f] \to [\Sigma^2 A, X] \to [Ci \wedge I_+, f] \to [B \wedge I_+, f] \to [\Sigma A, X]$$

Moreover, the image of $[\Sigma^2 A, X]$ in $[Ci \wedge I_+, f]$ is central.

Proof. Modify f so that it is radially constant in a neighbourhood of the boundary of the cone. Consider the sequence of based maps:

$$... \to \Omega Map(A, X) \to^\partial Map(Ci, X) \to^{j^*} Map(B, X) \to^{i^*} Map(A, X)$$

where the spaces are given basepoints f and the constant loop to f. The fact that f is radially constant in a neighbourhood of the boundary of the cone allows us to define a based map, which is also a weak equivalence, $Fi^* \to Map(Ci, X)$. The map ∂ is then induced by a comparison of the fibre sequences associated to j^* and $Fi^* \to Map(B, X)$. It follows that $[S^1, -]$ takes the above sequence of maps to an exact sequence of groups, since it does so for the homotopy fibre sequence induced by i^*. The fact that the image of $\pi_1(\partial)$ is central follows from [5, Lemma 1.4.7 v).]. □

Let $N(f, g)$ denote the double mapping path space associated to maps $f : X \to A$ and $g : Y \to A$. We will make use of the following result on homotopy classes of maps into a homotopy pullback:

Lemma 2.3.4: *Let K be a based G-CW complex. Then the natural map of pointed sets:*

$$[K, N(f, g)] \to [K, X] \times_{[K,A]} [K, Y]$$

is a surjection. Suppose that f is a fibration, so that every element of $[K, X] \times_{[K,A]} [K, Y]$ can be represented by a pair of maps $u : K \to X, v : K \to Y$ such that $fu = gv := w$. Then the preimage of (u, v) is isomorphic *to the set of orbits of* $[K \wedge I_+, w]$ *under the right action of the group* $[K \wedge I_+, u] \times [K \wedge I_+, v]$*. In particular, the map is injective iff each of the functions* $[K \wedge I_+, u] \times [K \wedge I_+, v] \to [K \wedge I_+, w]$ *is surjective.*

Proof. This follows from the same arguments as in [5, Proposition 2.2.2], where the result is proved in the special case when u and v are nullhomotopic. □

Next, we seek to understand how the groups $[K \wedge I_+, f]$ behave with respect to $\mathbf{T}$-localization.

Lemma 2.3.5: *Let* $\mathbf{T}$ *be a constant localization system. Let* K *be a finite based* G*-CW complex, let* X *be a nilpotent* G*-space, and let* $f : K \to X$ *be a map. Then:*

i) $[K \wedge I_+, f]$ *is a nilpotent group, which is finitely* T*-generated (see [5, Definition 5.6.3]) if, for each* $i \geq 2$*,* $\pi_i(X^H)$ *is finitely* T*-generated,*
ii) if the coefficient of $\mathbf{T}$ *is 0, then* $[K \wedge I_+, f] \to [K \wedge I_+, \phi_{\mathbf{T}} f]$ *is* T*-localization, where* $\phi_{\mathbf{T}}$ *is a* $\mathbf{T}$*-localization of* X*,*
iii) if the coefficient of $\mathbf{T}$ *is 1,* $\mathbb{H}_{\mathbf{T}} \underline{\pi}_1(X) = 0$*, and, for each* $i \geq 2$*,* $\underline{\pi}_i(X^H)$ *is finitely* T*-generated, then* $[K \wedge I_+, f] \to [K \wedge I_+, \phi_{\mathbf{T}} f]$ *is* T*-completion.*

Proof. This follows by induction using Lemma 2.3.3. In more detail, part i) follows from [5, Lemma 3.1.3] and the fact that a nilpotent group G is finitely T-generated iff G_T is $f\mathbb{Z}_T$-nilpotent. Part ii) follows from [5, Corollary 5.4.11]. Part iii) follows from [5, Corollary 10.4.5], and the condition that $\mathbb{H}_{\mathbf{T}} \underline{\pi}_1(X) = 0$ ensures that $\underline{\pi}_2(X) \to \underline{\pi}_2(X_{\mathbf{T}})$ is $\mathbf{T}$-localization, by Theorem 2.1.5. □

We can now state our main fracture theorem for homotopy classes of maps:

Theorem 2.3.6: *Let* $\mathbf{T}, \mathbf{S}$*, and, for each* i *in some indexing set* I*,* $\mathbf{T_i}$ *be constant localization systems such that* $\mathbf{T}$ *and* $\mathbf{S}$ *have coefficient 0,* $T = \cup_i T_i$ *and* $T_i \cap T_j = S$*, for all* $i \neq j$*. Let* K *be a finite based* G*-CW complex and let* X *be a* T*-local nilpotent* G*-space such that, if* $\mathbf{T_i}$ *has coefficient 1, then for every subgroup* H *of* G*,* $\mathbb{H}_{T_i} \pi_1(X^H) = 0$*, and for each* $i \geq 2$*,* $\pi_i(X^H)$ *is finitely* T_i*-generated. Then the following diagram is a pullback of sets:*

$$\begin{array}{ccc} [K, X] & \longrightarrow & [K, \prod_i X_{\mathbf{T_i}}] \\ \downarrow & & \downarrow \\ [K, X_{\mathbf{S}}] & \longrightarrow & [K, (\prod_i X_{\mathbf{T_i}})_{\mathbf{S}}] \end{array}$$

Proof. The map from $[K, X]$ to the pullback is surjective by Theorem 2.3.1 and Lemma 2.3.4. The map is injective by Lemma 2.3.4 and Lemma 2.3.5. In more detail, to see that the surjectivity hypothesis in the final sentence of Lemma 2.3.4 is satisfied, surjectivity tells us that we can find a map $\mu : K \to X$ projecting onto a representative (u, v) of any element of the pullback. Then we can apply Theorem 2.3.1 to give a fracture square for $K([K \wedge I_+, \mu], 1)$, and the fact that $K([K \wedge I_+, \mu], 1)$ is connected tells us that the required map is surjective, via use of Lemma 2.3.5. □

Note that the fracture theorems for nilpotent groups given in [5, Theorem 7.2.1 ii), Theorem 12.3.2], are both consequences of Theorem 2.3.6.

To finish this section, we give an example to show that the square:

$$\begin{array}{ccc} [K, X] & \longrightarrow & [K, X_{\mathbf{T}}] \\ \downarrow & & \downarrow \\ [K, X_0] & \longrightarrow & [K, (X_{\mathbf{T}})_0] \end{array}$$

need not be a pullback of sets if $\mathbf{T}$ is not a constant localization system, where K is a finite based G-CW complex and, for every H, X^H is $f\mathbb{Z}_{T([G/H])}$-nilpotent. Of course, the comparison map $[K, X] \to [K, X_{\mathbf{T}}] \times_{[K,(X_{\mathbf{T}})_0]} [K, X_0]$ is always a surjection by Theorem 2.3.1 and Lemma 2.3.4. We let $G = C_2$, and let $\mathbf{T}([G/G]) = (\{p, q\}, 1)$, and $\mathbf{T}([G/e]) = (\{p\}, 1)$. We let $X = K(\underline{\mathbb{Z}}, n+2)_T$, where $\underline{\mathbb{Z}}$ is the constant coefficient system to $\mathbb{Z}$, and so $X^G = K(\mathbb{Z}_{\{p,q\}}, n+2)$ and $X^e = K(\mathbb{Z}_{\{p\}}, n+2)$. We let K be the cofibre:

$$\Sigma S^n \wedge (G_+) \to \Sigma S^n \wedge (e_+) \to K$$

where the first map is induced by the constant map $G \to e$. The cofibre sequence implies that there is a map of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [\Sigma K, X_{\mathbf{T}}] \oplus [\Sigma K, X_0] & \longrightarrow & \hat{\mathbb{Z}}_{\{p,q\}} \oplus \mathbb{Q} & \longrightarrow & \hat{\mathbb{Z}}_p \oplus \mathbb{Q} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & [\Sigma K, (X_{\mathbf{T}})_0] & \longrightarrow & \hat{\mathbb{Q}}_{\{p,q\}} & \longrightarrow & \hat{\mathbb{Q}}_p & \longrightarrow & 0 \end{array}$$

It follows that the map $[\Sigma K, X_{\mathbf{T}}] \oplus [\Sigma K, X_0] \to [\Sigma K, (X_{\mathbf{T}})_0]$ can be identified with the rationalisation $\hat{\mathbb{Z}}_q \to \hat{\mathbb{Q}}_q$ which is not surjective. It follows that the square above is not a pullback of sets. In particular, there exist maps $f, g : K \to X$, such that $f_{\mathbf{T}} \simeq g_{\mathbf{T}}$ and $f_0 \simeq g_0$, but f and g are not homotopic.

2.4 Nilpotent G-spaces and Postnikov towers

We now discuss the relationship between nilpotent G-spaces and towers of principal fibrations. First, we will define what it means for a system of π-groups to be $\mathcal{B}$-nilpotent, where $\mathcal{B}$ is a class of coefficient systems. Then we will define the analogue of Postnikov towers in the equivariant setting, and we will show that a space, X, is equivalent to a *weak* Postnikov $\mathcal{B}$-tower iff its homotopy groups are $\mathcal{B}$-nilpotent systems of π-groups, where $\pi = \underline{\pi}_1(X)$. Finally, we will show that $\mathbf{T}$-local nilpotent G-spaces are equivalent to weak Postnikov $\mathcal{B}_{\mathbf{T}}$-towers, where $\mathcal{B}_{\mathbf{T}}$ is the class of $\mathbf{T}$-local coefficient systems.

Definition 2.4.1: *Let $\mathcal{B}$ be a class of coefficient systems of abelian groups. Let $\underline{\pi}$ be a coefficient system of groups and let $\mathbf{G}$ be a coefficient system of groups admitting an action of $\underline{\pi}$ by automorphisms. We say that $\mathbf{G}$ is a $\mathcal{B}$-nilpotent $\underline{\pi}$-group if there is a descending sequence of normal $\underline{\pi}$-subgroups:*

$$\mathbf{G} = \mathbf{G_0} \supseteq \mathbf{G_1} \supseteq \mathbf{G_2} \supseteq ...$$

such that:

i) $\underline{\pi}$ acts trivially on $\frac{\mathbf{G_{i-1}}}{\mathbf{G_i}}$,
ii) $\frac{\mathbf{G_{i-1}}}{\mathbf{G_i}} \in \mathcal{B}$,
iii) for every H, $\frac{\mathbf{G_{i-1}}}{\mathbf{G_i}}([G/H]) \to \frac{\mathbf{G}}{\mathbf{G_i}}([G/H])$ has central image,
iv) for every H, $\frac{\mathbf{G_{i-1}}}{\mathbf{G_i}}([G/H]) = 0$ for all but finitely many i.

Definition 2.4.2: *Call a $\mathcal{B}$-nilpotent $\underline{\pi}$-group bounded if the filtration in Definition 2.4.1 can be replaced by a finite filtration terminating at 1.*

Definition 2.4.3: *We call a G-space X $\mathcal{B}$-nilpotent if it is G-connected and, for all $i \geq 1$, $\underline{\pi}_i(X)$ is a $\mathcal{B}$-nilpotent $\underline{\pi}_1(X)$-group. We say that a $\mathcal{B}$-nilpotent G-space, X, is bounded if the homotopy groups $\underline{\pi}_i(X)$ are all bounded $\mathcal{B}$-nilpotent $\underline{\pi}_1(X)$-groups*

Note that a G-space is nilpotent iff it is $\mathcal{A}$-nilpotent, where $\mathcal{A}$ is the class of all coefficient systems of abelian groups. This follows from the fact that if X is a nilpotent space, then there are functorial filtrations of $\pi_i(X)$ satisying the conditions of the previous definition - the lower central series when $i = 1$, and the filtration induced by the augmentation ideal, $\{I^n \pi_i(X)\}$, for $i \geq 2$.

Definition 2.4.4: *A map of G-spaces, $f : X \to Y$, is called a principal $K(\underline{A}, n)$-fibration if it is the pullback of the path-space fibration along a map $k : Y \to K(\underline{A}, n+1)$. In particular, f is a fibration with fibre $K(\underline{A}, n)$.*

Definition 2.4.5: *Let $\mathbf{Q}$ be the totally ordered set consisting of pairs of natural numbers ordered by $(m, n) \leq (p, q)$ iff $m < p$ or $m = p$ and $n \leq q$. A weak Postnikov $\mathcal{B}$-tower is a functor $\mathbf{Q} \to G - Sp$, satisfying:*

i) $X_{1,1} = *$,
ii) $X_{n+1,1} \to \lim_i X_{n,i}$ is a weak equivalence,
iii) The map $X_{n,i+1} \to X_{n,i}$ is a principal $K(\underline{B}_{n,i}, n)$-fibration for some $\underline{B}_{n,i} \in \mathcal{B}$,
iv) for every n and H, $X^H_{n,i+1} \to X^H_{n,i}$ is a weak equivalence for all but finitely many i.

Definition 2.4.6: *A Postnikov $\mathcal{B}$-tower is a weak Postnikov $\mathcal{B}$-tower such that the maps $X_{n+1,1} \to lim_i X_{n,i}$ of condition ii) above are G-homeomorphisms.*

We have the principal fibration lemma:

Lemma 2.4.7: *Let $f : X \to Y$ be a map of well-pointed G-connected G-spaces with the homotopy type of a G-CW complex, such that $Ff \simeq K(\underline{A}, n)$ for some coefficient system $\underline{A}$ and $n \geq 1$. Then $\underline{\pi}_1(X)$ acts trivially on $\underline{\pi}_*(Ff)$ iff there is a weak equivalence $X \to Fk$ over Y, for some cofibration $k : Y \to K(\underline{A}, n+1)$.*

Proof. See [10, Lemma 2.2.2]. □

Lemma 2.4.8: *A G-space is $\mathcal{B}$-nilpotent iff it is weakly equivalent to a weak Postnikov $\mathcal{B}$-tower.*

Proof. To see that weak Postnikov $\mathcal{B}$-towers are $\mathcal{B}$-nilpotent, it suffices to show that $\underline{\pi}_n(X)$ is a $\mathcal{B}$-nilpotent $\underline{\pi}_1(X)$-group. Let $\mathbf{G}_i$ be the kernel of $\underline{\pi}_n(X) \to \underline{\pi}_n(X_{n,i+1})$. The quotients $\frac{\mathbf{G}_{i-1}}{\mathbf{G}_i}$ correspond to the coefficient systems $\underline{B}_{n,i}$ appearing in the tower, and so these are in $\mathcal{B}$ by assumption. $\underline{\pi}_1(X)$ acts trivially on $\frac{\mathbf{G}_{i-1}}{\mathbf{G}_i}$ by Lemma 2.4.7, and the required inclusions are central for the same reason as in the non-equivariant case, namely [5, Lemma 1.4.7 v)]. Finally, $\frac{\mathbf{G}_{i-1}}{\mathbf{G}_i}([G/H]) = 0$ for all but finitely many i, since $X^H_{n,i+1} \to X^H_{n,i}$ is a weak equivalence for all but finitely many i.

Next assume that a G-CW complex, X, is $\mathcal{B}$-nilpotent. So, for each n, we have a filtration of $\underline{\pi}_n(X)$, $\{\mathbf{G}^n_i\}$, satisfying the conditions of Definition 2.4.1. We define $X^0_{n,i+1}$ by first attaching cells to X along all possible maps $(\frac{G}{H})_+ \wedge S^n \to X$ representing an element of $\mathbf{G}^n_i([G/H])$, for some $H \leq G$. Then, inductively define $X^j_{n,i+1}$, for each $j \geq 1$, by attaching a cell to $X^{j-1}_{n,i+1}$ along every possible map $(\frac{G}{H})_+ \wedge S^{n+j} \to X^j_{n,i+1}$, for any $H \leq G$. Define $X_{n,i+1}$ as the union of the $X^j_{n,i+1}$. Then:

i) $\underline{\pi}_j(X_{n,i+1}) = \underline{\pi}_j(X)$ for $j < n$,
ii) $\underline{\pi}_n(X_{n,i+1}) = \frac{\underline{\pi}_n(X)}{\mathbf{G}_i}$,
iii) $\underline{\pi}_j(X_{n,i+1}) = 0$ for $j > n$.

Moreover, we have an inclusion $X_{n,i} \to X_{m,j}$, whenever $(m, j) \leq (n, i)$ in $\mathbf{Q}$. Since the action of $\underline{\pi}_1(X)$ is trivial on each $\frac{\mathbf{G}_{i-1}}{\mathbf{G}_i}$, each of the maps $X_{n,i+1} \to X_{n,i}$ is equivalent to a $K(\underline{B}_{n,i}, n)$-principal fibration, with $\underline{B}_{n,i} \in \mathcal{B}$. To define the weak Postnikov $\mathcal{B}$-tower, we keep each $X_{n,1}$ fixed, and inductively replace each $X_{n,i}$, for $i \geq 2$, using Lemma 2.4.7. Results of Waner, [11, Corollary 4.14], imply that by doing this we never leave the category of well-pointed G-spaces with the homotopy type of a G-CW complex. Unfortunately, we could leave this category by taking inverse limits, which is why we leave each $X_{n,1}$ fixed and only require a weak equivalence in Definition 2.4.5 ii). □

If we restrict attention to bounded $\mathcal{B}$-nilpotent G-spaces, such as pointwise simply connected G-spaces, then the same proof shows:

Lemma 2.4.9: *A bounded $\mathcal{B}$-nilpotent G-space is weakly equivalent to a Postnikov $\mathcal{B}$-tower.*

As promised, we have the following characterisation of $\mathbf{T}$-local nilpotent spaces:

Lemma 2.4.10: *A nilpotent G-space is $\mathbf{T}$-local iff it is $\mathcal{B}_{\mathbf{T}}$-nilpotent, where $\mathcal{B}_{\mathbf{T}}$ is the class of $\mathbf{T}$-local coefficient systems. A G-space is a $\mathbf{T}$-local bounded $\mathcal{A}$-nilpotent G-space iff it is bounded $\mathcal{B}_{\mathbf{T}}$-nilpotent.*

Proof. If X is $\mathcal{B}_{\mathbf{T}}$-nilpotent it is easily verified that all of the homotopy groups of X^H are $\mathbf{T}([G/H])$-local, which implies that X is $\mathbf{T}$-local. Suppose that X is $\mathbf{T}$-local. Recall that we have a central $\underline{\pi}_1(X)$-series for $\underline{\pi}_i(X)$ induced by the functorial lower central series when $i = 1$, or the functorial augmentation ideal series when $i \geq 2$. Localizing these series at $\mathbf{T}$, and considering their images in $\underline{\pi}_i(X)$, expresses each $\underline{\pi}_i(X)$ as a $\mathcal{B}_{\mathbf{T}}$-nilpotent $\underline{\pi}_1(X)$-series, and so X is $\mathcal{B}_{\mathbf{T}}$-nilpotent. If X is bounded $\mathcal{A}$-nilpotent, then the lower central series terminates after finitely many stages and so X is bounded $\mathcal{B}_{\mathbf{T}}$-nilpotent. □

2.5 Localization at equivariant cohomology theories

We end this paper by tying up the following loose end. Namely, in [1, Ch. II], localizations of nilpotent G-spaces were defined relative to equivariant cohomology theories, and we would like to compare this to our localizations relative to $\mathbf{T}$-equivalences.

Definition 2.5.1: *A map of G-spaces, $f : X \to Y$, is a cohomology $\mathbf{T}$-equivalence if for all $\mathbf{T}$-local coefficient systems $\underline{A}$, $f^* : H^*(Y; \underline{A}) \to H^*(X; \underline{A})$ is an isomorphism.*

We have:

Theorem 2.5.2: *A map of G-spaces, $f : X \to Y$, is a $\mathbf{T}$-equivalence iff it is a cohomology $\mathbf{T}$-equivalence.*

Proof. If f is a $\mathbf{T}$-equivalence, then each $K(\underline{A}, n)$ is $\mathbf{T}$-local for every $\mathbf{T}$-local coefficient system $\underline{A}$, so f is a cohomology $\mathbf{T}$-equivalence.

If f is a cohomology $\mathbf{T}$-equivalence, then we can assume that X, Y are well-pointed. Now, $\Sigma^2 f : \Sigma^2 X \to \Sigma^2 Y$ is a cohomology $\mathbf{T}$-equivalence between pointwise simply connected G-spaces. By the first part, $(\Sigma^2 f)_{\mathbf{T}}$ is also a cohomology $\mathbf{T}$-equivalence between pointwise simply connected $\mathbf{T}$-local G-spaces. Simply connected $\mathbf{T}$-local G-spaces are weakly equivalent to strict Postnikov $\mathcal{B}_{\mathbf{T}}$-towers by the previous section, so the equivariant analogue of co-HELP, [5, Theorem 3.3.7], implies that $(\Sigma^2 f)_{\mathbf{T}}$ is a weak equivalence and, therefore, that $\Sigma^2 f$ is a $\mathbf{T}$-equivalence. It follows that f is a $\mathbf{T}$-equivalence. □

Corollary 2.5.3: *Localization with respect to $\mathbf{T}$-equivalences is equivalent to localization with respect to cohomology $\mathbf{T}$-equivalences.*

References

[1] J. P. May et al. *Equivariant Homotopy and Cohomology Theory*, volume 91 of *CBMS Regional Conference Series in Mathematics*. AMS, 1996.

[2] J. P. May, J. McClure, and G. Triantafillou. Equivariant Localization. *Bulletin of the London Mathematical Society*, 14(3):223–230, 05 1982.

[3] J. P. May. Equivariant Completion. *Bulletin of the London Mathematical Society*, 14(3):231–237, 05 1982.

[4] A.K. Bousfield. Types of acyclicity. *Journal of Pure and Applied Algebra*, 4(3):293–298, 1974.

[5] J. P. May and K. Ponto. *More Concise Algebraic Topology: localization, completion, and model categories*. Chicago lectures in mathematics. University of Chicago Press, 2012.

[6] P. Hirschhorn. *Model Categories and Their Localizations*, volume 99 of *Mathematical Surveys and Monographs*. AMS, 2003.

[7] S. Illman. The equivariant triangulation theorem for actions of compact lie groups. *Mathematische Annalen*, 262:487–501, 1983.

[8] G. Bredon. *Introduction to Compact Transformation Groups*, volume 46 of *Pure annd Applied Mathematics*. Elsevier, 1972.

[9] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[10] A. Ronan. Completion preserves homotopy fibre squares of connected nilpotent spaces. `https://arxiv.org/abs/2209.00478`, 2022. https://arxiv.org/abs/2209.00478.

[11] Stefan Waner. Equivariant homotopy theory and Milnor's theorem. *Transactions of the American Mathematical Society*, 258(2):351–368, 1980.