Nilpotent Groups, Spaces and G-Spaces

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Contents

Prelude

Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy in Mathematics. It has been composed by myself and has not been submitted in any previous application for any degree. I declare that the work contained within this thesis is my own original work, except where otherwise indicated, cited, or commonly known. Content from Chapters 3, 4 and 5 has been submitted for publication.

Abstract

The purpose of this thesis is to study localisations and completions of nilpotent groups, spaces and G-spaces, and to contribute to the theory of each. We begin, in Chapter 2, with an introduction to nilpotent groups and spaces, based around a proof of the classical characterisation of finitely generated nilpotent spaces. In Chapter 3, we derive double coset formulae for the genus and extended genus of a finitely generated nilpotent group. In Chapter 4, we prove that T-completion preserves homotopy fibre squares of nilpotent spaces. In Chapter 5, we develop the equivariant generalisation of the theory of nilpotent spaces. We conclude the thesis, in Chapter 6, with a few more results, centred around the closure properties of the category of well pointed spaces with the homotopy type of a CW complex, that we have used in previous chapters. We invite the reader to read the Introduction, Chapter 1, where we describe the results of each chapter in more detail.

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Chapter 1

Introduction

The purpose of this thesis is to study localisations and completions of nilpotent groups, spaces and G-spaces, and to contribute to the theory of each. Nilpotent spaces arise in topology since they are precisely the spaces which are equivalent to Postnikov towers, and it is for this reason that they have particularly nice algebraic properties with respect to Bousfield localisation at the homology theories $H_*(-;\oplus_{p\in T}\mathbb{F}_p)$ and $H_*(-;\mathbb{Z}_T)$, where T is a set of primes. For example, the Serre spectral sequence can be used to prove, by induction up towers, that the homotopy groups of a nilpotent space are finitely generated iff its homology groups are, and similar arguments can be used to prove that a map of nilpotent spaces is a T-localisation iff it is a T-localisation at the level of homotopy or homology groups. Completion is more subtle, partially due to the existence of higher derived functors in the abelian group case, but even here we can inductively prove that a nilpotent space is T-complete iff its homotopy groups are, and that a T-completion of nilpotent spaces induces T-completion followed by the inclusion of a summand at the level of homotopy groups.

We begin the thesis, in Chapter 2, with an introduction to nilpotent groups and spaces, which is based around a proof of the following characterisation of f-nilpotent spaces ([MP12, Theorem 4.5.2]):

Theorem 1.0.1: *Let X be a nilpotent space. Then the following statements are equivalent:*

i) X is weakly equivalent to a CW complex with finite skeleta,

ii) X is f-nilpotent,

iii) $\pi_i(X)$ is finitely generated for each $i \geq 1$,

iv) $\pi_1(X)$ and, for $i \geq 2$, $H_i(\tilde{X})$ are finitely generated,

v) $H_i(X)$ is finitely generated for each $i \geq 1$.

We introduce nilpotent groups in Section 2.1, before deriving some properties of finitely generated nilpotent groups in Section 2.2. This allows us to prove the purely algebraic parts of Theorem 1.0.1. Moving onto the topology, we introduce nilpotent spaces in Section 2.3, and develop some of their basic theory, such as their relationship to Postnikov towers. In Section 2.4, we resume the proof of Theorem 1.0.1, by proving a theorem of Wall, [Wal65, Theorem A], which gives a criterion for when a space is equivalent to a CW complex with finite skeleta. The criterion applies to show that $K(G, 1)$ is equivalent to a CW complex with finite skeleta whenever G is a finitely generated nilpotent group, and this kick starts the proof of Serre's theorem, that the homotopy groups of a simply connected space are finitely generated iff its homology groups are, in Subsection 2.4.2. We complete the proof of Theorem 1.0.1 in Subsection 2.4.3, by proving that if each homology group of a nilpotent space is finitely generated, then so are the homology groups of its universal cover.

Other topics covered in Chapter 2 include the construction of relative Postnikov towers, in Theorem 2.3.11, for maps between connected spaces inducing a nilpotent action on a possibly disconnected fibre, cocellular maps and homotopies, in Subsection 2.3.2, and a brief review of the construction of localisations and completions of nilpotent spaces, in Section 2.5.

The purpose of Chapter 3, is to derive double coset formulae for the genus and extended genus of an $f\mathbb{Z}_T$-nilpotent group, where the genus and extended genus are defined as follows, relative to a decomposition of the set of primes, $T = \cup_i T_i$:

Definition 1.0.2: *Let G be an $f\mathbb{Z}_T$-nilpotent group. Define:*
i) the genus of G to be the set of isomorphism classes of $f\mathbb{Z}_T$-nilpotent groups, H, such that for every $i \in I$, $H_{T_i} \cong G_{T_i}$,
ii) the extended genus of G to be the set of isomorphism classes of T-local nilpotent groups, H, such that for every $i \in I$, $H_{T_i} \cong G_{T_i}$.

After quickly reviewing the results about localisations of nilpotent groups that we will use, in Section 3.2, we set about deriving the following two double coset formulae in Sections 3.3 and 3.4, respectively:

Theorem 1.0.3: *The extended genus of G is in 1-1 correspondence with the double coset:*

$$Aut(G_S)\backslash Aut_{b.a.}(\textstyle\prod_i G_S)/\prod_i Aut(G_{T_i})$$

where $Aut_{b.a.}(\prod_i G_S)$ is the monoid of automorphisms of the form $\prod_i \alpha_i$ which are S-bounded above, see Definition 3.3.1.

Theorem 1.0.4: *The genus of G is in 1-1 correspondence with the double coset:*

$$Aut(G_S)\backslash Aut_b(\textstyle\prod_i G_S)/\prod_i Aut(G_{T_i})$$

where $Aut_b(\prod_i G_S)$ is the subgroup of automorphisms of the form $\prod_i \alpha_i$ which are S-bounded, see Definition 3.4.1.

These results provide the correct generalisation of the double coset formula for the genus described in eg. [HY17, Example 2.3] to the case where I is an infinite indexing set, such as when $T_i = \{p_i\}$, where p_i is the ith prime.

The double coset formulae of Theorems 1.0.3 and 1.0.4 are formulated relative to a fixed fracture square with the diagonal map, $\Delta : G_S \to \prod G_S$, along the base. In Section 3.5, we investigate what happens if we try to derive double coset formulae with respect to a formal fracture square, with a localisation, $\omega : G_S \to (\prod G_{T_i})_S$, along the base. It turns out that such a fracture square only sees the genus of G, and not the extended genus, and we can derive the following double coset formula for the genus:

Theorem 1.0.5: *There is a 1-1 correspondence between the genus of G and the double coset:*

$$Aut(G_S) \setminus DAut((\prod G_{T_i})_S) \ / \ \prod Aut(G_{T_i})$$

where $DAut((\prod G_{T_i})_S)$ is the subgroup of diagonal automorphisms of $(\prod G_{T_i})_S$, see Definition 3.5.1.

The following satisfying result, which is at the heart of Section 3.5, explains why the formal fracture square only sees the genus, and allows us to relate the double coset formulae for the genus given in Theorems 1.0.4 and 1.0.5:

Lemma 1.0.6: *An automorphism, $\alpha \in \prod Aut(G_S)$, is the image of a diagonal automorphism iff α is S-bounded.*

We also discuss, in Subsection 3.1.1, some other notions of genus, such as the adelic genus of an $f\mathbb{Z}_T$-nilpotent group or space, and review some other double coset formulae which can be found in the literature.

Our convention in this thesis is that all nilpotent spaces are connected, and a disjoint union of nilpotent spaces is called a componentwise nilpotent space. With this in mind, the aim of Chapter 4 is to prove the following result, which is a generalisation of the connected fibre lemma of Bousfield and Kan, [BK72a, Ch. II Lemma 4.8]:

Theorem 1.0.7: *Let $f : X \to A$ and $g : Y \to A$ be maps between connected nilpotent spaces such that $N(f, g)$ is connected. If we have a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & A \arrow[d] & Y \arrow[l, "g"'] \arrow[d] \\ \widehat{X}_T \arrow[r, "\widehat{f}_T"'] & \widehat{A}_T & \widehat{Y}_T \arrow[l, "\widehat{g}_T"] \end{tikzcd}$$

such that the vertical maps are T-completions, then the induced map $N(f, g) \to N(\widehat{f}_T, \widehat{g}_T)$ is a T-completion.

It follows from Theorem 1.0.7, that a functorial T-completion preserves homotopy fibre squares of nilpotent spaces. As an application of Theorem 1.0.7, we deduce the following fracture square, [DFK77, Theorem 4.4], associated to a T-local nilpotent space:

Theorem 1.0.8: *Let X be a T-local nilpotent space. Then any commutative square:*

$$\begin{array}{ccc} X & \xrightarrow{\widehat{\phi}} & \widehat{X}_T \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\phi} \\ X_0 & \xrightarrow[\widehat{\phi}_0]{} & (\widehat{X}_T)_0 \end{array}$$

with $\widehat{\phi}$ a T-completion and ψ, ϕ rationalisations, is a homotopy fibre square.

We will see in Subsection 5.3.3, that further fracture squares for nilpotent spaces and groups follow in short order from Theorem 1.0.8, at least once one has got to grips with some general properties of homotopy pullbacks.

Along the way to proving Theorem 1.0.7, we make heavier use of a spectral sequence associated to a map between nilpotent groups, [BK72a, Ch. III, Lemma 5.8], than in the past, to derive the following useful criterion, which allows us to recognise when the kernel and, if the image is normal, the cokernel of a homomorphism between $\mathcal{C}$-nilpotent groups are $\mathcal{C}$-nilpotent:

Lemma 1.0.9: *Let $\mathcal{C}$ be a class of abelian groups which is closed under taking kernels and cokernels of abelian group homomorphisms between members of $\mathcal{C}$. Let $f : G \to H$ be a group homomorphism between $\mathcal{C}$-nilpotent groups. Then, the kernel and, if the image is normal, the cokernel of f are $\mathcal{C}$-nilpotent. Examples of such $\mathcal{C}$ include:*
i) the class of R-modules, where R is a solid ring ([BK72b, Definition 2.1]),
ii) the class, $\mathcal{B}_T$, of T-complete abelian groups,
iii) the class of $f\widehat{\mathbb{Z}}_T$-modules (that is, the class of finitely generated $\widehat{\mathbb{Z}}_T$-modules),
iv) the class of T-complete abelian groups A, such that, for every $p \in T$, $\widehat{A}_p$ is an $f\widehat{\mathbb{Z}}_p$-module.

As an application of Lemma 1.0.9, we deduce the following additional properties of the categories of T-complete and $f\widehat{\mathbb{Z}}_T$-nilpotent groups:

Lemma 1.0.10: *Let G be a T-complete nilpotent group, and H a T-complete subgroup. Then:*

i) there is a subnormal series $H = H_0 \leq H_1 \leq ... \leq H_k = G$, where each H_i is T-complete,
ii) if G is $f\widehat{\mathbb{Z}}_T$-nilpotent, then so is H,
iii) if T is a finite set of primes and G is $f\widehat{\mathbb{Z}}_T$-nilpotent, then G satisfies the ascending chain condition (ACC) for T-complete subgroups,
iv) if G is a T-torsion $f\widehat{\mathbb{Z}}_T$-nilpotent group, then G is finite.

In Chapter 5, we develop the theory of localisations and completions of nilpotent G-spaces, where G is a compact Lie group. Our treatment of the equivariant theory improves upon previous treatments, such as in [MMT82], [May82] and [M$^+$96, Ch. II], in the following ways. Firstly, we localise with respect to localisation systems, which allows us to localise or complete at different sets of primes at different fixed point spaces. Secondly, we develop the theory in both a based and unbased context, which allows us to extend the theory to G-spaces which are not G-connected, or have no G-fixed points. Thirdly, our approach allows us to use a more general definition of a nilpotent G-space than in previous use - that is, we do not require a common bound on the nilpotency class of $\pi_i(X^H)$, as H varies.

We take a homological approach to the theory, which allows us to use the Bousfield localisation process to reduce the equivariant theory to the non-equivariant theory at fixed point spaces. The definition of a localisation system is the starting point for such an approach - intuitively, it ensures that we invert more primes at X^K than X^H, whenever K is subconjugate to H in G, and so we have a map $X^H \to X^K$. Here, completing at a set of primes, T, 'inverts more primes' than localising at T.

Definition 1.0.11: *A localisation system is a functor* $\mathbf{T} : \mathcal{O}^{op} \to \mathbf{P}^{op} \times \mathbf{1}$*, where* $\mathbf{P}$ *is the poset of subsets of the set of prime numbers partially ordered by inclusion,* $\mathbf{1}$ *is the category with objects 0 and 1 and a single arrow from 0 to 1, and* $\mathcal{O}$ *is the orbit category of* G*.*

We can Bousfield localise with respect to the $\mathbf{T}$-equivalences, defined in Definition 5.2.4, and the definition of a localisation system ensures that the resulting model structure has the following properties:

Theorem 1.0.12: *Let* $\mathbf{T}$ *be a localisation system. Then:*
i) A based G*-space* Z *is* $\mathbf{T}$*-local iff* Z^H *is* $\mathbf{T}([{}^{G}\!/\!{}_{H}])$*-local for every* $H \leq G$*,*
ii) A map of based G*-spaces* $X \to Y$ *is a* $\mathbf{T}$*-localisation iff* $X^H \to Y^H$ *is a* $\mathbf{T}([{}^{G}\!/\!{}_{H}])$*-localisation for every* $H \leq G$*.*

Using Theorem 1.0.12, we deduce some fundamental properties of $\mathbf{T}$-localisations of nilpotent G-spaces in Subsections 5.3.1 and 5.3.2. For example, considering the appropriate constant localisation system, we have that a nilpotent G-space is p-complete iff all homotopy groups $\pi_i(X^H)$ are p-complete. In Subsection 5.3.3, we derive some new fracture squares for nilpotent G-spaces, using a general result on homotopy pullbacks, Lemma 5.3.10, to deduce fracture squares for homotopy classes, Theorem 5.3.12, from fracture squares for G-spaces, Theorem 5.2.12. We also use Lemma 5.3.10, to give an example of a non-constant localisation system, $\mathbf{T}$, and maps $f, g : K \to X$, where K is a finite based G-CW complex and each X^H is $f\mathbb{Z}_{T([{}^{G}\!/\!{}_{H}])}$-nilpotent, such that $f_{\mathbf{T}} \simeq g_{\mathbf{T}}$ and $f_0 \simeq g_0$, but f and g are not homotopic. Note that this cannot happen if $\mathbf{T}$ is a constant localisation system, by Theorem 5.2.12, and, therefore, cannot happen non-equivariantly.

To conclude Chapter 5, we introduce equivariant Postnikov towers, in Subsection 5.3.4, and use them to relate our homological approach to the theory to the classical cohomological approach of [MMT82] and [May82], in Subsection 5.3.5. In particular, we prove:

Theorem 1.0.13: *A map of* G*-spaces,* $f : X \to Y$*, is a* $\mathbf{T}$*-equivalence iff, for all* $\mathbf{T}$*-local coefficient systems* $\underline{A}$*,* $f^* : H^*(Y; \underline{A}) \to H^*(X; \underline{A})$ *is an isomorphism.*

In Chapter 6, we introduce the closure properties of the category of well-pointed spaces of the homotopy type of a CW complex that have been used in previous chapters. Note that, for these spaces, reduced cohomology is represented by based homotopy classes of maps into Eilenberg-MacLane spaces, which played a role in our inductive construction of Postnikov towers, Theorem 2.3.11. For some results, it is useful to view spaces of the (unbased) homotopy type of a CW complex as, precisely, the m-cofibrant objects in the mixed model structure of Cole, [Col06]. Therefore, we begin the chapter with an introduction to the q,h, and m-model structures on the category of spaces, taking the opportunity to showcase our favourite proof of the factorisation axiom for the h-model structure, Lemma 6.1.4, along the way. In Section 6.3, we use the m-model structure to prove one direction of a theorem of Stasheff, [Sta63], which is the non-equivariant case of Theorem 1.0.14, below. We prove the other direction by making use of the fact that geometric realisation takes Kan fibrations of simplicial sets to h-fibrations. In Section 6.4, we consider the equivariant generalisation of Stasheff's theorem:

Theorem 1.0.14: *Let $p : E \to B$ be an h-fibration, and suppose that B has the G-homotopy type of a G-CW complex. Then E has the G-homotopy type of a G-CW complex iff for every $b \in B$, $p^{-1}(b)$ has the H_b-homotopy type of an H_b-CW complex, where H_b is the isotropy group of b.*

Waner proved the more difficult direction in [Wan80, Corollary 4.14] - namely, that if E and B have the G-homotopy type of a G-CW complex, then each fibre has the H_b-homotopy type of an H_b-CW complex. In Theorem 6.4.6, we use the m-model structure to prove the reverse direction. Note that in this new direction we are not assuming any kind of continuity of CW-structures on the fibres, even in the compact Lie case.

We conclude the thesis, in Section 6.5, with an article defining locally F-trivial maps of simplicial spaces, and proving the following theorem about them:

Theorem 1.0.15: *If $p : X \to Y$ is a locally F-trivial map of simplicial spaces, and Y is a proper simplicial space, then $|p|$ is a Hurewicz fibration.*

The theorem provides a common generalisation of the results that minimal fibrations of simplicial sets realise to h-fibrations, [GJ09, Theorem 10.9], and that the realisation of the orbit map $EG \to BG$ is an h-fibration, whenever G is a topological group with a nondegenerate basepoint, [May75, Theorem 8.2]. The article is best viewed as paired with May's paper, [May90], in which he develops the basic properties of quasifibrations, and gives an inductive criterion for when a map of spaces is a quasifibration, [May90, Theorem 2.7]. These results can be used to conclude that the realisation of $EG \to BG$ is a quasifibration, whenever G is a grouplike topological monoid with a nondegenerate basepoint. The only difference is that there is a shortcut available to prove that $EG \to BG$ is an h-fibration, [May75, Theorem 8.2], in the topological group case, which means that the full generality of Theorem 1.0.15 is not needed in that context.

1.0.1 Prerequisites

The prerequisites for reading this thesis are fairly minimal - we assume the reader has taken a first course in homotopy theory, such as [May99], and is familiar with the first two chapters of [MP12], which discuss fundamental group actions in fibre sequences and some basic properties of homotopy limits, as well as chapter 24 of [MP12], which discusses the Serre spectral sequence. At times, we will refer the reader to other parts of [MP12] for topics which are already well-covered there. We do not assume any familiarity with nilpotent groups, and will derive the results we need about them from scratch. In Chapter 5, we assume familiarity with model categories and the Bousfield cardinality argument, as well as the basics of equivariant homotopy theory over a compact Lie group. Finally, some knowledge of simplicial sets, such as Chapter I of [GJ09], would be useful for parts of Chapter 6.

Chapter 2

Introduction to Nilpotent Groups and Spaces

In this introductory chapter, we will introduce the basic theory of nilpotent groups and spaces by proving the following theorem, which provides a characterisation of finitely generated nilpotent spaces:

Theorem 2.0.1: *Let X be a nilpotent space. Then the following statements are equivalent:*

i) X is weakly equivalent to a CW complex with finite skeleta,

ii) X is f-nilpotent,

iii) $\pi_i(X)$ is finitely generated for each $i \geq 1$,

iv) $\pi_1(X)$ and, for $i \geq 2$, $H_i(\tilde{X})$ are finitely generated,

v) $H_i(X)$ is finitely generated for each $i \geq 1$.

We note that a complete proof of the theorem is scattered throughout the literature, and a sketch proof, which pulls together the various sources, can be found in [MP12, Theorem 4.5.2].

2.1 Nilpotent groups

We will begin with the algebra, introducing nilpotent groups, as well as their basic properties. Our treatment is influenced, in particular, by the introduction to nilpotent groups in [CMZ17], and most of the results of this section can be found there. Firstly, we fix some notation:

Definition 2.1.1: *Let G be a group. If $x, y \in G$, define the commutator of x and y by the formula $[x, y] := x^{-1}y^{-1}xy$. Define $x^y := y^{-1}xy$. The n-fold commutator $[x_0, ..., x_n]$ of elements $x_i \in G$ is then defined inductively by the formula $[x_0, ..., x_n] := [[x_0, ..., x_{n-1}], x_n]$.*

The following identities relating to the commutator can be easily verified:

Lemma 2.1.2: *Let G be a group and $x, y, z \in G$. Then:*

i) $[x, y]^z = [x^z, y^z]$,

ii) $[xy, z] = [x, z]^y [y, z]$,

iii) $[x, yz] = [x, z][x, y]^z$,

iv) $[x^{-1}, y] = [x, y^{-1}]^{x^{-1}y}$.

Proof. See [CMZ17, Lemma 1.4]. □

We now consider subgroups generated by commutators:

Definition 2.1.3: *If S_0, S_1 are non-empty subsets of G, define $[S_0, S_1]$ to be the subgroup of G generated by the commutators $[s_0, s_1]$, with $s_0 \in S_0$ and $s_1 \in S_1$. If $S_0, ..., S_n$ are non-empty subsets of G, we inductively define $[S_0, ..., S_n]$ to be the subgroup generated by commutators of the form $[x, s_n]$, with $x \in [S_0, ..., S_{n-1}]$ and $s_n \in S_n$.*

It is clear that each n-fold commutator of the form $[s_0, ..., s_n]$, with $s_i \in S_i$, is an element of $[S_0, ..., S_n]$ - however, in general, $[S_0, ..., S_n]$ is not generated by such commutators. However, we do have:

Lemma 2.1.4: *If $H_0, ..., H_n$ are normal subgroups of G, then $[H_0, ..., H_n]$ is a normal subgroup of G which is generated by n-fold commutators $[h_0, ..., h_n]$, with $h_i \in H_i$.*

Proof. Firstly, we can inductively show that $[H_0, ..., H_n]$ is normal using the fact that $[x, y]^z = [x^z, y^z]$. Then, using the fact that $[xy, z] = [x, z]^y [y, z]$, we can reduce to showing that $[[h_0, ..., h_{n-1}]^{-1}, h_n]$ is a product of n-fold commutators and their inverses. This follows from the identity $[x^{-1}, y] = [x, y^{-1}]^{x^{-1}y}$. □

We can now introduce the lower and upper central series associated to a group G - we will see shortly that a group is nilpotent iff either of these series terminate at 1 or G, respectively.

Definition 2.1.5: *The lower central series of a group G, is the descending sequence of normal subgroups defined for $i > 0$ by $\Gamma^i G := [G_0, .., G_i]$, with each $G_j = G$. When $i = 0$, we take $\Gamma^i G = G$.*

Definition 2.1.6: *The upper central series of a group G, is the ascending sequence of normal subgroups defined by $Z_0 G = 1$, and, inductively, $Z_{i+1} G := \pi^{-1}(Z(\frac{G}{Z_i G}))$, where $\pi : G \to \frac{G}{Z_i G}$ is the quotient map. Therefore, $Z_1(G)$ is the centre of G.*

The next lemma provides a useful alternate characterisation of the component groups of the upper and lower central series:

Lemma 2.1.7: *i) $\Gamma^n G$ is the subgroup of G generated by the n-fold commutators $[g_0, ..., g_n]$, with $g_i \in G$,*
ii) $Z_n G$ is the subgroup of G consisting of elements z such that for any choice of $g_1, ..., g_n \in G$, $[z, g_1, ..., g_n] = 1$.

Therefore, $\Gamma^n G = 1$ iff $Z_n G = G$.

Proof. i) follows from Lemma 2.1.4 and ii) follows by induction from Definition 2.1.6. The remaining statement is an easy consequence. □

We can now define a nilpotent group, and relate the definition to the upper and lower central series:

Definition 2.1.8: *A group G is called nilpotent if there exists a finite series of subgroups of the form:*

$$1 = G_0 \subset G_1 \subset ... \subset G_k = G,$$

such that each G_i is normal in G and, for every i, $\frac{G_{i+1}}{G_i} \subset Z(\frac{G}{G_i})$ (equivalently, $[G_{i+1}, G] \subset G_i$).

Lemma 2.1.9: *If the series $\{G_i\}_{i=0}^k$ expresses G as a nilpotent group, then $G_i \leq Z_iG$ and $\Gamma^i G \leq G_{k-i}$.*

Proof. Suppose inductively that $G_i \leq Z_iG$. Then $G_{i+1} \leq \pi^{-1}(\frac{G}{G_i}) \leq \pi^{-1}(\frac{G}{Z_iG}) = Z_{i+1}G$. Next, suppose inductively that $\Gamma^i G \leq G_{k-i}$. Then, $\Gamma^{i+1}G = [\Gamma^i G, G] \leq [G_{k-i}, G] \leq G_{k-i-1}$, since $\frac{G_{k-i}}{G_{k-i-1}}$ is a central subgroup of $\frac{G}{G_{k-i-1}}$. □

Corollary 2.1.10: *A group G is nilpotent iff its upper central series terminates at G iff its lower central series terminates at 1. Moreover, if G is nilpotent, both the upper and lower central series represent G as a nilpotent group.*

Definition 2.1.11: *The nilpotency class of a nilpotent group G is the length of the lower central series of G, which is equal to the length of the upper central series of G, and is the minimal possible length of any central series representing G as a nilpotent group. Equivalently, c is the minimal integer such that $\Gamma^c G = 1$.*

Nilpotent groups are preserved under quotients, subgroups and central extensions. The exact formulation of the following lemma will be useful for proving that the homotopy pullback of nilpotent spaces is componentwise nilpotent:

Lemma 2.1.12: *If G is a nilpotent group, then any subgroup and any quotient group of G is nilpotent. Moreover, if $1 \to K \to G \to H \to 1$ is an exact sequence of groups such that K and H are nilpotent and K is contained in Z_iG for some i, then G is nilpotent.*

Proof. This is a consequence of Lemma 2.1.7. □

Recall that a subgroup H of G is subnormal if there is a finite chain of subgroups $H = H_0 \subset H_1 \subset ... \subset H_k = G$, with each H_i normal in H_{i+1}. Another important property of nilpotent groups is that all subgroups are subnormal - this can often be used to inductively extend proofs that only work for normal subgroups to include all subgroups.

Lemma 2.1.13: *If H is a subgroup of a nilpotent group G, then H is subnormal in G.*

Proof. The sequence of subgroups given by $H\Gamma^i G$ is a subnormal series for H, since if $g \in \Gamma^i G$, then g commutes with any element of G up to an element of $\Gamma^{i+1}G$. □

Finally, for this section, we discuss bilinear maps induced by taking commutators:

Lemma 2.1.14: *If $\{G_i\}$ is a nilpotent series for G, then taking commutators defines a bilinear map:*

$$\frac{G_{i+1}}{G_i} \otimes Ab(G) \to \frac{G_i}{G_{i-1}}$$

Proof. For fixed $g \in G$, the map $G_{i+1} \to \frac{G_i}{G_{i-1}}$ defined by $a \to [a,g]$ is a group homomorphism, since $[ab,g] = [a,g]^b[b,g]$, and $\frac{G_i}{G_{i-1}}$ is central in $\frac{G}{G_{i-1}}$. If $a \in G_i$, then $[a,g] \in G_{i-1}$, so we get a map $G \to Hom(\frac{G_{i+1}}{G_i}, \frac{G_i}{G_{i-1}})$. Since $[a,gh] = [a,h][a,g]^h$, we get a group homomorphism $Ab(G) \to Hom(\frac{G_{i+1}}{G_i}, \frac{G_i}{G_{i-1}})$, as desired. □

Applied to the lower central series, we get a surjective bilinear map:

Lemma 2.1.15: *There is an epimorphism:*

$$Ab(G) \otimes ... \otimes Ab(G) \to \frac{\Gamma^n G}{\Gamma^{n+1} G},$$

sending $(g_0, ..., g_n)$ to $[g_0, ..., g_n]$.

Proof. This follows from Lemma 2.1.14 and Lemma 2.1.7. □

The next result, which is [War76, Theorem 3.25], will be used repeatedly in Chapter 3:

Lemma 2.1.16: *Let G be a nilpotent group of nilpotency class c, H a subgroup of G, and A a set of elements of G such that there exists an $s \in \mathbb{N}$ such that $a \in A \implies a^s \in H$. Then, if $g \in G$ is in the subgroup generated by elements of A and H, $g^{s^d} \in H$ where $d = \frac{1}{2}c(c+1)$.*

Proof. Let K be the subgroup of G generated by elements of A and H. Then K has nilpotency class $e \leq c$. Let:

$$1 = \Gamma^e K \subset ... \subset \Gamma^1 K \subset \Gamma^0 K = K$$

be the lower central series of K. Suppose that $k \in K$ is of the form xh where $x \in \Gamma^i K$ and $h \in H$. By Lemma 2.1.15, $\frac{\Gamma^i K}{\Gamma^{i+1} K}$ is an abelian group generated by commutators of the form $[z_0, ..., z_i]$ where each $z_i \in A \cup H$. Since the commutators are bilinear, and $\frac{\Gamma^i K}{\Gamma^{i+1} K}$ is a central subgroup of $\frac{K}{\Gamma^{i+1} K}$, it follows that $k^{s^{i+1}} = yh^{'}$ for some $y \in \Gamma^{i+1} K$, $h^{'} \in H$. □

We can also use Lemma 2.1.14 to study the upper central series. Let $\tau_p(G)$ denote the p-torsion subgroup of a nilpotent group G, which is a subgroup by Lemma 2.1.16. We have:

Lemma 2.1.17: *Let G be a nilpotent group and r a non-negative integer. If $\tau_p(Z(G))^{p^r} = 1$, then, for all $i \geq 0$, $\tau_p(\frac{Z_{i+1} G}{Z_i G})^{p^r} = 1$.*

Proof. Let $z \in \frac{Z_{i+1}G}{Z_iG}$ be p-torsion, and suppose that $z^{p^r} \neq 1$. Then, by Lemma 2.1.7, there exist $g_1, ..., g_i$ in $Ab(G)$, such that $[z^{p^r}, g_1, ..., g_i] = p^r[z, g_1, ..., g_i] \neq 1$ in $Z(G)$. However, $[z, g_1, ..., g_i]$ is p-torsion, a contradiction. □

Corollary 2.1.18: *Let G be a nilpotent group. Then, G is p-torsion-free iff $Z(G)$ is p-torsion-free. Also, G has bounded p-torsion iff $Z(G)$ has bounded p-torsion.*

2.2 Finitely generated nilpotent groups

We now restrict attention to finitely generated nilpotent groups. We will use our previous work to develop their basic properties, culminating in a proof that finitely generated nilpotent groups are finitely presented and have Noetherian group rings. We begin with the following definition:

Definition 2.2.1: *An f-nilpotent group, or $f\mathbb{Z}$-nilpotent group, is a nilpotent group G such that there exists a series $\{G_k\}$ representing G as a nilpotent group, for which each quotient $\frac{G_{i+1}}{G_i}$ is a finitely generated abelian group.*

Our previous results allow us to prove the bulk of the following theorem. The remainder of this section will be devoted to proving the final sentence.

Theorem 2.2.2: *Let G be a nilpotent group. Then the following are equivalent:*

i) G is finitely generated,
ii) $Ab(G)$ is finitely generated,
iii) G is f-nilpotent,
iv) Every subgroup of G is finitely generated.

Moreover, if these conditions are satisfied, then G is finitely presented, and $\mathbb{Z}[G]$ is a (left and right) Noetherian ring.

Proof. The implications $iv) \implies i) \implies ii)$ are trivial. The implication $ii) \implies iii)$ follows from the epimorphism of Lemma 2.1.15. It remains to prove that $iii) \implies iv)$. For this we start with the fact that all subgroups of a finitely generated abelian group are finitely generated. Suppose that the series:

$$1 = G_0 \to G_1 \to ... \to G_m = G$$

expresses G as an f-nilpotent group. Assume that every subgroup of G_k is finitely generated and consider the short exact sequence:

$$1 \to G_k \to G_{k+1} \to \frac{G_{k+1}}{G_k} \to 1$$

in which both G_k and the finitely generated abelian group $\frac{G_{k+1}}{G_k}$ satisfy 'Max', which means that every subgroup of them is finitely generated. This is the terminology used in [Hal54, Theorem 1], which is the source of Lemma 2.2.5 below. Now if H is a subgroup of G_{k+1}, then the image of H in $\frac{G_{k+1}}{G_k}$ is finitely generated, as well as $H \cap G_k$. It follows that H is finitely generated, and so G_{k+1} satisfies Max. It follows, inductively, that G satisfies Max. □

Note that since every subgroup of an f-nilpotent group G is finitely generated, any series representing G as a nilpotent group has finitely generated quotients.

For the remainder of this section, assume that G is a finitely generated nilpotent group. The fact that $\mathbb{Z}[G]$ is Noetherian holds more generally for polycyclic groups, and we begin by showing that G is polycyclic.

Definition 2.2.3: *A group G is called polycyclic if it has a subnormal series of the form:*

$$1 = G_0 \to G_1 \to ... \to G_m = G$$

in which each quotient is a cyclic group.

By the structure theorem for finitely generated abelian groups, we know that finitely generated abelian groups are polycyclic. The fact that G is polycyclic follows from this fact and an induction up a series expressing G as an f-nilpotent group, using the next lemma.

Lemma 2.2.4: *If K and H are polycyclic in the short exact sequence of groups:*

$$1 \to K \to G \to H \to 1,$$

then G is polycyclic.

Proof. Let the subnormal series $\{K_i\}$ and $\{H_i\}$ express K and H as polycyclic groups. Then we define a subnormal series on G by:

$$1 = K_0 \to K_1 \to ... \to K_m = K = f^{-1}(H_0) \to f^{-1}(H_1) \to ... \to f^{-1}(H_n) = G$$

The fact that this series expresses G as a polycyclic group follows from the third isomorphism theorem for groups. □

A G-module, M, is said to satisfy Max-G if all G-submodules of M are finitely generated as G-modules. The following lemma, due to P.Hall, [Hal54, Theorem 1], is precisely what's needed to conclude that $\mathbb{Z}[G]$ is Noetherian.

Lemma 2.2.5: *Suppose that H is a normal subgroup of G such that $\frac{G}{H}$ is either finite or infinite cyclic, and that N is an H-submodule of the right G-module M such that $M = NG$. Then if N satifies Max-H, M satisfies Max-G.*

Proof. If $\frac{G}{H}$ is finite, then let $g_0, ..., g_n$ be elements of G representing each element of $\frac{G}{H}$. Then Ng_i is an H-submodule of M for each i and we have an epimorphism $\oplus_i Ng_i \to M$, since $M = NG$. If A is an H-submodule of Ng_i, then $A' = \{n \in N | ng_i \in A\}$ is an H-submodule of N and so is finitely generated. It follows that Ng_i satisfies Max-H and, therefore, so do $\oplus_i Ng_i$ and M. It follows that M satisfies Max-G.

If $\frac{G}{H} \cong \mathbb{Z}$, let g be an element of G representing $1 \in \frac{G}{H}$. Then, since $M = NG$, every element $m \in M$ is of the form:

$$m = \sum_{k \in \mathbb{Z}} n_k g^k$$

where $n_k \in N$ for all k, and all but finitely many of the n_k are 0. Let A be a G-submodule of M. If p, q are integers such that $p < q$, define the H-submodule $N_{p,q}$ of N by:

$$N_{p,q} = \{n \in N | \text{ there exists } \textstyle\sum_k n_k g^k \in A \text{ such that } n_i = 0 \text{ if } i < p \text{ or } i > q \text{ and } n_p = n\}$$

Multiplication by g and its inverse shows that $N_{p,q}$ depends only on the value of $q - p$, so define $N_i = N_{0,i}$ for $i \in \mathbb{N}$. Then we have an ascending chain of H-submodules of N:

$$N_1 \subset N_2 \subset ...$$

Since N satisfies Max-H, this sequence must terminate after finitely many steps, say at $N_j,\ j \in \mathbb{N}$.

For each $i \leq j$, let $\{m_{i,0}, m_{i,1}, ..., m_{i,t_i}\}$ be a set of elements of A representing a generating set for N_i. We claim that this is a generating set for A as a G-module. Suppose that:

$$m = \sum_{k \in \mathbb{Z}} n_k g^k$$

is an element of A. By subtracting elements of the form $m_{j,t} g^k h$ and multiplying by some g^s, we may assume that $n_k = 0$ for $k < 0$ and $k > j - 1$. Then, by subtracting elements of the form $m_{i,t} h$, where $i < j$, we can reduce all the way to 0. Hence, A is a finitely generated G-module, as desired. □

Corollary 2.2.6: *If G is a polycyclic group, then $\mathbb{Z}[G]$ is a Noetherian ring.*

Proof. This follows from the Lemma 2.2.5, the definition of a polycyclic group and the observation that if $H \leq G$, then $\mathbb{Z}[H]$ is an H-submodule of $\mathbb{Z}[G]$, and $\mathbb{Z}[G] = \mathbb{Z}[H]G$. Also, $\mathbb{Z}[1] = \mathbb{Z}$ is Noetherian. □

This shows that f-nilpotent groups have Noetherian group rings. The final task of Theorem 2.2.2 is to show that they are also finitely presented. We know that finitely generated abelian groups are finitely presented, and so the result will follow by inducting up a series expressing G as an f-nilpotent group, using the following lemma:

Lemma 2.2.7: *If K and H are finitely presented in a short exact sequence of groups:*

$$1 \to K \to G \to H \to 1$$

then G is also finitely presented.

Proof. Let $H = \langle S \mid R \rangle$ be a presentation of H, where S is a finite set and $R \subset W(S, S^{-1})$ is a finite subset of the words in S and their inverses. Similarly, let $K = \langle P \mid Q \rangle$ be a finite presentation of K. Then we have an epimorphism $\phi : \langle S \cup P \mid \ \rangle \to G$ defined by sending elements of P to their images in G and elements of S to a chosen preimage in G.

We will define three finite subsets of the kernel of ϕ. Firstly, we have

$$Q \subset W(P, P^{-1}) \subset W((S \cup P), (S \cup P)^{-1}).$$

For each $r \in R$, we have $\phi(r) \in K$. Let w_r be a word in P and P^{-1} representing $\phi(r)^{-1}$. Then we define:

$$R' = \{rw_r \mid r \in R\} \subset W(S \cup P, (S \cup P)^{-1})$$

Now let (s, p) be a pair consisting of an element $s \in S \cup S^{-1}$ and an element $p \in P \cup P^{-1}$. The image of the conjugate p^s is in K and so let $w_{(s,p)}$ be a word in P and P^{-1} representing $\phi(p^s)^{-1}$. Define:

$$T = \{p^s w_{(s,p)} \mid s \in S \cup S^{-1}, p \in P \cup P^{-1}\}.$$

We claim that $G = \langle S \cup P \mid Q \cup R' \cup T \rangle$.

Suppose that:

$$w_1 v_1 w_2 v_2 ... w_k v_k$$

is a product of words in P and P^{-1} (the w_i) and words in S and S^{-1} (the v_i), which is in the kernel of ϕ. Then $v_1 ... v_k$ is a product of conjugates of elements of R. Therefore, by multiplying by conjugates of elements of R', we may assume that $v_1 ... v_k = 1$. In this case our word is of the form:

$$w_1^{v_1} ... w_k^{v_k}$$

where the w_i are words in P and the v_i are words in S. In fact we may assume that each v_i is an element of $S \cup S^{-1}$, and each w_i is an element of $P \cup P^{-1}$. Then this word is of the form:

$$t_{(s_1,p_1)} w_{(s_1,p_1)}^{-1} ... t_{(s_k,p_k)} w_{(s_k,p_k)}^{-1}$$

where $t_{(s_i,p_i)}$ is the element of T corresponding to (s_i, p_i). Hence, by multiplying by conjugates of elements of T, we may assume our original word is of the form w_1, a word in P and P^{-1}. This case is then dealt with by multiplying by conjugates of elements of Q to reduce to 1. □

Corollary 2.2.8: *If G is an f-nilpotent group, then G is finitely presented.*

2.3 Nilpotent Spaces

In this section, we introduce nilpotent spaces, and derive some of their basic properties. When working with spaces, the homotopy groups involved are equipped with an action of the fundamental group of the space. Therefore, it is necessary to generalise the definitions of Section 2.1 slightly, by defining a $\mathcal{C}$-nilpotent π-group. To do this, we recall the following definitions from [MP12]:

Definition 2.3.1: *Let π be a group. A π-group is another group, G, equipped with an action of π on G via automorphisms.*

If G is abelian, we recover the notion of a π-module. Now, let $\mathcal{C}$ be a class of abelian groups.

Definition 2.3.2: *Let G be a π-group. We say that G is a $\mathcal{C}$-nilpotent π-group, if there exists a finite normal series:*

$$1 = G_0 \subset G_1 \subset ... \subset G_q = G$$

such that, for every i, G_i is a π-subgroup of G, $\frac{G_{i+1}}{G_i} \in \mathcal{C}$, π acts trivially on $\frac{G_{i+1}}{G_i}$, and $\frac{G_{i+1}}{G_i}$ is a central subgroup of $\frac{G}{G_i}$.

By convention, if $\mathcal{C} = Ab$ or $\pi = 1$, we can drop the $\mathcal{C}$ or π from the notation, as appropriate. In this case, our new definition of a nilpotent group, agrees with the old one, Definition 2.1.8. Note that a π-module, M, is nilpotent iff there exists a natural number n such that $I^n M = 0$, where I is the augmentation ideal of $\mathbb{Z}[\pi]$.

We have the following generalisation of Lemma 2.1.12:

Lemma 2.3.3: *Let $1 \to K \to G \to L \to 1$ be an exact sequence of π-groups. If the extension is central, and K, L are $\mathcal{C}$-nilpotent π-groups, then G is a $\mathcal{C}$-nilpotent π-group. Conversely, if $\mathcal{C}$ is closed under passage to subgroups and quotient groups and G is a $\mathcal{C}$-nilpotent π-group, then K and L are $\mathcal{C}$-nilpotent π-groups.*

Proof. This is [MP12, Lemma 3.1.3]. □

We can now define a nilpotent space:

Definition 2.3.4: *A space X is said to be $\mathcal{C}$-nilpotent if it is connected, and, for every $i \geq 1$, $\pi_i(X)$ is a $\mathcal{C}$-nilpotent π-group, where $\pi = \pi_1(X)$.*

Note that the condition on the fundamental group, π, is equivalent to π being a $\mathcal{C}$-nilpotent group. Using this algebraic definition of a nilpotent space, it is reasonably straightforward to prove that each component of a homotopy pullback of nilpotent spaces is nilpotent:

Proposition 2.3.5: *If $f : X \to B$ and $g : Y \to B$ are maps between nilpotent spaces, then every component of $N(f, g)$ is a nilpotent space.*

Proof. Pick any basepoint for $N(f, g)$. We will consider the long exact sequence of $\pi_1(N(f, g))$-modules associated to the fibre sequence:

$$\Omega B \to^{\partial} N(f, g) \to X \times Y$$

To show that $\pi_1(N(f, g))$ is nilpotent, it suffices to show that the image of ∂_* is contained in some $Z_i\pi_1(N(f, g))$, by Lemma 2.1.12. By comparison with the fibre sequence associated to $B^{I_+} \to B \times B$, it is clear that $\pi_1(N(f, g))$ acts nilpotently on $\pi_2(B)$, since B is nilpotent. Now, ∂ is a map of $\pi_1(N(f, g))$-modules which means that if $g \in \pi_1(N(f, g)), b \in \pi_2(B)$, then $\partial(g^{-1} \cdot b) = g^{-1}\partial(b)g$. Therefore, $\partial(-b + g^{-1} \cdot b) = \partial(b)^{-1}g^{-1}\partial(b)g = [\partial(b), g]$. So considering $(g^{-1}-1)$ as an element of the augmentation ideal in $\mathbb{Z}[\pi_1(N(f, g))]$, we have $\partial((g^{-1} - 1) \cdot b)) = [\partial(b), g]$. Iterating this, we have for any $g_1, ..., g_k \in \pi_1(N(f, g))$,

$$[\partial(b), g_1, ..., g_k] = \partial((g_k^{-1} - 1)...(g_1^{-1} - 1) \cdot b)$$

Since $\pi_2(B)$ is a nilpotent $\pi_1(N(f, g))$-module, this implies there is some k such that $[\partial(b), g_1, ..., g_k] = 1$ for any choice of elements in $\pi_1(N(f, g))$. This is equivalent to saying $\partial(b) \in Z_k\pi_1(N(f, g))$, as desired.

The fact that $\pi_i(N(f, g))$ is a nilpotent $\pi_1(N(f, g))$-group for $i \geq 2$, follows from the fact that $\pi_i(X) \times \pi_i(Y)$ is a nilpotent $\pi_1((N(f, g))$-group and Lemma 2.3.3. ∎

2.3.1 Relative Postnikov towers

Nilpotent spaces arise in topology, since they are precisely the spaces which can be approximated by Postnikov towers. Roughly speaking, a Postnikov tower inductively constructs the homotopy groups of the space by coattaching cocells of the form $PK(A, n) \to K(A, n)$, where A is an abelian group. In this way, Postnikov towers can be viewed as 'dual' to CW-complexes, and, therefore, many results concerning CW complexes, such as Whitehead's theorem, have analogues for nilpotent spaces. Moreover, since a nilpotent space is equivalent to a tower of principal fibrations, the Serre spectral sequence can be used to inductively extend properties of the Eilenberg MacLane spaces that build the tower, to the nilpotent space itself.

We will now prove a relative version of the result that nilpotent spaces are equivalent to Postnikov towers, since the relative version is well-known and used, eg in [Far03], but it is hard to find a proof in the literature.

We begin with the definition of a relative Postnikov tower:

Definition 2.3.6: *A map of spaces, $f : X \to Y$, is called a principal $K(A, n)$-fibration if it is the pullback of the path-space fibration along a map $k : Y \to K(A, n+1)$. In particular, f is a fibration with fibre $K(A, n)$.*

Definition 2.3.7: *Let $\mathcal{B}$ be a class of abelian groups. Let $\mathbf{Q}$ be the totally ordered set consisting of pairs of natural numbers ordered by $(m, n) \leq (p, q)$ iff $m < p$ or $m = p$ and $n \leq q$. A relative Postnikov $\mathcal{B}$-tower is a functor $\mathbf{Q} \to Sp$, satisfying:*

i) the map $X_{n,i+1} \to X_{n,i}$ is a principal $K(B_{n,i}, n-1)$-fibration for some $B_{n,i} \in \mathcal{B}$,
ii) for every n, $X_{n,i+1} \to X_{n,i}$ is a homeomorphism for all but finitely many i,
iii) $X_{n+1,1} \cong \lim_i X_{n,i}$.

Definition 2.3.8: *Let $f : X \to Y$ be a map of spaces. A relative Postnikov $\mathcal{B}$-tower for f, is a relative Postnikov $\mathcal{B}$-tower, $\{X_{n,i}\}$, and a weak equivalence $X \to \lim X_{n,i}$, such that $X_{1,1} = Y$ and the composite $X \to \lim X_{n,i} \to Y$ is equal to f.*

As usual, we can drop the $\mathcal{B}$ from the notation if $\mathcal{B} = Ab$. For the proof, we need a criterion to tell us whether a fibration is a principal $K(A, n)$-fibration. For this, we will use the following reformulation of the relative Hurewicz theorem:

Lemma 2.3.9: *Let $n \geq 1$. If $f : X \to Y$ is a map of connected spaces such that Ff is $(n-1)$-connected and $\pi_1(X)$ acts trivially on $\pi_n(Ff)$, then $\eta : Ff \to \Omega Cf$ is an n-equivalence which induces an isomorphism on π_n.*

Our criterion is the same as [MP12, Lemma 3.4.2] in the case where $n \geq 1$:

Lemma 2.3.10: *Let $n \geq 0$. Let $f : X \to Y$ be a map between connected well-pointed spaces of the homotopy type of a CW complex such that $Ff \simeq K(A, n)$. If $n = 0$, suppose that the image of $\pi_1(X)$ in $\pi_1(Y)$ is normal. If $n \geq 1$, suppose that $\pi_1(X)$ acts trivially on A. Then there exists an equivalence $X \to Fk$ over Y, for some cofibration $k : Y \to K(A, n+1)$.*

Proof. Suppose first that $n \geq 1$. We know that $\eta : Ff \to \Omega Cf$ induces an isomorphism on π_i for $i \leq n$, by Lemma 2.3.9. Therefore, there is a cofibration $j : Cf \to K(A, n+1)$ which induces an isomorphism on π_{n+1}. We now consider the diagram:

$$\begin{tikzcd}
X \arrow[r, "\nu"] \arrow[dr, "f"'] & Fi \arrow[r, "w"] \arrow[d] & Fk \arrow[dl] \\
& Y \arrow[d, tail, "i"'] \arrow[dr, tail, "k"] & \\
& Cf \arrow[r, tail, "j"'] & K(A,n+1)
\end{tikzcd}$$

where w is induced by j and ν is the canonical map which induces η on fibres. By construction, $w\nu$ is a weak equivalence, as desired.

If $n = 0$, Cf is a connected space with $\pi_1(Cf) = A = \frac{\pi_1(Y)}{\pi_1(X)}$, and so we can construct the same diagram as in the $n \geq 1$ case. This time, we can observe that η induces a bijection on π_0, and the result now follows similarly. $\square$

We can now prove:

Theorem 2.3.11: *Let X and Y be well-pointed, connected spaces of the homotopy type of a CW complex. If $f : X \to Y$ is a map such that, for all $n \geq 1$, $\pi_n(Ff)$ is a $\mathcal{B}$-nilpotent $\pi_1(X)$-module, and the image of $\pi_1(X)$ in $\pi_1(Y)$ is $\mathcal{B}$-subnormal (that is subnormal with quotients in $\mathcal{B}$), then there is a relative Postnikov $\mathcal{B}$-tower for f, with coattaching maps that are cofibrations.*

Proof. Assume that $n \geq 1$. Let $\{G_{n,i}\}$ denote a $\mathcal{B}$-nilpotent filtration of the $\pi_1(X)$-group $\pi_n(Ff)$. Denote also by $G_{n,i}$ the image of $G_{n,i}$ in $\pi_n(X)$ and the preimage of $G_{n,i}$ in $\pi_{n+1}(Y)$. Define $X_{n,i}$ by first adding $(n+k)$-cells for all triples (ϵ, δ, k), where $k \geq 1$, $\epsilon : S^{n+k-1} \to X$ is a map, and $\delta : C\epsilon \to Y$ is a map with $\delta i = f$, where $i : X \to C\epsilon$ is the inclusion, and, if $k = 1$, the induced element of $\pi_n(Ff)$ being in $G_{n,i}$. Call this space L_0 and note that we can canonically extend f to L_0. Now for each $p \geq 1$, inductively define L_p by attaching $(n+p+1)$-cells to L_{p-1} along all pairs (ϵ, δ), where $\epsilon : S^{n+p} \to L_{p-1}$ is a map and $\delta : C\epsilon \to Y$ is a map such that $\delta i = f$. At each stage, there is a canonical extension of f to L_p. We define $X_{n,i}$ to be the union of the L_p and we have a canonical extension of f to $X_{n,i}$. We have:

$$\pi_j(X_{n,i}) = \begin{cases} \pi_j(X) & \text{if } j < n, \\ \frac{\pi_n(X)}{G_{n,i}} & \text{if } j = n, \\ G_{n,i} & \text{if } j = n+1, \\ \pi_j(Y) & \text{if } j > n+1. \end{cases}$$

We also have that $X_{n,i}$ is a relative subcomplex of $X_{m,j}$ if $n > m$ or $n = m$ and $i \leq j$. Furthermore, the map $X_{n,\max} \to X_{n-1,0}$ is a weak equivalence, where $X_{n,\max}$ corresponds to the final group in the filtration of $\pi_n(Ff)$, and so we can redefine $X_{n,\max} := X_{n-1,0}$ for $n \geq 2$. Our construction implies that the homotopy fibre of the map $g : X \to X_{n,i}$ satisfies:

$$\pi_j(Fg) = \begin{cases} 0 & \text{if } j \leq n-1, \\ G_{n,i} & \text{if } j = n, \\ \pi_j(Ff) & \text{if } j \geq n+1. \end{cases}$$

Moreover, the $\pi_1(X)$-action on $G_{n,i}$ agrees with the $\pi_1(X)$-action on $\pi_n(Ff)$. It follows easily that the homotopy fibre of $X_{n,i} \to X_{n,i+1}$ is $K(\frac{G_{n,i+1}}{G_{n,i}}, n)$, and the action of $\pi_1(X_{n,i})$ on $\frac{G_{n,i+1}}{G_{n,i}}$ is trivial, by assumption.

We now have a map $X_{1,\max} \to Y$ which induces an isomorphism of π_n for $n \geq 2$ and is the inclusion of the image of $\pi_1(X)$ on π_1. Using the same methods as above, we can refine this map into a composition of maps $X_{0,i} \to X_{0,i+1}$ corresponding to the inclusion of successive groups in a $\mathcal{B}$-subnormal series for the image of $\pi_1(X)$ in $\pi_1(Y)$. The theorem is now a consequence of successive applications of Lemma 2.3.10, with each stage of the construction being a well-pointed space of the homotopy type of a CW complex by the results of Section 6.3. $\square$

The fact that we can take the coattaching maps to be cofibrations is a convenient technical detail that we will make use of in Chapter 4. Specialising to the case $Y = *$, yields:

Theorem 2.3.12: *A connected space, X, is nilpotent iff it is equivalent to a Postnikov tower with cottaching maps that are cofibrations.*

Proof. If X is nilpotent, then Theorem 2.3.11 implies that there is a relative Postnikov tower for $X \to *$, with coattaching maps that are cofibrations. Since X is connected, $B_{1,i} = 0$ for all i. Therefore, the relative Postnikov tower in this case agrees with the usual definition of a Postnikov tower. In the other direction, a straightforward inductive argument using the long exact sequence of a fibration, implies that the limit of a Postnikov tower is a nilpotent space, see eg [MP12, page 63]. $\square$

2.3.2 Cocellular maps and homotopies

Finally, for this section, we take the analogy between Postnikov towers and CW complexes further, by defining a cocellular map between Postnikov towers, and proving that any map between Postnikov towers can be approximated by a cocellular map. We begin with the following well-known result, which is the heart of the co-HELP lemma of [MP12, Theorem 3.3.7]. We will assume that all spaces in sight are well-pointed with the homotopy type of a CW complex, so that cohomology is represented by maps into Eilenberg-MacLane spaces.

Lemma 2.3.13: *Let A be a fixed abelian group. A map $f : X \to Y$ induces an injection $H^m(f; A)$ and a surjection $H^{m-1}(f; A)$ iff there always exist dashed lifts in diagrams of the form:*

$$\begin{tikzcd}
PK(A,m) \arrow[dd] & & PK(A,m)^{I_+} \arrow[ll, "\pi_0"'] \arrow[rr, "\pi_1"] \arrow[dd] & & PK(A,m) \arrow[dd] \\
 & X \arrow[ul] \arrow[ur, dashed] \arrow[rr] \arrow[dr] & & Y \arrow[ur, dashed] \arrow[dr] & \\
K(A,m) & & K(A,m)^{I_+} \arrow[ll, "\pi_0"] \arrow[rr, "\pi_1"'] & & K(A,m)
\end{tikzcd}$$

where $PK(A, m) \to K(A, m)$ is the path-space fibration.

Corollary 2.3.14: *If f is an n-equivalence, then we can find lifts of diagrams of the form in Lemma 2.3.13, for arbitrary A, provided that $m \leq n$.*

As in [May83], Lemma 2.3.13 is the starting point for proving the dual Whitehead theorems, and we refer the reader either [May83] or [MP12, Theorem 3.3.8] for their proofs. In particular, we have that a homology isomorphism between nilpotent spaces is a weak equivalence.

Recall that a Postnikov tower is a sequence of maps $X_{n,i+1} \to X_n$ which are each principal $K(B, n)$-fibrations for some abelian group B, where for each fixed natural number n, i ranges from 1 to some natural number i_n, and $X_{n+1,1} = X_{n,i_n}$. This differs, notationally, from the definition of a relative Postnikov tower we gave in Definition 2.3.7. We will now define a cocellular maps between Postnikov towers, as well as cocellular homotopies:

Definition 2.3.15: *A cocellular map between Postnikov towers, $f : X \to Y$, is a collection of maps $f_n : X_{n,1} \to Y_{n,1}$ such that the squares:*

$$\begin{array}{ccc} X_{n+1,1} & \xrightarrow{f_{n+1}} & Y_{n+1,1} \\ \downarrow & & \downarrow \\ X_{n,1} & \xrightarrow[f_n]{} & Y_{n,1} \end{array}$$

commute. A cocellular homotopy between cocellular maps $f, g : X \to Y$ is a collection of maps $H_n : X_{n,1} \to Y_{n,1}^{I_+}$ such that $\pi_0 H_n = f_n$, $\pi_1 H_n = g_n$, and the squares:

$$\begin{array}{ccc} X_{n+1,1} & \xrightarrow{H_{n+1}} & Y_{n+1,1}^{I_+} \\ \downarrow & & \downarrow \\ X_{n,1} & \xrightarrow[H_n]{} & Y_{n,1}^{I_+} \end{array}$$

commute.

We've seen that a nilpotent space is weakly equivalent to a Postnikov tower, and we will now prove the following naturality statement:

Lemma 2.3.16: *If $f : X \to Y$ is a map between nilpotent spaces, then there is a cocellular map g making the following square commute up to homotopy:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\gamma", "\simeq"'] & Y \arrow[d, "\simeq"] \\
\lim X_{n,i} \arrow[r, "g"'] & \lim Y_{n,i}
\end{tikzcd}$$

Moreover, g is unique up to cocellular homotopy.

Proof. The existence of g follows by inductively forming maps $g_n : X_n \to Y_n$ and homotopies $H_n : X \to Y_n^{I_+}$ using co-HELP as in the following diagram:

$$\begin{tikzcd}
Y_{n+1,1} \arrow[dd] & & Y_{n+1,1}^{I_+} \arrow[ll] \arrow[rr] \arrow[dd] & & Y_{n+1,1} \arrow[dd] \\
& X \arrow[ul, "df"] \arrow[ur, dashed, "H_{n+1}"] \arrow[dr, "H_n"'] \arrow[rr] & & X_{n+1,1} \arrow[ur, dashed, "g_{n+1}"] \arrow[dr, "g_n d"] & \\
Y_{n,1} & & Y_{n,1}^{I_+} \arrow[ll] \arrow[rr] & & Y_{n,1}
\end{tikzcd}$$

The map $Y_{n+1,1} \to Y_{n,1}$ is a composite of pullbacks of maps of the form $PK(A, n+1) \to K(A, n+1)$, and the map $X \to X_{n+1,1}$ is an $(n+1)$-equivalence, so the existence of the lifts follows from Lemma 2.3.13 and Corollary 2.3.14. For the uniqueness statement, suppose that g^1 and g^2 are two cocellular maps making the square commute. We inductively construct maps $L_n : X \to (Y_{n,1}^{I_+})^{I_+}$ and $K_n : X_{n,1} \to Y_{n,1}^{I_+}$ using the following diagram:

$$\begin{tikzcd}
Y_{n+1,1}^{I_+} \arrow[dd] & & (Y_{n+1,1}^{I_+})^{I_+} \arrow[ll] \arrow[rr] \arrow[dd] & & Y_{n+1,1}^{I_+} \arrow[dd] \\
& X \arrow[ul, "H"] \arrow[ur, dashed, "L_{n+1}"] \arrow[dr] \arrow[rr] & & X_{n+1,1} \arrow[ur, dashed, "K_{n+1}"] \arrow[dr] & \\
Y_{n,1}^{I_+} \times_p (Y_{n+1,1} \times Y_{n+1,1}) & & (Y_{n,1}^{I_+} \times_p (Y_{n+1,1} \times Y_{n+1,1}))^{I_+} \arrow[ll] \arrow[rr] & & Y_{n,1}^{I_+} \times_p (Y_{n+1,1} \times Y_{n+1,1})
\end{tikzcd}$$

The maps H_n are defined using some fixed homotopy $H : X \to \lim Y_{n,i}^{I_+}$ between $g^1\gamma$ and $g^2\gamma$. The map $X \to (Y_{n,1}^{I_+} \times_p (Y_{n+1,1} \times Y_{n+1,1}))^{I_+}$ is defined using L_n, as well as the constant homotopies on g^1_{n+1} and g^2_{n+1}. The lifts exist since $X \to X_{n+1,1}$ is an $(n+1)$-equivalence and $Y_{n+1,1}^{I_+} \to Y_{n,1}^{I_+} \times_p (Y_{n+1,1} \times Y_{n+1,1})$ is a composite of pullbacks of maps of the form $PK(A, n) \to K(A, n)$. Then $K = \lim K_n$ defines a cocellular homotopy between g^1 and g^2. □

The proof of Lemma 2.3.16 highlights one of the convenient features of cocell complexes when compared with cell complexes. Since Ω reduces the dimension of $K(A, n+1)$, it was more comfortable to beat the bound $m \leq n$ of Corollary 2.3.14 in the second half of the proof, where we were considering composites of pullbacks of maps of the form $PK(A, n) \to K(A, n)$, when we could have got away with considering the maps $PK(A, n+1) \to K(A, n+1)$ as in the first half of the proof. On the other hand, if looping increased the dimension, the second half of the proof would fail. This is the case cellularly, where taking suspensions increases the dimension of S^n. This has real implications. For instance, suppose that $f : X \to Y$ is a map between CW-complexes, and g_1, g_2 are cellular maps homotopic to f. Then $g_1^{(n)}, g_2^{(n)} : X^{(n)} \to Y^{(n)}$ are not necessarily homotopic, although they become homotopic after composing with the inclusion $Y^{(n)} \to Y^{(n+1)}$. However, working cocellularly we have the following observation:

Lemma 2.3.17: *Let $f : X \to Y$ be a map between simple spaces. Approximate X and Y by Postnikov towers with a single cocell of each dimension. Then f induces a well-defined homotopy class of maps $f_n : X_n \to Y_n$.*

It follows that if a group acts up to homotopy on a simple space X, there is an induced action up to homotopy on each X_n of a Postnikov tower associated to X with a single cocell in each each dimension. Therefore, if G acts up to homotopy on a nilpotent space X, it also acts up to homotopy on the universal cover $\tilde{X}$ of X, and on the spaces $\tilde{X}_n$ of a Postnikov tower for $\tilde{X}$ with a single cocell in each dimension. These observations were exploited by Hilton in his proof of the following result:

Theorem 2.3.18: *Let G act up to homotopy on a nilpotent space X. Then the following are equivalent:*

i) G acts nilpotently on $\pi_i(X)$ for all $i \geq 1$,

ii) G acts nilpotently on $H_i(X)$ for all $i \geq 1$,

iii) G acts nilpotently on $\pi_1(X)$ and on $H_i(\tilde{X})$ for all $i \geq 1$.

Proof. This is [Hil76, Theorem 2.1]. □

Corollary 2.3.19: *Let $F \to E \to B$ be a fibre sequence with all spaces connected. Then $\pi_1(E)$ acts nilpotently on $\pi_*(F)$ iff F is nilpotent and $\pi_1(B)$ acts nilpotently on $H_*(F)$.*

Proof. This is [Hil76, Corollary 2.2]. □

2.4 Finitely generated nilpotent spaces

2.4.1 A Theorem of Wall

We now return to our proof of Theorem 2.0.1. We define an f-nilpotent space to be a $\mathcal{B}$-nilpotent space, where $\mathcal{B}$ is the class of finitely generated abelian groups. Using the algebraic results of Section 2.2, it is an easy matter to prove:

Lemma 2.4.1: *Let X be a nilpotent space. Then X is an f-nilpotent space iff $\pi_i(X)$ is finitely generated for all $i \geq 1$.*

Proof. If X is f-nilpotent, then $\pi_i(X)$ is an f-nilpotent group for all $i \geq 1$. It follows that $\pi_i(X)$ is finitely generated for $i \geq 1$, by Theorem 2.2.2.

Let $i \geq 1$. If $\pi_i(X)$ is finitely generated, then all subgroups are finitely generated by Theorem 2.2.2. In particular, any series expressing $\pi_i(X)$ as a nilpotent π-group will have finitely generated quotients. Therefore, X is f-nilpotent. □

Next, we will prove a theorem of Wall concerning when a space is weakly equivalent to a CW complex with finite skeleta ([Wal65, Theorem A]). This will allow us to immediately prove the implication $iv) \iff i)$ of Theorem 2.0.1, and the theorem will continue to be useful for the remaining implications. A key point of the proof is the following observation, which is closely related to a result of J.H.C. Whitehead, [Whi50, Lemma 15]:

Lemma 2.4.2: *Suppose that (Y, B) is an n-connected CW pair such that Y has finite skeleta. Then there exists a weak equivalence of CW pairs $(Y, B) \to (\widehat{Y}, \widehat{B})$ such that $\widehat{Y}$ has finite skeleta and $\widehat{B}$ contains the n-skeleton of $\widehat{Y}$.*

Proof. Suppose, inductively, that B contains that k-skeleton of Y, where $-1 \leq k < n$. Let $\alpha : S^k \to B$ be the attaching map of a $(k+1)$-cell of Y. Since (Y, B) is n-connected, an application of HELP yields an extension of α, $\beta : D^{k+1} \to B$, and a homotopy rel S^k between β and the inclusion of the cell corresponding to α into Y. Construct $\widehat{B}_\alpha$ by considering D^{k+1} as the lower hemisphere of S^{k+1} and forming the pushout:

$$\begin{array}{ccc} D^{k+1} & \xrightarrow{\beta} & B \\ \downarrow & & \downarrow \\ D^{k+2} & \longrightarrow & \widehat{B}_\alpha \end{array}$$

Construct $\widehat{Y}_\alpha$ by first attaching an $(k+2)$-cell to Y along the map $\gamma : S^{k+1} \to Y$ defined on the lower hemisphere by β, and on the upper hemisphere by the inclusion of the cell corresponding to α. By assumption, γ is nullhomotopic. Call the space constructed so far Y'_α. To define $\widehat{Y}_\alpha$, attach an $(k+3)$-cell to Y'_α along a map $D^{k+2} \to Y'_\alpha$, defined on the lower hemisphere by a nullhomotopy of γ, and on the upper hemisphere by the inclusion of the cell corresponding to γ. Then, $(Y, B) \to (\widehat{Y}_\alpha, \widehat{B}_\alpha)$ is a weak equivalence, the $(k+1)$-skeleton of Y equals the $(k+1)$-skeleton of $\widehat{Y}_\alpha$, and $\widehat{B}_\alpha$ contains the $(k+1)$-cell α. Repeating the process for all the cells in the $(k+1)$-skeleton of Y, results in a weak equivalence $(Y, B) \to (\widehat{Y}, \widehat{B})$ such that $\widehat{Y}$ has finite skeleta, the $(k+1)$-skeleton of Y equals the $(k+1)$-skeleton of $\widehat{Y}$, and $\widehat{B}$ contains the $(k+1)$-skeleton of $\widehat{Y}$. The result now follows by induction. □

Theorem 2.4.3: *A space X is weakly equivalent to a CW complex with finite skeleta iff each of the following conditions is satisfied:*

i) $\pi = \pi_1(X)$ is finitely presented,

ii) for every map $\sigma : K \to X$ from a connected finite CW complex K which induces an isomorphism on fundamental groups, $\pi_1(F\sigma)$ is finitely generated as a π-module,

iii) for every n-connected ($n \geq 2$) map $\sigma : K \to X$ from a finite CW complex K, $\pi_n(F\sigma)$ is finitely generated as a π-module.

Proof. ($\Longleftarrow$) We'll first show that if each of the conditions is satisfied, then X is weakly equivalent to a CW complex with finite skeleta. Since π is finitely presented, we can construct, using the van Kampen theorem, a finite CW complex K, with cells of dimension ≤ 2, equipped with a map $\sigma : K \to X$ inducing an isomorphism on fundamental groups. Now assume inductively that we have constructed a finite CW complex K and, either a map inducing an isomorphism on fundamental groups, $\sigma : K \to X$, for $n = 1$, or an n-connected map $\sigma : K \to X$ for $n \geq 2$. Then, picking a finite number of maps $S^n \to F\sigma$ which generate $\pi_n(F\sigma)$ as a π-module specifies an extension $i : K \to K^{'}$ induced by attaching $(n+1)$-cells, and an extension of σ to $\sigma^{'} : K^{'} \to X$. We have the fibre sequence:

$$Fi \to F\sigma \to F\sigma^{'}$$

Note that if $n = 1$, then $\pi_1(F\sigma)$ is abelian due to the assumption that σ induces an isomorphism on π_1. Moreover, for all n, the long exact sequence of homotopy groups associated to the fibration above is a long exact sequence of π-modules. Due to our construction of the map $\sigma^{'}$, it follows that $\sigma^{'}$ is an $(n+1)$-connected map from a finite CW complex $K^{'}$. This completes the proof of one direction.

($\Longrightarrow$) It suffices to prove the converse by assuming that X is a CW complex with finite skeleta. By the van Kampen theorem, it is clear that π is finitely presented. Let $\sigma : K \to X$ be a map from a finite CW complex K, as in condition ii) or iii). Then we may as well assume that σ is a cellular map and then replace σ by the inclusion $i : K \to M\sigma$, where $M\sigma$ is a CW complex with finite skeleta and K is a subcomplex. Using Lemma 2.4.2, we can replace $(M\sigma, K)$ by a CW pair (Y, B), where Y has finite skeleta and B contains the n-skeleton of Y. We have:

$$\pi_n(Fi) = \pi_{n+1}(Y, B) = \pi_{n+1}(\tilde{Y}, \tilde{B}) = H_{n+1}(\tilde{Y}, \tilde{B}),$$

where the final equality follows from the relative Hurewicz theorem and the middle equality follows, when $n = 1$, from the fact that σ induces an isomorphism on fundamental groups. Considering the cellular chain complex of $(\tilde{Y}, \tilde{B})$, we see that it is zero at degree n and is a finitely generated π-module at degree $n + 1$. It follows that the quotient group $H_{n+1}(\tilde{Y}, \tilde{B})$ is a finitely generated π-module, as desired. □

The following corollary is useful for applications:

Corollary 2.4.4: *If X is a space such that $\pi_1(X)$ is finitely presented, $\mathbb{Z}[\pi]$ is a Noetherian ring and $H_i(\tilde{X})$ is a finitely generated π-module for all i, then X is weakly equivalent to a CW complex with finite skeleta.*

Proof. Suppose that $\sigma : K \to X$ is an n-connected map ($n \geq 1$) from a finite CW complex K which induces an isomorphism on fundamental groups. Then we have $\pi_i(F\sigma) \cong H_{i+1}(\tilde{X}, \tilde{K})$. Consider the exact sequence:

$$... \to H_{i+1}(\tilde{K}) \to H_{i+1}(\tilde{X}) \to H_{i+1}(\tilde{X}, \tilde{K}) \to H_i(\tilde{K}) \to H_i(\tilde{X}) \to 0$$

$H_{i+1}(\tilde{X})$ is a finitely generated π-module by assumption and $H_i(\tilde{K})$ is too since K is a finite cell complex. Since $\mathbb{Z}[\pi]$ is Noetherian, any submodule of $H_i(\tilde{K})$ is also a finitely generated π-module. It follows that $H_{i+1}(\tilde{X}, \tilde{K})$ is a finitely generated π-module, which means that X satisfies the conditions i) - iii) of Theorem 2.4.3. $\square$

In light of Theorem 2.2.2, we have the following consequence:

Corollary 2.4.5: *If G is a finitely generated nilpotent group, then $K(G, 1)$ is weakly equivalent to a CW complex with finite skeleta.*

Finally, for this section, we prove the implication $iv) \iff i)$ of Theorem 2.0.1. For the proof, we will use the fact that, if X is a nilpotent space, then $H_i(\tilde{X})$ is a nilpotent π-module for all i ([MP12, Proposition 4.2.1]). We will also prove this in Lemmas 4.2.7 and 4.2.8, with no intermediate theory required.

Lemma 2.4.6: *If X is a nilpotent space, then X is weakly equivalent to a CW complex with finite skeleta iff $\pi_1(X)$ and $H_i(\tilde{X})$ are finitely generated for all i.*

Proof. If X is a nilpotent space which is weakly equivalent to a finite CW complex, then π is finitely presented, and, hence, a finitely generated nilpotent group. Moreover, $H_i(\tilde{X})$ is finitely generated as a π-module. Since $\mathbb{Z}[\pi]$ is Noetherian, it follows that all quotients of a series expressing $H_i(\tilde{X})$ as a nilpotent π-module are finitely generated π-modules on which π acts trivially. This means they are finitely generated abelian groups. Inducting up the series, we see that this implies that $H_i(\tilde{X})$ is also a finitely generated abelian group, as desired.

Now assume that $\pi_1(X)$ and $H_i(\tilde{X})$ are finitely generated. Then π is f-nilpotent and, therefore, π is finitely presented and $\mathbb{Z}[\pi]$ is Noetherian. Since $H_i(\tilde{X})$ is also finitely generated viewed as a π-module, Corollary 2.4.4 implies that X is weakly equivalent to a CW complex with finite skeleta. $\square$

2.4.2 A Theorem of Serre

We will now move on to a classical result of Serre - that for simply connected spaces X, all homotopy groups of X are finitely generated iff all homology groups of X are ([Ser53]). The strategy of proof will be to replace the simply connected space X by a Postnikov tower and induct up it using the Serre spectral sequence. From

Lemma 2.4.5, we know that if A is a finitely generated abelian group, then $K(A, 1)$ is weakly equivalent to a CW complex with finite skeleta, and so, in particular, has finitely generated homology groups. The next lemma allows us to extend this result to $K(A, n)$ for $n > 1$:

Lemma 2.4.7: *If X is simply connected, then $H_i(X)$ is finitely generated for all i iff $H_i(\Omega X)$ is finitely generated for all i.*

Proof. Consider the Serre spectral sequence for the fibration:

$$\Omega X \to PX \to X$$

where the local coefficient system is trivial since X is simply connected. Since PX is contractible, the only non-zero term of the E^∞-page is $E^\infty_{0,0} \cong \mathbb{Z}$. We have exact sequences:

$$E^r_{r,q-r+1} \to E^r_{0,q} \to E^{r+1}_{0,q} \to 0$$

Suppose that $H_i(X)$ is finitely generated for all i, and that $H_i(\Omega X)$ is finitely generated for $i < q$. Then, for $r \geq 2$:

$$E^2_{r,q-r+1} = H_r(X, H_{q-r+1}(\Omega X))$$

is finitely generated, since if A, B are finitely generated abelian groups, then $Tor_1(A, B)$ is finitely generated since it can be expressed as a homology group of a chain complex of finitely generated abelian groups. It follows that $E^r_{r,q-r+1}$ is finitely generated, and by induction using the exact sequence above, that $E^2_{0,q} = H_q(\Omega X)$ is finitely generated. Therefore, by induction, if $H_i(X)$ is finitely generated for all i, then $H_i(\Omega X)$ is finitely generated for all i. The proof of the reverse implication is entirely analogous. □

Corollary 2.4.8: *If A is a finitely generated abelian group and $n \geq 1$, then $K(A, n)$ is weakly equivalent to a CW complex with finite skeleta.*

Proof. Inductively, Lemma 2.4.7 implies that $H_i(K(A, n))$ is finitely generated for all i, so the result follows from Corollary 2.4.4. □

We can now prove:

Theorem 2.4.9: *If X is simply connected, then $\pi_i(X)$ is finitely generated for all i iff $H_i(X)$ is finitely generated for all i.*

Proof. Since X is simple, which means that the fundamental group acts trivially on the homotopy groups of X, we can assume that X is the limit of a Postnikov tower:

$$... \to X_{n+1} \to X_n \to ... \to X_1 = \star$$

induced by fibre sequencex:

$$X_{n+1} \to X_n \to K(\pi_{n+1}(X), n+2)$$

If $H_i(X_n)$ is finitely generated for all i, then inspection of the argument given in Lemma 2.4.7 shows that we only used the fact that the E^∞ page was finitely generated. Therefore, the argument can be generalised to show that $H_i(X_{n+1})$ is finitely generated for all i iff $H_i(K(\pi_{n+1}(X), n+2))$ is finitely generated for all i. Therefore, if $\pi_i(X)$ is finitely generated for all i, an inductive argument using the fact that $H_i(K(\pi_{n+1}(X), n+2))$ is finitely generated for all i and n, shows that $H_i(X_n)$ is finitely generated for all i and n. It follows that $H_i(X)$ is finitely generated for all i, since the groups $H_i(X_n)$ eventually stabilise at $H_i(X)$ for large n.

Now suppose that $H_i(X)$ is finitely generated for all i. The map $X \to X_n$ is an $(n+1)$-equivalence, since $\pi_{n+1}(X_n) = 0$, and so $H_i(X) \to H_i(X_n)$ is an isomorphism for $i \leq n$ and a surjection when $i = n+1$. It follows that $H_i(X_n)$ is finitely generated whenever $i \leq n+1$. Suppose that we have proved that $\pi_i(X)$ is finitely generated for $i \leq n$. Then inductively, similarly to our previous work, we can show that $H_j(X_i)$ is finitely generated for $i \leq n$ and all j. Consider the Serre spectral sequence for the fibration:

$$X_{n+1} \to X_n \to K(\pi_{n+1}(X), n+2)$$

Then $E^\infty_{p,q}$ is finitely generated for all p and q, and $E^r_{0,q}$ is finitely generated whenever $q \leq n+2$. We also have $E^2_{n+2,0} = \pi_{n+1}(X)$. We have exact sequences:

$$0 \to E^{r+1}_{n+2,0} \to E^r_{n+2,0} \to E^r_{n-r+2,r-1}$$

When $r = n+2$, $E^{r+1}_{n+2,0} = E^\infty_{n+2,0}$ and $E^r_{n-r+2,r-1} = E^{n+2}_{0,n+1}$ and so both of these are finitely generated. It follows that $E^{n+2}_{n+2,0}$ is finitely generated. For $2 \leq r < n+2$, $E^r_{n-r-2,r-1} = 0$ and so, inductively, it follows that $E^2_{n+2,0} = \pi_{n+1}(X)$ is finitely generated, as desired. It follows, again inductively, that $\pi_i(X)$ is finitely generated for all i. □

We can now prove another implication of Theorem 2.0.1:

Corollary 2.4.10: *If X is a connected space, then $\pi_i(X)$ is finitely generated for every $i \geq 1$ iff $\pi_1(X)$ and, for $i \geq 2$, $H_i(\tilde{X})$ are finitely generated.*

Proof. This follows from applying Theorem 2.4.9 to the universal cover of X. □

2.4.3 Completing the proof

We have already shown that the first four conditions of Theorem 2.0.1 are equivalent and that $i) \implies v)$. Therefore, to complete the proof of Theorem 2.0.1, we just need to show that $v)$ implies any one of $i) - iv)$.

We will show that $v) \implies iv)$. We will, again, use that, if X is nilpotent, then $\pi_1(X)$ acts nilpotently on $H_i(\tilde{X})$.

Theorem 2.4.11: *Let X be a nilpotent space. If $H_i(X)$ is finitely generated for each $i \geq 1$, then $\pi_1(X)$ and, for $i \geq 2$, $H_i(\tilde{X})$ are finitely generated.*

Proof. By the Hurewicz theorem, the abelianisation of the nilpotent group $\pi = \pi_1(X)$ is finitely generated, and so it follows that π itself is finitely generated by our algebraic result, Theorem 2.2.2. Consider the Serre spectral sequence of the fibration:

$$\tilde{X} \to X \to K(\pi, 1)$$

Since π is finitely presented and $\mathbb{Z}[\pi]$ is Noetherian, it follows that $K(\pi, 1)$ is weakly equivalent to a CW complex with finite skeleta. The E^{∞} page is finitely generated by assumption, and we have that $E^2_{p,0} = H_p(K(\pi, 1); \mathcal{H}_0(\tilde{X})) = H_p(K(\pi, 1))$ is finitely generated. Suppose that we have shown $H_i(\tilde{X})$ is finitely generated for $i \leq n$. We have exact sequences:

$$E^r_{r,n-r+2} \to E^r_{0,n+1} \to E^{r+1}_{0,n+1} \to 0$$

Now, $E^2_{r,n-r+2} = H_r(K(\pi, 1); \mathcal{H}_{n-r+2}(\tilde{X}))$ and, if $r \geq 2$, this is finitely generated since it is the homology of a complex of finitely generated abelian groups, namely the cellular chain complex, with finitely generated local coefficients, of a CW complex with finite skeleta. It follows, inductively, that $E^2_{0,n+1}$ is finitely generated. We have:

$$E^2_{0,n+1} = H_0(K(\pi, 1); \mathcal{H}_{n+1}(\tilde{X})) = H_{n+1}(\tilde{X})/\pi$$

The rest our proof will be analogous to the proof that if G is a nilpotent group such that $Ab(G)$ is finitely generated, then G is finitely generated. Note that if G is a group, and $I \trianglelefteq \mathbb{Z}[G]$ is the augmentation ideal, it is easily verified that I is finitely generated as a $\mathbb{Z}[G]$-module iff G is finitely generated. We also have an isomorphism of abelian groups, $\frac{I}{I^2} \cong Ab(G)$. Now, let $M = H_{n+1}(\tilde{X})$ and $G = \pi$, and observe that there is an epimorphism:

$$\frac{I}{I^2} \otimes ... \otimes \frac{I}{I^2} \otimes \frac{M}{IM} \to \frac{I^n M}{I^{n+1} M}$$

defined by the $\mathbb{Z}[\pi]$-action on M. Since $Ab(G)$ and $\frac{M}{IM} = E^2_{0,n+1}$ are finitely generated, it follows that all $\frac{I^n M}{I^{n+1} M}$ are finitely generated. Since M is a nilpotent π-module, it follows that $M = H_{n+1}(\tilde{X})$ is finitely generated. Inductively, it now follows that $H_i(\tilde{X})$ is finitely generated for all i, as desired. $\square$

2.5 Localisations and completions of nilpotent spaces

In this section, we will review the construction of localisations and completions of nilpotent spaces via induction up Postnikov towers, following [MP12, Sections 5.3 and 10.3]. We will assume the reader is familiar with the basic theory of localisations and completions of abelian groups, as outlined in [MP12, Sections 5.1 and 10.1]. As in Section 2.4, the Serre spectral sequence will play a key role in the arguments, and we record here the refinement of the Zeeman comparison theorem due to Hilton and Roitberg, [HR76]:

Theorem 2.5.1: *Consider a map of fibrations:*

$$\begin{tikzcd} F \arrow[r] \arrow[d, "f_1"] & E \arrow[r, two heads] \arrow[d, "f"] & B \arrow[d, "f_2"] \\ F' \arrow[r] & E' \arrow[r, two heads] & B' \end{tikzcd}$$

in which all spaces are connected, $\pi_1(B)$, $\pi_1(B')$ are nilpotent, and act nilpotently on $H_*(F)$ and $H_*(F')$, respectively. Take all homology to have coefficients in a fixed abelian group. The following conclusions hold:

i) Let $P \geq 2$ and $Q \geq 0$ be fixed integers. Suppose that:

a) $H_q(f_1)$ is an isomorphism for $q < Q$, and $H_Q(f_1)$ is surjective,

b) $H_p(f_2)$ is an isomorphism for $p < P$, and $H_P(f_2)$ is surjective.

Then, $H_n(f)$ is an isomorphism for $n < N := min(P, Q)$, and $H_N(f)$ is surjective.

ii) Let $P \geq 2$ and $N \geq 0$ be fixed integers. Suppose that:

a) $H_n(f)$ is an isomorphism for $n < N$, and $H_N(f)$ is surjective,

b) $H_p(f_2)$ is an isomorphism for $p < P$, and $H_P(f_2)$ is surjective.

Then, $H_q(f_1)$ is an isomorphism for $q < Q := min(N, P-1)$, and $H_Q(f_1)$ is surjective.

iii) Let $N \geq 0$ and $Q \geq 0$ be fixed integers. Suppose that:

a) $H_n(f)$ is an isomorphism for $n < N$, and $H_N(f)$ is surjective,

b) $H_q(f_1)$ is an isomorphism for $q < Q$, and $H_Q(f_1)$ is surjective,

c) $\pi_1(B) \to \pi_1(B')$ is an isomorphism.

Then, $H_p(f_2)$ is an isomorphism for $p < P := min(N, Q+1)$, and $H_P(f_2)$ is surjective.

The proof is a carefully arranged inductive argument, which makes repeated use of the Four Lemma for abelian groups. We are mostly interested in the case where $P, Q, N = \infty$, and the fundamental groups act trivially on the homology of the fibres. However, we will have occasion to consider nilpotent actions on the homology of the fibres in Chapter 4.

Next, we introduce the definitions of localisation and completion at a set of primes T:

Definition 2.5.2: *Let $f : X \to Y$ be a map of spaces. We call f a $\mathbb{Z}_T$-equivalence if it induces an isomorphism on homology with coefficients in $\mathbb{Z}_T$, where $\mathbb{Z}_T$ denotes the integers localised at T. We call f an $\mathbb{F}_T$-equivalence if it induces an isomorphism on homology with coefficients in $\mathbb{F}_p$, for all $p \in T$.*

Definition 2.5.3: *Let Z be a space. We say that Z is T-local if $f^* : [Y, Z] \to [X, Z]$ is a bijection for all $\mathbb{Z}_T$-equivalences, $f : X \to Y$, between cofibrant objects. We say that Z is T-complete if $f^* : [Y, Z] \to [X, Z]$ is a bijection for all $\mathbb{F}_T$-equivalences, $f : X \to Y$, between cofibrant objects.*

Definition 2.5.4: *A map $f : X \to Y$ is called a T-localisation (of X) if it is a $\mathbb{Z}_T$-equivalence to a T-local space. A map $f : X \to Y$ is called a T-completion (of X) if it is an $\mathbb{F}_T$-equivalence to a T-complete space.*

Assuming that all spaces are cofibrant, a map from X to a T-local space will factor uniquely, up to homotopy, through its T-localisation, and completion has an analogous universal property.

The following result is crucial for our inductive construction of localisations and completions of nilpotent spaces, and the proof is surprisingly computational. The difficulty with giving a more conceptual proof is that multiplication by p on A does not necessarily induce multiplication by p on all homology groups of $K(A, n)$:

Theorem 2.5.5: *Let T be a set of primes. If A is a T-local abelian group, then, for all $n, m \geq 1$, $H_n(K(A, m))$ is T-local.*

Proof. We can reduce to the case where T is the set of primes not equal to a prime p. Then $K(\mathbb{Z}_T, 1)$ is the sequential colimit of the maps $K(\mathbb{Z}, 1) \to K(\mathbb{Z}, 1)$ which induce multiplication by p on π_1. Therefore, $H_1(K(\mathbb{Z}_T, 1), 1) = \mathbb{Z}_T$ and, as for S^1, the higher homology groups of $K(\mathbb{Z}_T, 1)$ are zero. The Künneth theorem allows us to deduce the $m = 1$ result for $A = \oplus_I \mathbb{Z}_T$, where I is a finite indexing set, and transfinite induction allows us to deduce the result even when I is infinite. Use of the Serre spectral sequence applied to path-space fibrations allows us to deduce the result for arbitrary m. Finally, an arbitrary T-local A, has a presentation of the form $0 \to \oplus_J \mathbb{Z}_T \to \oplus_I \mathbb{Z}_T \to A \to 0$, and the result follows by use of the Serre spectral sequence applied to the fibration $K(\oplus_I \mathbb{Z}_T, m) \to K(A, m) \to K(\oplus_J \mathbb{Z}_T, m + 1)$. □

From now on, we concentrate on the construction of completions of nilpotent spaces, since the construction of localisations is similar, but slightly easier. We refer the interested reader to [MP12, Section 5.3], for the construction of localisations. We begin by classifying the T-complete, connected, Eilenberg-MacLane spaces:

Lemma 2.5.6: *A is T-complete iff $K(A, 1)$ is T-complete iff $K(A, m)$ is T-complete for all $m \geq 0$.*

Proof. Firstly, $\tilde{H}_*(X; \oplus_{p\in T}\mathbb{F}_p) = 0$ iff $\tilde{H}_*(X)$ is a $\mathbb{Z}[T^{-1}]$-module. Moreover, A is T-complete iff $Ext(\mathbb{Z}[T^{-1}], A) = Hom(\mathbb{Z}[T^{-1}], A) = 0$ iff, for all $\mathbb{Z}[T^{-1}]$-modules, B, $Ext(B, A) = Hom(B, A) = 0$. So, if A is T-complete, then $K(A,m)$ is T-complete for all $m \geq 0$. We still need to prove that if $K(A,1)$ is T-complete, then A is T-complete. For this, represent $\mathbb{Z}[T^{-1}]$ as a sequential colimit of copies of $\mathbb{Z}$. Then, we get an induced injective homomorphism $\oplus\mathbb{Z} \to \oplus Z$ with cokernel $\mathbb{Z}[T^{-1}]$, representing the colimit. Let ϕ be a map $\vee S^1 \to \vee S^1$ representing this homomorphism on homology. Then, ϕ is an $\mathbb{F}_{\mathbb{T}}$-equivalence, and it follows that $\tilde{H}^*(C\phi; A) = 0$, since $K(A,1)$ is T-complete. Therefore, $Ext(\mathbb{Z}[T^{-1}], A) = Hom(\mathbb{Z}[T^{-1}], A) = 0$, so A is T-complete. □

Recall that $\mathcal{B}_T$ denotes the category of T-complete abelian groups. We can deduce:

Lemma 2.5.7: *If X is a $\mathcal{B}_T$-nilpotent space, then X is T-complete.*

Proof. This now follows from Theorem 2.3.11, which tells us we can replace X by a Postnikov $\mathcal{B}_T$-tower, and co-HELP, which is Lemma 2.3.13. □

We can now recognise T-complete spaces, and once we can complete Eilenberg-MacLane spaces, it will be an easy matter to complete nilpotent spaces by induction up Postnikov towers:

Lemma 2.5.8: *Let A be an abelian group and $n \geq 1$. Then there exists a T-completion $K(A,n) \to (\widehat{K(A,n)})_T$.*

Proof. Firstly, let I be a, possibly infinite, indexing set, and let $n \geq 2$. Then, the homotopy fibre of $K(\oplus_I\mathbb{Z}, n) \to K((\widehat{\oplus_I\mathbb{Z})}_T, n)$ is equal to $K(Ext(\mathbb{Z}[T^{-1}], \oplus_I\mathbb{Z}), n-1)$, which has homology groups which are $\mathbb{Z}[T^{-1}]$-modules, by Theorem 2.5.5. Therefore, $\tilde{H}^*(F\phi; \mathbb{F}_p) = 0$, for all $p \in T$. Applying the Serre spectral sequence, we see that ϕ is an $\mathbb{F}_T$-equivalence to a T-complete space, by Lemma 2.5.7, and, therefore, a T-completion.

For arbitrary A and $n \geq 1$, we consider an exact sequence $0 \to \oplus_J\mathbb{Z} \to \oplus_I\mathbb{Z} \to A \to 0$. We get a fibration $K(A,n) \to K(\oplus_J\mathbb{Z}, n+1) \to K(\oplus_I\mathbb{Z}, n+1)$, and we define the T-completion $\psi : K(A,n) \to \widehat{K(A,n)}_T$, to be the homotopy fibre of the T-completions constructed in the previous paragraph. Then, ψ is an $\mathbb{F}_T$-equivalence by the Zeeman comparison theorem, Theorem 2.5.1, and $\widehat{K(A,n)}_T$ is T-complete, by Lemma 2.5.7. □

We can now prove:

Theorem 2.5.9: *If X is a nilpotent space, then there exists a T-completion $X \to \widehat{X}_T$.*

Proof. As in the proof of Lemma 2.5.8, we can define the T-completion of X_{n+1}, which lives in a fibre sequence of the form $X_{n+1} \to X_n \to K(A,m)$, with $m \geq 2$, to be the homotopy fibre of the T-completions of X_n and $K(A,m)$, which exist, by induction. □

This inductive construction of localisations and completions allows us to study their algebraic properties, via induction up towers. For example, arguments similar to those of Section 2.4, can be used to prove that a nilpotent space is T-local iff its homology groups are T-local iff its homotopy groups are T-local. More generally, we have the following two results, whose proofs can be found in [MP12]:

Theorem 2.5.10: *For a nilpotent space X, the following properties of a map $\phi : X \to Y$, where Y is a connected T-local space, are equivalent:*

i) ϕ is a T-localisation,

ii) $\phi_* : \pi_n(X) \to \pi_n(Y)$ is a T-localisation for all $n \geq 1$,

iii) $\phi_* : H_n(X) \to H_n(Y)$ is a T-localisation for all $n \geq 1$.

iv) $\phi^* : [Y, Z] \to [X, Z]$ is a bijection for all T-local spaces Z.

Proof. This is [MP12, Theorem 6.1.2]. □

Theorem 2.5.11: *If $\phi : X \to Y$ is a T-completion of a nilpotent space, then, for every $n \geq 1$, there is a splittable exact sequence:*

$$1 \to \mathbb{E}_T \pi_n(X) \to \pi_n(Y) \to \mathbb{H}_T \pi_{n-1}(X) \to 1$$

such that the composite $\pi_n(X) \to \mathbb{E}_T \pi_n(X) \to \pi_n(Y)$ is ϕ_*. Moreover, a nilpotent space is T-complete iff its homotopy groups are T-complete.

Proof. This is [MP12, Theorem 11.1.2]. □

Chapter 3

A double coset formula for the genus of a nilpotent group

Abstract: We derive double coset formulae for the genus and extended genus of a finitely generated nilpotent group G, using the notions of bounded and bounded above automorphisms of $\prod G_S$, which are defined relative to a fixed fracture square for G.

Assume, in this introductory paragraph, that all groups in sight are finitely generated and nilpotent, and if, in addition, a group is torsion free, we refer to it as an N-group. Given a finitely generated nilpotent group, there are fracture theorems which exhibit G as a pullback of its rational and p-local parts, or as a pullback of its rational and p-complete parts, [MP12, Theorem 7.2.1 ii), Theorem 12.3.2]. However, in general, knowing the p-localisations of a group is not sufficient to determine its isomorphism class, [PH75, pg. 32], and, therefore, the notion of the genus of G arises, defined as the set of isomorphism classes of groups, H, such that $H_p \cong G_p$ for all p. Similarly, there is a notion of the adelic genus of G, defined as the set of isomorphism classes of groups, H, such that $Z_pH \cong Z_pG$, for all p, and $H_0 \cong G_0$, where Z_pG is the p-adic completion of G, and G_0 is the rationalisation of G. Clearly, the genus is contained within the adelic genus. What is known about the (adelic) genus, in general? Well, perhaps surprisingly, it is known that the adelic genus is a finite set, and, therefore, so is the genus. In fact, this result was a key step in proving a theorem of Pickel, [Pic71, Theorem pg. 327], and we briefly outline the story below. We begin with a couple of definitions. Two groups, G and H, are said to be commensurable if there are subgroups G_1 and H_1 of finite index in G and H, respectively, such that $G_1 \cong H_1$. Two groups, G and H, are said to have isomorphic finite quotients if whenever F is a finite group, there exists a surjective homomorphism $G \to F$ iff there exists a surjective homomorphism $H \to F$. We have the following two facts, both of which follow, after a little work to set up the theory, from the relevant definitions. Firstly, G and H have isomorphic finite quotients iff for every prime p, Z_pG and Z_pH are isomorphic, [Pic71, Lemma 1.2]. Secondly, if G and H

are N-groups, then G and H are commensurable iff $G_0 = H_0$, [Mal49]. The main result of [Pic71] is now that the set of isomorphism classes of groups with isomorphic finite quotients to G is finite. Previously, in unpublished work, Borel had shown that if H has isomorphic finite quotients to G, then H lies in one of only finitely many commensurability classes, [Pic71, Theorem 3.1]. Therefore, after a little work to reduce to the torsion-free case, Pickel is able to reduce the problem to showing that the adelic genus of an N-group G is finite. An N-group G can be studied by considering associated Lie algebras, and the adelic genus can then be attacked with the theory of arithmetic groups, as developed by Borel in [Bor63], thereby completing the proof of the theorem. We also have notions of genus in the theory of nilpotent spaces, and we will discuss these some more in Subsection 3.1.1, along with their associated double coset formulae. More recently, in [HY17], members of the same genus of G were viewed as an example of conjugate objects in an ∞-category, and, in the case of a finite partition of the set of primes, a double coset formula for the genus was derived as an application of a general formula for calculating conjugates in an ∞-category.

3.1 Introduction

Let T, S and, for each i in some indexing set I, T_i be sets of primes such that $T = \cup_i T_i$ and $T_i \cap T_j = S$ for all $i \neq j$. Suppose also that $T \neq S$. Recall that an $f\mathbb{Z}_T$-nilpotent group is a nilpotent group which can be represented by a central series with quotients that admit the structure of finitely generated $\mathbb{Z}_T$-modules. Throughout, we let G be an $f\mathbb{Z}_T$-nilpotent group and consider a fixed reference diagram:

$$\begin{tikzcd}
G \arrow[r, "(\psi_i)"] \arrow[d, "\sigma"'] & \prod G_{T_i} \arrow[d, "\phi"] \arrow[dr, "\prod \phi_i"] & \\
G_S \arrow[r, "\omega"'] & (\prod G_{T_i})_S \arrow[r, "\tilde{\pi}"'] & \prod G_S
\end{tikzcd}$$

where each ψ_i is a localisation at T_i, ϕ is a localisation at S, σ is a localisation at S, ϕ_i is the unique localisation at S such that $\phi_i \psi_i = \sigma$, ω is the localisation of (ψ_i) and $\tilde{\pi}$ is the unique map making the triangle on the right commute. It follows from these definitions that $\tilde{\pi}\omega = \Delta$.

The purpose of this chapter is to derive double coset formulae for the genus and extended genus of G, and we begin by recalling the relevant definitions from [MP12]:

Definition 3.1.1: *i) the genus of G is the set of isomorphism classes of $f\mathbb{Z}_T$-nilpotent groups H such that for every $i \in I$, $H_{T_i} \cong G_{T_i}$,*
ii) the extended genus of G is the set of isomorphism classes of T-local nilpotent groups H such that for every $i \in I$, $H_{T_i} \cong G_{T_i}$.

We remark that these definitions depend on G being $f\mathbb{Z}_T$-nilpotent, and the sets of primes T_i. From this point of view, the genus should really be referred to as the $\{T_i\}$-genus, although we will not use this notation in this chapter, since the sets of primes are fixed. The fact that the extended genus is a set is a consequence of the fracture theorem, [MP12, Theorem 7.2.1 ii)], for T-local nilpotent groups.

In [MP12, Section 7.5], a map was defined which sends an automorphism $\alpha \in \prod Aut(G_S)$ to the pullback of $\alpha \circ \prod \phi_i$ along Δ, and it was claimed that this map was a surjection onto the extended genus of G. However, it is not necessarily true that the image of this map is contained within the extended genus of G. To see this consider the following fracture square for $\mathbb{Z}$, where the product is indexed over the natural numbers, p_i is the ith prime number, and each of the undefined maps is the inclusion sending 1 to 1:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "(\psi_i)"] \arrow[d, "\sigma"'] & \prod \mathbb{Z}_{\{p_i\}} \arrow[d, "\prod \phi_i"] \\ \mathbb{Q} \arrow[r, "\Delta"'] & \prod \mathbb{Q} \end{tikzcd}$$

Consider the automorphism $\alpha = \prod p_i$ of $\prod \mathbb{Q}$, where p_i also denotes multiplication p_i. Then the image of $\alpha \circ \prod \phi_i$ consists of elements $(q_i) \in \prod \mathbb{Q}$ such that if $q_i = \frac{a_i}{b_i}$ with a_i, b_i coprime, then a_i is divisible by p_i. In particular, the image of $\alpha \circ \prod \phi_i$ intersects the image of Δ only at 0. Therefore, the pullback group of $\alpha \circ \prod \phi_i$ along Δ is 0, which does not localise to $\mathbb{Z}_{\{p_i\}}$ for any i.

Nevertheless, this example turns out to be instructive. Suppose, instead, that $\alpha = \prod(\frac{u_i}{v_i})$ with u_i and v_i coprime non-zero integers. Suppose that α is 'bounded' in the sense that there are only finitely many primes which divide some u_i or v_i. Then the induced pullback is isomorphic to $\mathbb{Z}$, which is the unique abelian group in the genus of $\mathbb{Z}$. If, instead, α is only 'bounded above' in the sense that there are only finitely many primes which divide some u_i, then the pullback turns out to be in the extended genus of $\mathbb{Z}$ and the maps ψ_i are localisations at T_i. In fact, we will see that the pullback group is not finitely generated unless α is 'bounded'. Note that in the counterexample we formulated, the map α was neither 'bounded' nor 'bounded above'.

With this in mind, the purpose of this chapter will be to prove the following pair of double coset results, relating to the genus and extended genus of G respectively:

Theorem 3.1.2: *The genus of G is in 1-1 correspondence with the double coset:*

$$Aut(G_S)\backslash Aut_b(\textstyle\prod_i G_S)/\prod_i Aut(G_{T_i})$$

where $Aut_b(\prod_i G_S)$ is the subgroup of automorphisms of the form $\prod_i \alpha_i$ which are S-bounded, see Definition 3.4.1.

Theorem 3.1.3: *The extended genus of G is in 1-1 correspondence with the double coset:*

$$Aut(G_S)\backslash Aut_{b.a.}(\textstyle\prod_i G_S)/\prod_i Aut(G_{T_i})$$

where $Aut_{b.a.}(\prod_i G_S)$ is the monoid of automorphisms of the form $\prod_i \alpha_i$ which are S-bounded above, see Definition 3.3.1.

Here, the definitions of S-bounded and S-bounded above automorphisms, and the correct classification of the (extended) genus are new contributions.

We will begin in Section 3.2, with a review of the results about localisations of nilpotent groups that we will use in this chapter. Then, in Sections 3.3 and 3.4, we will derive our double coset formulae for the extended genus and genus, respectively. We then conclude the chapter in Section 3.5, by relating our results to the formal fracture square, and deriving a double coset formula in that context.

3.1.1 Review of other types of genus and double coset formulae

Finally, for this introductory section, we will review some other notions of genus, and corresponding double coset formulae, that can be found in the literature. Firstly, we adopt the following definition from [MP12, Definition 12.4.6], where recall that a subscript 0 denotes rationalisation:

Definition 3.1.4: *Let G be an $f\mathbb{Z}_T$-nilpotent group. The adelic genus of G is defined to be the set of isomorphism classes of $f\mathbb{Z}_T$-nilpotent groups, H, such that $H_0 = G_0$ and $\widehat{H}_p = \widehat{G}_p$ for every $p \in T$.*

We will restrict attention to the case where T is the set of all primes. In this case, Pickel has shown that the adelic genus of a torsion free finitely generated nilpotent group is in 1-1 correspondence with a subset of the double coset $\mathcal{G}_A^\infty \backslash \mathcal{G}_A / \mathcal{G}_Q$, in [Pic71, Proposition 3.2]. Our proofs of Theorems 3.1.2 and 3.1.3 will make heavy use of the universal property of localisations; however, in the case of the adelic genus, proofs of double coset formulae do not seem to hinge on the universal property of completion, but, instead, on the universal property of extensions of scalars. To justify this, note that if A is abelian, then $(\widehat{A}_p)_0$ is not p-complete, but is a $\mathbb{Q} \otimes \widehat{\mathbb{Z}}_p$-module. In the nilpotent setting, the notion of an R-module is replaced by the notion of a nilpotent R-group, [War76, Definition 10.4], and Warfield has shown that, if R is a binomial domain (eg $\mathbb{Q}, \widehat{\mathbb{Q}}_p, \widehat{\mathbb{Z}}_p$), then we can define the tensor product of a nilpotent group with R, with the universal property that a group homomorphism from G to a nilpotent R-group H factors uniquely through $G \to G \otimes R$, via an R-map $G \otimes R \to H$ ([War76, Theorem 10.14]).

Moving on to spaces, there is an entirely analogous definition of the genus and adelic genus of an $f\mathbb{Z}_T$-nilpotent space, and Wilkerson has derived a double coset formula for the adelic genus of a simply connected CW-complex of finite type, in [Wil76, Theorem 3.8]. Here, Sullivan's formal completion, [Sul05, page 76], takes the place of the extension of scalars functor $- \otimes \widehat{\mathbb{Z}}_p$. In order to generalise Wilkerson's double coset formula to nilpotent spaces, it would be interesting if a homotopical adjoint functor theorem, such as those of [NRS20] or [BLV23], could be used to construct the tensor product of a nilpotent space with a ring, with an appropriate universal property. In theory, we want that, if X is an $f\mathbb{Z}$-nilpotent space, then $X \to X \otimes \widehat{\mathbb{Z}}_p$ would be a p-completion of X, and $X \to X \otimes \mathbb{Q}$ would be a rationalisation.

We are not aware of a double coset formula for the extended adelic genus, or fully general double coset formulae for the genus or extended genus of an $f\mathbb{Z}_T$-nilpotent space. However, the special case where I is a finite indexing set is worth mentioning. In this case, the genus and extended genus of an $f\mathbb{Z}_T$-nilpotent space are the same, and the formal arguments of [MP12, Proposition 7.5.2] go through to yield a double coset formula for both. Moreover, in [HY17], this double coset formula is derived as an application of a general formula for calculating conjugates in an ∞-category. Finally, we remark that some other double coset formulae for the (adelic) genus of a nilpotent group/space are claimed in [Wil76, Theorem 1.2] and [MP12, Proposition 8.5.10, Remark 12.4.8, Theorem 13.6.6], but the proofs are incorrect, or missing in detail.

3.2 Review of nilpotent groups and their localisations

In this short, introductory section, we recall some definitions and results about nilpotent groups and their localisations which will help us on our way. The following result, which is an easy generalisation of a theorem of Warfield, [War76, Theorem 3.25], is used repeatedly throughout this chapter:

Lemma 3.2.1: *Let G be a nilpotent group of nilpotency class c, H a subgroup of G, and A a set of elements of G such that there exists an $s \in \mathbb{N}$ such that $a \in A \implies a^s \in H$. Then, if $g \in G$ is in the subgroup generated by elements of A and H, $g^{s^d} \in H$ where $d = \frac{1}{2}c(c+1)$.*

Proof. This is Lemma 2.1.16 □

Moving on to localisations, recall that a nilpotent group is T-local iff it is uniquely p-divisible for all $p \in T$. Recall, also, the following definitions from [MP12]:

Definition 3.2.2: *If R is a set of primes, then an R-number is a natural number which is a product of primes not in R.*

Definition 3.2.3: *Let $f : G \to H$ be a homomorphism between nilpotent groups. Then, we call f an:*

i) R-monomorphism if $f(g) = 1 \implies$ there is an R-number, r, such that $g^r = 1$,

ii) R-epimorphism if, for all $h \in H$, there exists an R-number, r, such that $h^r \in im(f)$,

iii) R-isomorphism if it is both an R-monomorphism and an R-epimorphism.

Unsurprisingly, we have:

Lemma 3.2.4: *A homomorphism between nilpotent groups, f, is an R-monomorphism/R-epimorphism/R-isomorphism iff f_R is a monomorphism/epimorphism/isomorphism, respectively.*

Proof. This is [MP12, Proposition 5.5.4]. □

We will also use:

Lemma 3.2.5: *R-localisation preserves pullbacks.*

Proof. This is [MP12, Lemma 5.5.7]. □

Recall that our reference group, G, is $f\mathbb{Z}_T$-nilpotent. The next few results record some consequences of this, starting with the observation that G is finitely T-generated in the following sense:

Definition 3.2.6: *A nilpotent group G is said to be finitely T-generated if there exists a finite subset A of G such that, for every $g \in G$, there exists a T-number, t, such that g^t is in the subgroup generated by A.*

Lemma 3.2.7: *A nilpotent group G is finitely T-generated iff G_T is $f\mathbb{Z}_T$-nilpotent.*

Proof. Firstly, if G is finitely T-generated, then the images of a finite T-generating set for G give a finite T-generating set for G_T. Conversely, if G_T is finitely T-generated, we can assume that the finite T-generating set is contained in the image of G, by Lemma 3.2.1. Then, we can form a finite T-generating set for G by picking an element in the preimage of each element of the finite T-generating set for G. The fact that this is a finite T-generating set for G again follows from Lemma 3.2.1. So we can assume that G is T-local, and this case is already proved in [MP12, Proposition 5.6.5]. To sketch how the argument goes, if G is $f\mathbb{Z}_T$-nilpotent, it is straightforward use a central series to inductively show that G is finitely T-generated, using Lemma 3.2.1. Conversely, if G is finitely T-generated, then it is clear that $Ab(G) = \frac{G}{[G,G]}$ is an $f\mathbb{Z}_T$-module, and so we can use the epimorphisms $Ab(G) \otimes ... \otimes Ab(G) \to \frac{\Gamma^i G}{\Gamma^{i+1} G}$ onto the quotients of the lower central series, [CMZ17, Corollary 2.10], to conclude that the lower central series expresses G as an $f\mathbb{Z}_T$-nilpotent group. □

Lemma 3.2.8: *Let G be an $f\mathbb{Z}_T$-nilpotent group with reference diagram as in the introduction. Then:*
i) G is T-Noetherian; that is G satisfies the ascending chain condition for T-local subgroups,
ii) $\tilde{\pi}$ is a monomorphism,
iii) G_{T_i} has no $(T_i - S)$-torsion for all but finitely many i. Equivalently, ϕ_i is a monomorphism for all but finitely many i.

Proof. i) This follows in the abelian case from the fact that $\mathbb{Z}_T$ is Noetherian, and the general nilpotent case follows via induction up a central series.
ii) It suffices to prove that $\prod \phi_i : \prod G_{T_i} \to \prod G_S$ is an S-monomorphism. This will follow from iii) and the fact that each ϕ_i is an S-monomorphism,
iii) Let $P = \{p_1, ..., p_k\}$ be a finite set of prime numbers and define:

$$G^P = \{g \in G | \ g^p = 1 \text{ for some product p of primes in P}\}$$

Then G^P is a T-local subgroup of G, by Lemma 3.2.1. Since G is T-Noetherian it follows that there is a finite set of primes Q such that if $g^n = 1$ for some $n \in \mathbb{N}$, then $g^q = 1$ for some product of primes in Q. Now suppose that T_i does not contain any primes in Q. If $a \in G_{T_i}$ is such that $a^s = 1$ for some product of primes in $(T_i - S)$, let t_1 be a T_i-number such that $a^{t_1} = \psi_i(g)$ for some $g \in G$. We have that $\psi_i(g^s) = 1$ and so there is a T_i-number t_2 such that $g^{st_2} = 1$. Since s is coprime to each of the primes in Q, it follows that $g^{t_2} = 1$ and, therefore, that $a^{t_1 t_2} = 1$. Since G_{T_i} is T_i-local, it follows that $a = 1$, as desired. $\square$

3.3 A double coset formula for the extended genus of G

Recall the notation from the reference diagram at the beginning of the introduction, which we will use throughout the remaining sections:

$$\begin{tikzcd}
G \arrow[r, "(\psi_i)"] \arrow[d, "\sigma"'] & \prod G_{T_i} \arrow[d, "\phi"] \arrow[dr, "\prod \phi_i"] & \\
G_S \arrow[r, "\omega"'] & (\prod G_{T_i})_S \arrow[r, "\tilde{\pi}"'] & \prod G_S
\end{tikzcd}$$

The aim of this section is to show that if $\alpha = \prod \alpha_i$ is an automorphism of $\prod G_S$, then in the pullback diagram below:

$$\begin{tikzcd}
H \arrow[r, "(\kappa_i)"] \arrow[d, "\mu"'] & \prod G_{T_i} \arrow[d, "\prod \alpha_i \phi_i"] \\
G_S \arrow[r, "\Delta"'] & \prod G_S
\end{tikzcd}$$

κ_i is a T_i-localisation for all i iff α is S-bounded above in the following sense:

Definition 3.3.1: *An automorphism $\alpha = \prod \alpha_i \in \prod Aut(G_S)$ is said to be S-bounded above if there exists an S-number s such that for all i and for all $g_i \in G_{T_i}$, $\alpha_i^{-1}\phi_i(g_i^s) \in im(\phi_i)$.*

From this, the double coset formula for the extended genus will follow in the expected manner. We start with:

Lemma 3.3.2: *If κ_i is a T_i-localisation for all i, then α is S-bounded above.*

Proof. Let A be a finite T-generating set for G. Since the κ_i are T_i-localisations, μ is an S-localisation. It follows that there exists an S-number s such that for all $a \in A$, $\sigma(a^s) \in im(\mu) \subset im(\alpha_i \phi_i)$ for all i. Since $\psi_i(A)$ is a finite T_i-generating set for G_{T_i}, it follows from Lemma 3.2.1 that if $g_i \in G_{T_i}$, then $\phi_i(g_i^{s^d}) \in im(\alpha_i \phi_i)$, where $d = \frac{1}{2}c(c+1)$, for c the nilpotency class of G. $\square$

For the reverse direction, we start with the following observation which does not require α to be S-bounded above:

Lemma 3.3.3: *μ is an S-monomorphism.*

Proof. By Lemma 3.2.8, $\prod \phi_i$ is an S-monomorphism, and, therefore, so is $\prod \alpha_i\phi_i$. The result follows since the pullback of an S-monomorphism is an S-monomorphism, by Lemmas 3.2.4 and 3.2.5. □

Lemma 3.3.4: *If α is S-bounded above, then μ is an S-epimorphism, hence an S-localisation.*

Proof. If $x \in G_S$, then since σ is an S-localisation, there exists an S-number r such that $x^r \in im(\sigma) \subset im(\phi_i)$ for all i. Since α is S-bounded, there exists an S-number s such that $x^{rs} \in im(\alpha_i\phi_i)$ for all i. It follows that x^{rs} is in the image of μ by the definition of a pullback. □

We now have:

Lemma 3.3.5: *If α is S-bounded above, then κ_i is a T_i-localisation for all i.*

Proof. If $h \in H$ and $\kappa_i(h) = 1$, then $\mu(h) = 1$ and so there exists an S-number s such that $h^s = 1$. Write s as a product of a T_i-number t and a product of primes in T_i, r. Then, if $j \neq i$, $\kappa_j(h^t) = 1$, since G_{T_j} is T_j-local and $T_i \cap T_j = S$. Clearly $\kappa_i(h^t) = 1$, so it follows that κ_i is a T_i-monomorphism.

Now suppose that $g_i \in G_{T_i}$ and let $x = \alpha_i\phi_i(g_i)$. Since μ is an S-localisation, there exists an S-number s and $h \in H$ such that $x^s = \mu(h)$. Write s as a product of a T_i-number t and a product of primes in T_i, r. If $j \neq i$, then the image of $\alpha_j\phi_j$ is a T_j-local subgroup of G_S and so $x^t \in im(\alpha_j\phi_j)$ for all $j \neq i$. Since x^t is also in $im(\alpha_i\phi_i)$, it follows that g_i^t is in the image of κ_i by the definition of a pullback. □

In order to state a double coset formula for the extended genus, we need to show that $\prod \alpha$ is S-bounded above, for $\alpha \in Aut(G_S)$. In fact, we will prove the stronger result that $\prod \alpha$ is S-bounded, and the reader is invited to skip ahead and read the definition of an S-bounded automorphism in Definition 3.4.1.

Lemma 3.3.6: *If $\alpha \in Aut(G_S)$, then $\prod \alpha$ is S-bounded.*

Proof. Let A be a finite set of T-generators for G. Since σ is an S-localisation, there exists an S-number s such that for all $a \in A$, $\alpha\sigma(a^s), \alpha^{-1}\sigma(a^s) \in im(\sigma)$. Since, for all i, $\sigma = \phi_i\psi_i$ and $\psi_i(A)$ is a finite set of T_i generators for G_{T_i}, this implies, by Lemma 3.2.1, that for all $g_i \in G_{T_i}$, $\alpha\phi_i(g_i^{s^d})$ and $\alpha^{-1}\phi_i(g_i^{s^d}) \in im(\phi_i)$, where $d = \frac{1}{2}c(c+1)$ is independent of i. It follows that $\prod \alpha$ is S-bounded. □

It is clear than an automorphism of the form $\prod_i \beta_i \in \prod_i Aut(G_{T_i})$ also induces an S-bounded automorphism of $\prod_i G_S$, and we can now prove:

Theorem 3.3.7: *The extended genus of G is in 1-1 correspondence with the double coset:*

$$Aut(G_S)\backslash Aut_{b.a.}(\textstyle\prod_i G_S)/\prod_i Aut(G_{T_i})$$

where $Aut_{b.a.}(\prod_i G_S)$ is the monoid of automorphisms of the form $\prod_i \alpha_i$ which are S-bounded above. The correspondence sends an S-bounded above automorphism α to the pullback group of $\alpha \circ (\prod \phi_i)$ along Δ.

Proof. We have a map from $Aut_{b.a.}(\prod_i G_S)$ to the extended genus of G, defined by sending an S-bounded above automorphism α to the pullback group of $\alpha \circ (\prod \phi_i)$ along Δ. The fact that the map factors through the double coset follows from the following commutative diagram, in which all vertical maps are isomorphisms:

$$\begin{tikzcd}
G_S \arrow[r, "\Delta"] \arrow[d, "\alpha"'] & \prod G_S \arrow[d, "\prod \alpha"] & \prod G_{T_i} \arrow[l, "\prod \alpha_i \phi_i"'] \arrow[d, "\prod \beta_i"] \\
G_S \arrow[r, "\Delta"'] & \prod G_S & \prod G_{T_i} \arrow[l, "\prod \alpha\alpha_i (\beta_i)_S^{-1} \phi_i"]
\end{tikzcd}$$

where $\prod \alpha_i \in Aut_{b.a.}(\prod_i G_S)$, $\alpha \in Aut(G_S)$ and, for every i, $\beta_i \in Aut(G_{T_i})$.

For surjectivity, if H is in the extended genus of G, then we can form a diagram:

$$\begin{tikzcd}
H \arrow[r, "(\epsilon_i)"] \arrow[d, "\mu"'] & \prod G_{T_i} \arrow[d, "\phi"] \arrow[dr, "\prod \phi_i"] & \\
G_S \arrow[r, "\omega_H"'] & (\prod G_{T_i})_S \arrow[r, "\tilde{\pi}"'] & \prod G_S
\end{tikzcd}$$

Here, each ϵ_i is a T_i-localisation, μ is a S-localisation, and ω_H is then defined as the localisation of (ϵ_i). By [MP12, Theorem 7.2.1ii)], the left hand square is a pullback, and, since $\tilde{\pi}$ is a monomorphism, by Lemma 3.2.8ii), so is the larger square with base $\tilde{\pi}\omega_H$. Now $\tilde{\pi}\omega_H \neq \Delta$, in general. Instead, it is the product of localisations of each ϵ_i - that is, $\tilde{\pi}\omega_H = (\alpha_i)$, where each α_i is an automorphism of G_S. Let $\alpha := \prod \alpha_i$. Rearranging the pullback, we see that H is isomorphic to the pullback of $\alpha^{-1} \circ \prod \phi_i$ along Δ, and α^{-1} is S-bounded above by Lemma 3.3.2.

For injectivity, suppose that $\alpha = \prod \alpha_i, \beta = \prod \beta_i$ are S-bounded above automorphisms and we have pullbacks:

$$\begin{tikzcd}
P \arrow[r, "(\kappa_i)"] \arrow[d, "\mu"'] & \prod G_{T_i} \arrow[d, "\prod \alpha_i \phi_i"] \\
G_S \arrow[r, "\Delta"'] & \prod G_S
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "(\bar{\kappa}_i)"] \arrow[d, "\bar{\mu}"'] & \prod G_{T_i} \arrow[d, "\prod \beta_i \phi_i"] \\
G_S \arrow[r, "\Delta"'] & \prod G_S
\end{tikzcd}$$

By uniqueness of localisations, there is an automorphism $\gamma = \prod \gamma_i$ of $\prod G_{T_i}$ such that $\gamma(\kappa_i) = (\bar{\kappa}_i)$. Therefore, since we only care about equivalence classes in the double coset we may assume that $\kappa_i = \bar{\kappa}_i$ for all i. Similarly, there is an automorphism $\gamma^{'}$ of G_S such that $\gamma^{'}\mu = \bar{\mu}$, and so we can reduce to the case $\mu = \bar{\mu}$. Now, for all i, $\alpha_i\phi_i$ and $\beta_i\phi_i$ are both the unique factorisation of μ through κ_i. By uniqueness of factorisation through

ϕ_i, we must have $\alpha_i = \beta_i$, as desired. $\square$

3.4 A double coset formula for the genus of G

The purpose of this section is to prove that if we restrict the map of Theorem 3.3.7 to the S-bounded automorphisms, defined as follows, then its image is precisely the genus of G.

Definition 3.4.1: *An automorphism $\alpha = \prod \alpha_i \in \prod Aut(G_S)$ is said to be S-bounded if there exists an S-number s such that for all $(g_i) \in \prod G_{T_i}$, $\alpha \circ (\prod \phi_i)(g_i^s) \in im(\prod(\phi_i))$ and $\alpha^{-1} \circ (\prod \phi_i)(g_i^s) \in im(\prod(\phi_i))$.*

We begin with the following observation, which is also [MP12, Proposition 7.4.3]:

Lemma 3.4.2: *If I is a finite indexing set and H is a T-local nilpotent group such that H_{T_i} is finitely T_i-generated for all $i \in I$, then H is finitely T-generated.*

Proof. Let $H_0 \subset H_1 \subset ...$ be an ascending chain of T-local subgroups of H. For each i, let κ_i denote a T_i-localisation of H and let H_j^i denote the T_i-local subgroup of H_{T_i} generated by $\kappa_i(H_j)$. Choose an integer N such that $H_0^i \subset H_1^i \subset ...$ terminates at H_N^i for all i. Now let $n \geq N$; we claim that $H_n = H_N$. If $h \in H_n$, then there exists a T_i-number t_i and a $k \in H_N$ such that $\kappa_i(h^{t_i}) = \kappa_i(k)$. It follows that there is a T_i-number s_i such that $(h^{t_i}k^{-1})^{s_i} = 1$. Since the set of $g \in H$ such that there exists a T_i-number s such that $g^s \in H_N$ is a subgroup of G which contains $h^{t_i}k^{-1}$ and k, it follows that there is a T_i-number r_i such that $h^{r_i} \in H_N$. Now any common factor of each of the r_i lies outside of T and so there is a T-number r such that $h^r \in H_N$. Since H_N is T-local, it follows that $h \in H_N$ as desired. So H is T-Noetherian, which implies that H is finitely T-generated. $\square$

Now suppose that H is in the image of an S-bounded automorphism, α, under the map of Theorem 3.3.7. We consider the finite subset, F, of I consisting of i such that ϕ_i is not a monomorphism. Then H fits into a diagram of the form:

$$\begin{tikzcd}
H \arrow[r] \arrow[d] & (\prod_{i \notin F} G_{T_i}) \times (\prod_{j \in F} G_{T_j}) \arrow[d, "1 \times (\prod(\alpha_j \phi_j))"] & \\
P \arrow[r] \arrow[d] & (\prod_{i \notin F} G_{T_i}) \times (\prod_{j \in F} G_S) \arrow[r] \arrow[d, "(\prod(\alpha_i \phi_i)) \times 1"] & \prod_{i \notin F} G_{T_i} \arrow[d] \\
G_S \arrow[r, "\Delta \times \Delta"'] & (\prod_{i \notin F} G_S) \times (\prod_{j \in F} G_S) \arrow[r] & \prod_{i \notin F} G_S
\end{tikzcd}$$

where $\alpha = \prod \alpha_i$ is S-bounded, and each of the squares is a pullback, which defines P. Consider the localisation of the diagram at $T' = \cup_{i \notin F} T_i$. The groups in the bottom two rows are all T'-local. If $j \in F$, $T_j \cap T' = S$, so $\alpha_j \phi_j$ is a T'-localisation. It follows that P is a T'-localisation of H. In light of Lemma 3.4.2, if we want to show that H is finitely T-generated, it suffices to show that P is finitely T'-generated. Note also that P is the image of an S-bounded automorphism in the extended genus of $G_{T'}$. In this way we can reduce the next lemma to the case where ϕ_i is a monomorphism for all i.

Lemma 3.4.3: *If H is the image of an S-bounded automorphism, then H is finitely T-generated.*

Proof. As discussed above, we can reduce to the case where ϕ_i is a monomorphism for all i. Let $\alpha = \prod \alpha_i$ be an S-bounded automorphism such that we have a pullback square:

$$\begin{tikzcd} H \arrow[r] \arrow[d] & \prod G_{T_i} \arrow[d, "\prod \alpha_i \phi_i"] \\ G_S \arrow[r, "\Delta"'] & \prod G_S \end{tikzcd}$$

Let K be the T-local subgroup of H consisting of pairs $(x,(g_i))$ with $x \in G_S, g_i \in G_{T_i}$, such that, for all i, $\alpha_i\phi_i(g_i) = x$ and $x \in im(\phi_i)$, say $x = \phi_i(a_i)$. Then there is an injective group homomorphism $K \to G$ sending $(x,(g_i))$ to $(x,(a_i))$. Since G is finitely T-generated so is K, and since α is S-bounded there exists an S-number s such that if $h \in H$, then $h^s \in K$. Consider a T-subnormal series for K:

$$K = K_0 \subset K_1 \subset ... \subset K_m = H$$

If we localise at T_i, then all of the groups in the chain become finitely T_i-generated. Moreover, $(\frac{K_{j+1}}{K_j})_{T_i}$ is a finitely T_i-generated nilpotent group such that if $k \in (\frac{K_{j+1}}{K_j})_{T_i}$, then $k^s = 1$. For all but finitely many i this implies that $(\frac{K_{j+1}}{K_j})_{T_i}$ is trivial. For the remaining i, $(\frac{K_{j+1}}{K_j})_{T_i}$ is finitely T_i-generated (in fact it is finite). Therefore, using the fracture square [MP12, Theorem 7.2.1ii)], we see that $\frac{K_{j+1}}{K_j}$ is finitely T-generated (in fact it is finite). Inductively, it follows that H is finitely T-generated (and K is a subgroup of finite index in H). □

It remains to prove that every element of the genus is the image of an S-bounded automorphism. We start with the following observation:

Lemma 3.4.4: *If H is in the genus of G, then there is a finite subset F of I such that if $T' = \cup_{i \notin F} T_i$, then $G_{T'} \cong H_{T'}$.*

Proof. By [PH75, Theorem I.3.3], since $G_S \cong H_S$, there is a finitely T-generated nilpotent group P equipped with S-isomorphisms $f : P \to G$ and $g : P \to H$. In fact, we just need to consider the pullback:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & G \arrow[d, "\phi_S"] \\ H \arrow[r, "\phi_S"'] & G_S \end{tikzcd}$$

to get the desired maps, where ϕ_S denotes a localisation at S. Since G, H and P are finitely T-generated, we can use Lemma 3.2.1 to show that there exists an S-number s such that if $p \in ker(f)$ or $p \in ker(g)$, then $p^s = 1$ and, if $g \in G, h \in H$, then $g^s \in im(f), h^s \in im(g)$. This implies that if we take $T^{'}$ to be the union of the T_i which don't contain any prime factors of s, then both f and g are $T^{'}$-isomorphisms, which implies the result. □

We can now prove:

Lemma 3.4.5: *If H is in the genus of G, then H is the image of an S-bounded automorphism.*

Proof. Let F be a finite subset of I such that if $T' = \cup_{i \notin F} T_i$, then $H_{T'} \cong G_{T'}$. Let $\mu : H \to G_{T'}$ and $\epsilon : G \to G_{T'}$ be $T^{'}$-localisations. Then there are unique factorisations of σ and ψ_i, for $i \notin F$, through ϵ; denote them by $\sigma^{'}, \psi_i^{'}$. Note that $\phi_i \psi_i^{'} = \sigma_i^{'}$. Since H is finitely T-generated, we can form a global to local fracture square, [MP12, Theorem 7.2.1 iii)], of the form:

$$\begin{tikzcd} H \arrow[r, "(\psi_i^{'}\mu)\times(\kappa_j)"] \arrow[d, "\sigma^{'}\mu"'] & (\prod_{i \notin F} G_{T_i}) \times (\prod_{j \in F} G_{T_j}) \arrow[d, "(\phi_i)\times(\alpha_j\phi_j)"] \\ G_S \arrow[r, "\Delta\times\Delta"'] & (\prod_{i \notin F} G_S) \times (\prod_{j \in F} G_S) \end{tikzcd}$$

where κ_j is any T_j-localisation of H and $\alpha_j \in Aut(G_S)$. Since F is finite, $1 \times (\alpha_j)$ is S-bounded, which can be seen directly or via Lemma 3.5.2 of Section 3.5, as desired. □

We can now prove our double coset formula for the genus of G:

Theorem 3.4.6: *The genus of G is in 1-1 correspondence with the double coset:*

$$Aut(G_S) \backslash Aut_b(\textstyle\prod_i G_S) / \prod_i Aut(G_{T_i})$$

where $Aut_b(\prod_i G_S)$ is the subgroup of automorphisms of the form $\prod_i \alpha_i$ which are S-bounded. The correspondence sends an S-bounded automorphism α to the pullback group of $\alpha \circ (\prod \phi_i)$ along Δ.

Proof. We have already shown that the correspondence is well-defined and surjective in Lemmas 3.4.3 and 3.4.5. It is injective by Theorem 3.3.7. □

3.5 Relationship to the formal fracture square

So far we have phrased our results in terms of the fracture square [MP12, Theorem 7.2.1 iii)] with the diagonal map as the base. In this section, we investigate what happens if we try to define a double coset formula relative to the fracture square:

$$\begin{array}{ccc} G & \xrightarrow{(\psi_i)} & \prod G_{T_i} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\phi} \\ G_S & \xrightarrow[\omega]{} & (\prod G_{T_i})_S \end{array}$$

It turns out that this fracture square only sees the genus of G, and not the whole of the extended genus. Recall that we have previously considered 'diagonal' automorphisms of $\prod G_S$ of the form $\prod \alpha_i$, for $\alpha_i \in Aut(G_S)$. We first define the analogue of a diagonal automorphism in $Aut((\prod G_{T_i})_S)$:

Definition 3.5.1: *$DAut((\prod G_{T_i})_S)$ is the subgroup of $Aut((\prod G_{T_i})_S)$ consisting of automorphisms α such that, for every $j \in I$, under the identification of $(\prod G_{T_i})_S$ with $G_{T_j} \times (\prod_{i \neq j} G_{T_i})_S$ determined by ϕ and any S-localisation of $G_{T_j} \times (\prod_{i \neq j} G_{T_i})$ of the form $\phi_j \times \phi_j^{'}$, $\alpha = \alpha_j \times \beta$ for some automorphisms α_j of G_S and β of $(\prod_{i \neq j} G_{T_i})_S$.*

Note that the subgroup of diagonal automorphisms, $DAut((\prod G_{T_i})_S)$, is independent of the choice of the collection $\{\phi_j^{'}\}_{j \in I}$. Also, note that if α is a diagonal automorphism of $(\prod G_{T_i})_S$, then there is a commutative diagram:

$$\begin{array}{ccc} (\prod G_{T_i})_S & \xrightarrow{\tilde{\pi}} & \prod G_S \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\prod \alpha_i} \\ (\prod G_{T_i})_S & \xrightarrow[\tilde{\pi}]{} & \prod G_S \end{array}$$

Since $\tilde{\pi}$ is a monomorphism, it follows that there is an injective homomorphism $DAut((\prod G_{T_i})_S) \to \prod Aut(G_S)$. We will now show that the image of this map is the subgroup of S-bounded automorphisms of $\prod G_S$. It follows, from Lemma 3.3.6, that $Aut(G_S)$ defines a subgroup of $DAut((\prod G_{T_i})_S)$.

Lemma 3.5.2: *An automorphism $\alpha \in \prod Aut(G_S)$ is the image of a diagonal automorphism β iff α is S-bounded.*

Proof. First suppose that α is the image of a diagonal automorphism β. Let A be a finite set of T-generators for G. Since ϕ is an S-epimorphism, there exists an S-number s such that for all $a \in A$, $\beta\omega\sigma(a^s)$ and $\beta^{-1}\omega\sigma(a^s) \in im(\phi)$. Now $\psi_i(A)$ is a finite set of T_i generators for G_{T_i} and $im(\phi_i)$ is a T_i-local subgroup of

G_S. It follows, by Lemma 3.2.1, that if $g_i \in G_{T_i}$ then $\alpha_i\phi_i(g_i^{s^d}), \alpha_i^{-1}\phi_i(g_i^{s^d}) \in im(\phi_i)$, where $d = \frac{1}{2}c(c+1)$, for c the nilpotency class of G. Since d is independent of i, it follows that α is S-bounded as desired.

Now suppose that α is S-bounded. Let $F \subset I$ be the finite subset of I such that ϕ_i is not a monomorphism for $i \in F$. If $i \notin F$, let H_i be the subgroup of G_{T_i} consisting of g_i such that $\alpha_i\phi_i(g_i) \in im(\phi_i)$. Define unique homomorphisms, f_i, such that the following square commutes:

$$\begin{tikzcd}
\prod_{i\notin F} H_i \arrow[r, "\prod f_i"] \arrow[d, "(\prod \phi_i)\circ\iota"'] & \prod_{i\notin F} G_{T_i} \arrow[d, "\prod \phi_i"] \\
\prod_{i\notin F} G_S \arrow[r, "\prod \alpha_i"'] & \prod_{i\notin F} G_S
\end{tikzcd}$$

where ι is the inclusion of $\prod H_i$ into $\prod G_{T_i}$. Since α is S-bounded, ι is an S-isomorphism. Similarly, the image of the monomorphism $f := \prod f_i$ is $\{(g_i) \mid \forall i\ \alpha_i^{-1}\phi_i(g_i) \in im(\phi_i)\}$ and so, since α is S-bounded, f is also an S-isomorphism. Let $\phi_{I/F}$ be an S-localisation of $\prod_{i\notin F} G_{T_i}$, so there is an induced isomorphism $(\prod G_{T_i})_S \cong_\tau \prod_{i\in F} G_S \times (\prod_{i\notin F} G_{T_i})_S$ induced by ϕ and $(\prod_{i\in F}\phi_i)\times\phi_{I/F}$. Since the vertical arrows in the diagram below are S-localisations, there is a unique map f_S making the diagram commute:

$$\begin{tikzcd}
\prod_{i\notin F} H_i \arrow[r, "f"] \arrow[d, "\phi_{I/F}\iota"'] & \prod_{i\notin F} G_{T_i} \arrow[d, "\phi_{I/F}"] \\
(\prod_{i\notin F} G_{T_i})_S \arrow[r, "f_S"'] & (\prod_{i\notin F} G_{T_i})_S
\end{tikzcd}$$

Since f is an S-isomorphism, $(\prod \alpha_i)\times f_S$ defines an automorphism of $\prod_{i\in F} G_S \times (\prod_{i\notin F} G_{T_i})_S$. Noting that the S-localisation of f_i with respect to $\phi_i \circ \iota_i$ and ϕ_i is α_i, it follows that if we define $\beta \in Aut((\prod G_{T_i})_S)$ to correspond to $(\prod \alpha_i)_{i\in F} \times f_S$ under the isomorphism, τ, given above, then β is a diagonal automorphism whose image is α. □

It is now an easy matter to reformulate our double coset formula for the genus of G in terms of the formal fracture square:

Theorem 3.5.3: *There is a 1-1 correspondence between the genus of G and the double coset:*

$$Aut(G_S) \setminus DAut((\textstyle\prod G_{T_i})_S) \ / \ \textstyle\prod Aut(G_{T_i})$$

The correspondence sends a diagonal automorphism α to the pullback group of $\alpha\phi$ along ω.

Proof. By Theorem 3.4.6 and Lemma 3.5.2, it is immediate that there is a 1-1 correspondence between the double coset and the genus of G, sending α to the pullback group of $\prod \alpha_i\phi_i$ along Δ. This is equivalent to sending α to the pullback group of $\alpha\phi$ along ω, since $\tilde{\pi}$ is a monomorphism. □

Our final result tells us that a nilpotent group, H, in the extended genus of G, is finitely T-generated iff the S-localisation of the map $H \to \prod G_{T_i}$ is equivalent to ω. To make this precise, we have:

Definition 3.5.4: *Define $Orb(G_S, (\prod G_{T_i})_S)$ to be the set of orbits of $Hom(G_S, (\prod G_{T_i})_S)$ under the action of the group $Aut(G_S) \times DAut((\prod G_{T_i})_S)$.*

If $E(G)$ denotes the extended genus of G, then we have a map $L : E(G) \to Orb(G_S, (\prod G_{T_i})_S)$ defined by sending H to the S-localisation of some product of T_i-localisations, $(\epsilon_i) : H \to \prod G_{T_i}$, with respect to some S-localisation $\mu : H \to G_S$, and ϕ. By definition, $L(H)$ is independent of the choices of μ and ϵ_i. We have:

Lemma 3.5.5: *The genus of G is equal to $L^{-1}(Orb(\omega))$.*

Proof. If $H \in L^{-1}(Orb(\omega))$, then the fracture square, [MP12, Theorem 7.2.1], exhibits H as the pullback of $\alpha\phi$ along ω, for some diagonal automorphism α. So H is in the genus of G, by Theorem 3.5.3. Conversely, if H is in the genus of G, then, by Theorem 3.5.3, we can view H as the pullback of $\alpha\omega$ along ϕ, for some diagonal automorphism α, so $L(H) = Orb(\omega)$. $\square$

Chapter 4

Completion preserves fibre squares of nilpotent spaces

Abstract: We prove that completion preserves homotopy fibre squares of nilpotent spaces. As an application, we deduce the Hasse fracture square associated to a nilpotent space.

4.1 Introduction

Let T be a non-empty set of primes. By a functorial T-completion, we mean a pair (F, α), where F is a functor that takes nilpotent spaces to nilpotent spaces, and $\alpha : 1 \to F$ is a natural transformation such that, for all X, $\alpha_X : X \to F(X)$ is a T-completion of X, see Definition 2.5.4. By a homotopy fibre square, we mean a strictly commutative square such that the canonical map to the double mapping path space, which we denote by $N(f,g)$, is a weak equivalence. The main result of this chapter now states:

Theorem 4.1.1: *Let $f : X \to A$ and $g : Y \to A$ be maps between connected nilpotent spaces such that $N(f,g)$ is connected. If we have a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & A \arrow[d] & Y \arrow[l, "g"'] \arrow[d] \\ \widehat{X}_T \arrow[r, "\widehat{f}_T"'] & \widehat{A}_T & \widehat{Y}_T \arrow[l, "\widehat{g}_T"] \end{tikzcd}$$

such that the vertical maps are T-completions, then the induced map $N(f,g) \to N(\widehat{f}_T, \widehat{g}_T)$ is a T-completion.

It follows from Theorem 4.1.1 that any functorial T-completion preserves homotopy fibre squares of connected nilpotent spaces, where by a homotopy fibre square we mean a strictly commutative square such that the canonical map to the double mapping path space is a weak equivalence. For example, this applies to the functorial completions obtained via Bousfield localisation at the homology theory $H_*(-;\bigoplus_{p\in T}\mathbb{F}_p)$, or via use of the Bousfield-Kan completion functor.

Similar results to Theorem 4.1.1 can be found in the literature. In particular, in [Far03, Theorem 1.1], Farjoun proves an analogue of Theorem 4.1.1 in the case of disconnected spaces. However, one can only conclude directly from Theorem 1.1 of [Far03] that the comparison map $\widehat{N(f,g)}_T \to N(\widehat{f}_T,\widehat{g}_T)$ has homotopically discrete fibre. Our result can also be viewed as a generalisation of the connected fibre lemma of Bousfield and Kan, [BK72a, Ch. II Lemma 4.8], which is the special case $Y = Y^{'} = *$ of Theorem 4.1.1.

Our main reason for being interested in Theorem 4.1.1 is that it provides a natural context in which to deduce the following well-known fracture theorem, sometimes known as the Hasse square:

Theorem 4.1.2: *Let X be a T-local connected nilpotent space. Then any commutative square:*

$$\begin{tikzcd} X \arrow[r, "\widehat{\phi}_T"] \arrow[d, "\psi"'] & \widehat{X}_T \arrow[d, "\phi"] \\ X_0 \arrow[r, "(\widehat{\phi}_T)_0"'] & (\widehat{X}_T)_0 \end{tikzcd}$$

with $\widehat{\phi}_T$ a T-completion and ψ, ϕ rationalisations, is a homotopy fibre square.

For the standard proof, see [DFK77, Theorem 4.4]. We explain how the Hasse square can be deduced from Theorem 4.1.1 at the end of this chapter.

Finally, at the beginning of the chapter, we prove some basic results about the category of T-complete nilpotent groups, thereby allowing us to clarify some misleading statements in the literature. For example, we will show that the kernel and, if the image is normal, the cokernel of a homomorphism between T-complete nilpotent groups are T-complete, whereas it was claimed that this is not necessarily the case on page 218 of [MP12]. It goes without saying that such a flexible result is useful when dealing with exact sequences of T-complete nilpotent groups.

4.1.1 Counterexamples

We now say a few words about the hypotheses of Theorem 4.1.1. Consideration of the path-space fibration associated to $K(G,1)$ for any G with $\pi_2(\widehat{K(G,1)}_T) \neq 0$, shows that connectivity assumptions on $N(f,g)$ are necessary.

We also have the following counterexample, due to Sullivan, when the spaces involved are not nilpotent. The counterexample is based on a non-nilpotent space Z satisfying the following properties - see [Sul05, pg. 104] and [BK72a, Ch. VII. 3.6] for details of the construction:

1. $\pi_1(Z) = \frac{\mathbb{Z}}{n\mathbb{Z}}$, where n can be any integer dividing $p-1$,
2. $\pi_2(Z) = \widehat{\mathbb{Z}}_p$,
3. $\pi_i(Z) = 0$ for $i \geq 3$,
4. $\Omega \widehat{Z}_p \simeq \widehat{S}_p^{2n-1}$.

For large values of n and p, it is clear from these properties that p-completion cannot preserve the fibre sequence:

$$K(\widehat{\mathbb{Z}}_p, 2) \to Z \to K(\frac{\mathbb{Z}}{n\mathbb{Z}}, 1)$$

4.1.2 Notation

We use throughout the notations and conventions of [MP12]. Indeed, anybody who has read [MP12] has more than enough background to understand this chapter. For example, we have the definitions $\mathbb{E}_T G := \pi_1(\widehat{K(G,1)}_T), \mathbb{H}_T G := \pi_2(\widehat{K(G,1)}_T)$. If G is abelian, then $\mathbb{H}_T G$ and $\mathbb{E}_T G$ are the first and zeroth derived functors of T-adic completion, respectively. In the abelian case, $\mathbb{H}_T G = Hom(\mathbb{Z}[T^{-1}]/\mathbb{Z}, G)$ and $\mathbb{E}_T G = Ext(\mathbb{Z}[T^{-1}]/\mathbb{Z}, G)$, which justifies the notation.

4.2 Completion and Fibre Squares

4.2.1 Properties of T-complete nilpotent groups

We begin the chapter with some basic results about the category $\mathcal{B}_T$ of T-complete abelian groups, and their nilpotent analogues. Let G and H be nilpotent groups equipped with central series of the same length, $\{G_i\}$ and $\{H_i\}$. Let $f : G \to H$ be a group homomorphism such that, for all i, $f(G_i) \subset H_i$, and let K denote the kernel of f and, if the image is normal, let $\pi : H \to Q$ denote the cokernel of f. Then, there are induced central series $\{G_i \cap K\}$ and $\{\pi(H_i)\}$ expressing K and, if the image is normal, Q as nilpotent groups. Recall from [BK72a, Ch. III, Lemma 5.8], that there is a singly graded spectral sequence which inductively computes the induced central series for K and Q, starting from the maps $f : \frac{G_i}{G_{i-1}} \to \frac{H_i}{H_{i-1}}$. Its E^0-page is given by:

$$E_i^0 = \frac{G_i}{G_{i-1}} \oplus \frac{H_i}{H_{i-1}}$$

and has differential defined by $d^0(g, h) = (0, f(g))$. It converges after finitely many pages to its E^∞-page, which is defined by:

$$E_i^{\infty} = \frac{G_i \cap K}{G_{i-1} \cap K} \oplus \frac{H_i}{(f(G) \cap H_i)H_{i-1}}$$

Of course, when the image of f is normal, the second term of the sum can be identified with $\frac{\pi(H_i)}{\pi(H_{i-1})}$. If $\mathcal{C}$ is a class of abelian groups, then we call a nilpotent group, G, $\mathcal{C}$-nilpotent if there exists a central series expressing G as a nilpotent group, such that each quotient $\frac{G_i}{G_{i-1}}$ is in $\mathcal{C}$. We have:

Lemma 4.2.1: *Let $\mathcal{C}$ be a class of abelian groups which is closed under taking kernels and cokernels of abelian group homomorphisms between members of $\mathcal{C}$. Let $f : G \to H$ be a group homomorphism between $\mathcal{C}$-nilpotent groups. Then, the kernel, K, and, if the image is normal, the cokernel, Q, of f are $\mathcal{C}$-nilpotent. Moreover, if $\{G_i\}$ expresses G as a $\mathcal{C}$-nilpotent group, then $\{G_i \cap K\}$ expresses K as a $\mathcal{C}$-nilpotent group. Similarly, if the image is normal, and $\{H_i\}$ expresses H as a $\mathcal{C}$-nilpotent group, then $\{\pi(H_i)\}$ expresses Q as a $\mathcal{C}$-nilpotent group.*

Proof. Let $\{G_i\}$ and $\{H_i\}$ express G and H as $\mathcal{C}$-nilpotent groups. We can reindex these central series so that they have the same length, $f(G_i) \subset H_i$, and, for any i, either $\frac{G_i}{G_{i-1}}$ or $\frac{H_i}{H_{i-1}}$ is 0. The result now follows directly from the spectral sequence discussed above. □

A word of caution is required regarding the hypotheses of Lemma 4.2.1. Namely, if R is a ring, we cannot, in general, take $\mathcal{C}$ to be a category of R-modules, even if we require that each $f : \frac{G_i}{G_{i-1}} \to \frac{H_i}{H_{i-1}}$ is an R-module homomorphism. This is because, even though the differentials d^0 are R-module homomorphisms, there is no guarantee that the differentials d^1 are R-module homomorphisms, as the following example shows:

Example 4.2.2: Suppose that $\{G_i\}_{i=0}^2$ and $\{H_i\}_{i=0}^2$ are central series of length 2 representing G and H as nilpotent groups, and that we have a commutative diagram of group homomorphisms:

$$\begin{tikzcd}
1 \arrow[r] & G_1 \arrow[r] \arrow[d, "f_1"] & G_2 \arrow[r] \arrow[d, "f"] & G_2/G_1 \arrow[r] \arrow[d, "f_2"] & 1 \\
1 \arrow[r] & H_1 \arrow[r] & H_2 \arrow[r] & H_2/H_1 \arrow[r] & 1
\end{tikzcd}$$

Then a summand of the differential d^1 can be identified with the connecting homomorphism $\partial : ker(f_2) \to coker(f_1)$ induced by the Snake Lemma, assuming that all images are normal. Taking $\mathcal{C} = \mathbb{C}$-modules, let $\psi : \mathbb{C} \to \mathbb{C}$ denote complex conjugation, and consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{C} \arrow[r, "\psi"] \arrow[d, "1"] & \mathbb{C} \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \mathbb{C} \arrow[r, "1"'] & \mathbb{C} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Then the connecting homomorphism $\partial : \mathbb{C} \to \mathbb{C}$ can be identified with complex conjugation which is not a map of $\mathbb{C}$-modules, even though $f_1 = f_2 = 0$ are. Note that we could swap the position of ψ with the vertical identity map to produce a counterexample.

If $\mathcal{C} = \widehat{\mathbb{Z}}_p$-modules, we can replace complex conjugation by the identity map $\widehat{\mathbb{Z}}_p \otimes \widehat{\mathbb{Z}}_p \to \widehat{\mathbb{Z}}_p \otimes \widehat{\mathbb{Z}}_p$, viewing the domain as a $\widehat{\mathbb{Z}}_p$-module via multiplication in the left factor, and the codomain as a $\widehat{\mathbb{Z}}_p$-module via multiplication in the right factor. Then $\psi = 1$ is not a map of $\widehat{\mathbb{Z}}_p$-modules, since $\widehat{\mathbb{Z}}_p$ is not a solid ring, [BK72b, Definition 2.1]. To see this, note that including $\widehat{\mathbb{Z}}_p \otimes \widehat{\mathbb{Z}}_p$ into $\widehat{\mathbb{Q}}_p \otimes \widehat{\mathbb{Q}}_p$ shows that $r \in \widehat{\mathbb{Z}}_p$ is in the core $c(\widehat{\mathbb{Z}}_p) := \{r \in \widehat{\mathbb{Z}}_p \mid r \otimes 1 = 1 \otimes r \in \widehat{\mathbb{Z}}_p \otimes \widehat{\mathbb{Z}}_p\}$ iff $mr \in \mathbb{Z}$ for some non-zero integer m. This shows that $c(\widehat{\mathbb{Z}}_p) = \mathbb{Z}_p$.

Therefore, [MP12, Lemma 4.3.4] and preceding discussion are incorrect as stated, where it was claimed that the kernel and, if the image is normal, the cokernel of a '$\widehat{\mathbb{Z}}_p$-map' between '$\widehat{\mathbb{Z}}_p$-nilpotent groups' are $\widehat{\mathbb{Z}}_p$-nilpotent. For a counterexample, consider the map $\widehat{\mathbb{Z}}_p \xrightarrow{1\otimes\widehat{\phi}_p - \widehat{\phi}_p\otimes 1} \widehat{\mathbb{Z}}_p \otimes \widehat{\mathbb{Z}}_p$, which can be viewed as a $\widehat{\mathbb{Z}}_p$-map with $f_1 = f_2 = 0$ as in Example 4.2.2. Its kernel is $c(\widehat{\mathbb{Z}}_p) = \mathbb{Z}_p$, which is not $\widehat{\mathbb{Z}}_p$-nilpotent, since no non-trivial subgroup admits a $\widehat{\mathbb{Z}}_p$-module structure. This is because no non-trivial maps $\widehat{\mathbb{Z}}_p \to \widehat{\mathbb{Z}}_p$ factor through the core $\mathbb{Z}_p$.

We note, however, that there are alternative definitions of a $\widehat{\mathbb{Z}}_p$-nilpotent group in the literature, such as [War76, Definition 10.4], which is a generalisation of the notion of a $\widehat{\mathbb{Z}}_p$-module. For our purposes, we have the following result:

Lemma 4.2.3: *The following classes of abelian groups are closed under kernels and cokernels of abelian group homomorphisms between members of the class:*

i) the class of R-modules, where R is a solid ring (such as $\mathbb{Z}[T^{-1}]$ or $\mathbb{F}_p$, see [BK72b, Definition 2.1]),
ii) the class, $\mathcal{B}_T$, of T-complete abelian groups,
iii) the class of $f\widehat{\mathbb{Z}}_T$-modules (that is, the class of finitely generated $\widehat{\mathbb{Z}}_T$-modules),
iv) the class of T-complete abelian groups A, such that, for every $p \in T$, $\widehat{A}_p$ is an $f\widehat{\mathbb{Z}}_p$-module.

Proof. i) This is the case originally dealt with in [BK72a, Ch. III, Lemma 5.8]. The key points are that, if R is solid, then an abelian group has at most one R-module structure, and any homomorphism of abelian groups between R-modules is an R-module homomorphism, [BK72b, 2.4].
ii) An abelian group is T-complete iff $\mathrm{Hom}(\mathbb{Z}[T^{-1}], A) = \mathrm{Ext}(\mathbb{Z}[T^{-1}], A) = 0$, [MP12, Prop. 10.1.18], and it follows from this that the image and, therefore, the kernel and cokernel of an abelian group homomorphism between T-complete abelian groups are T-complete.
iii) The universal property of T-completion implies that any T-complete abelian group has a $\widehat{\mathbb{Z}}_T$-module structure, and that this $\widehat{\mathbb{Z}}_T$-module structure is unique. The universal property also implies that an abelian group homomorphism between T-complete abelian groups is a $\widehat{\mathbb{Z}}_T$-module homomorphism - in fact, that it is a product of $\widehat{\mathbb{Z}}_p$-module homomorphisms between the individual p-completions. Now, an $f\widehat{\mathbb{Z}}_T$-module is equivalent to a $q \in \mathbb{N}$, and a product of $f\widehat{\mathbb{Z}}_p$-modules which can each be generated by less than q elements.

Therefore, since each $\widehat{\mathbb{Z}}_p$ is a PID, the kernel and cokernel of a homomorphism between $f\widehat{\mathbb{Z}}_T$-modules are $f\widehat{\mathbb{Z}}_T$-modules.
iv) Similarly to iii), this follows from the fact that each $\widehat{\mathbb{Z}}_p$ is a PID, and the fact that a homomorphism of abelian groups between T-complete abelian groups is a product of $\widehat{\mathbb{Z}}_p$-module maps between the individual p-completions. □

Remark 4.2.4: If R is a ring with the property that any abelian group homomorphism between R-modules is an R-module homomorphism, then R is solid. To see this, consider the two R-module structures on $R \otimes_{\mathbb{Z}} R$ given by multiplication on the left and right factors, and the identity map as the homomorphism of abelian groups. The fact that the identity is an R-module homomorphism implies that $r \otimes_{\mathbb{Z}} 1 = 1 \otimes_{\mathbb{Z}} r$ for all $r \in R$.

Recall that a nilpotent group is T-complete iff it is $\mathcal{B}_T$-nilpotent, [MP12, Lemma 10.4.1]. We briefly sketch how the argument goes. If G is $\mathcal{B}_T$-nilpotent, then it is T-complete due to the construction of a Postnikov tower for $K(G,1)$ from a $\mathcal{B}_T$-central series for G, as well as the fact that such towers are T-complete via co-HELP, [MP12, Theorem 3.3.7]. Conversely, if G is T-complete, then we can inductively T-complete any Postnikov tower for $K(G,1)$. The fact that the abelian homotopy groups of the building blocks $\widehat{K(B,2)}_T$ are T-complete is the starting point for an inductive proof that G is $\mathcal{B}_T$-nilpotent, using Lemma 4.2.3ii) and the closure of $\mathcal{B}_T$-nilpotent groups under central extensions, [MP12, Lemma 3.1.3]. Therefore, we have:

Corollary 4.2.5: *The kernel and, if the image is normal, the cokernel of a homomorphism between T-complete nilpotent groups are T-complete.*

Before moving on to the proof of our main result, we record some further consequences of Lemmas 4.2.1 and 4.2.3 to the theory of T-complete nilpotent groups:

Lemma 4.2.6: *Let G be a T-complete nilpotent group, and H a T-complete subgroup. Then:*

i) there is a subnormal series $H = H_0 \leq H_1 \leq ... \leq H_k = G$, where each H_i is T-complete,
ii) if G is $f\widehat{\mathbb{Z}}_T$-nilpotent, then so is H,
iii) if T is a finite set of primes and G is $f\widehat{\mathbb{Z}}_T$-nilpotent, then G satisfies the ascending chain condition (ACC) for T-complete subgroups,
iv) if G is a T-torsion $f\widehat{\mathbb{Z}}_T$-nilpotent group, then G is finite.

Proof. i) We will induct on the nilpotency class of G, noting that the result is trivial if G is abelian. Let $e = G_0 \leq ... \leq G_q = G$ represent G as a $\mathcal{B}_T$-nilpotent group. Let:

$$\frac{H}{H \cap G_1} = K_0 \leq K_1 \leq ... \leq K_k = \frac{G}{G_1}$$

be a subnormal series as is guaranteed to exist by the inductive hypothesis. Note, for example, that $H \cap G_1$ is T-complete since it is the kernel of $H \to \frac{G}{G_1}$. Let $\pi : G \to \frac{G}{G_1}$ denote the quotient, and define $H_{i+1} = \pi^{-1}(K_i)$.

Then, each H_i is T-complete and H_i is a normal subgroup of H_{i+1}. Moreover, H is a normal subgroup of H_1, since G_1 is central in G.

ii) If $\{G_i\}$ represents G as an $f\widehat{\mathbb{Z}}_T$-nilpotent group, then $\{H \cap G_i\}$ represents H as an $f\widehat{\mathbb{Z}}_T$-nilpotent group, since a T-complete submodule of an $f\widehat{\mathbb{Z}}_T$-module is an $f\widehat{\mathbb{Z}}_T$-module, by the discussion in the proof of Lemma 4.2.3 iii). Here, the fact that each $H \cap G_i$ is T-complete follows from i).

iii) When G is abelian, this follows from the fact that $\widehat{\mathbb{Z}}_T$ is Noetherian when T is a finite set of primes. In general, we can induct on the length q of a central series, $\{G_i\}_{i=0}^q$, expressing G as an $f\widehat{\mathbb{Z}}_T$-nilpotent group. If $\{H_i\}_{i=0}^\infty$ is an ascending chain of T-complete subgroups of G, then $\{H_i \cap G_1\}$ and $\{\pi(H_i)\}$ are ascending chains of T-complete subgroups of G_1 and $\frac{G}{G_1}$, respectively, where $\pi : G \to \frac{G}{G_1}$ is the quotient map. These chains both terminate by the inductive hypothesis, and this implies that $\{H_i\}$ also terminates, as desired.

iv) Since each $\widehat{G}_p$ is $\{p\}$-local, and $G = \prod_{p \in T} \widehat{G}_p$, we must have $\widehat{G}_p = 1$ for all but finitely many primes, in order for G to be T-torsion. Therefore, we can reduce to the case where T is a finite set of primes. When G is abelian, the ACC implies that there is a product of primes in T, r, such that $rg = 0$ for all $g \in G$. Tensoring $\frac{\mathbb{Z}}{r\mathbb{Z}}$ with a suitable $\widehat{\mathbb{Z}}_T$-free resolution of G, we conclude that G is finite. The general case then follows by induction. $\square$

4.2.2 Action of fundamental groups on fibres

If $f : X \to Y$ is a map between well-pointed spaces, then Ff is well-pointed and we can define an action of $\pi_1(X)$ on $\pi_n(Ff)$ in the usual way. For more general maps f, we can define an action of $\pi_1(X)$ on $\pi_n(Ff)$ by using a Reedy cofibrant approximation. For our proof of Theorem 4.2.9, we would like to understand how the action on fibres behaves in a composite of fibrations:

Lemma 4.2.7: *Consider a triangle of fibrations with well-pointed fibres:*

$$\begin{tikzcd} X \arrow[d, two heads, "f"'] \arrow[dr, two heads, "h"] & \\ Y \arrow[r, two heads, "g"'] & Z \end{tikzcd}$$

Let F_1, F_2, F_3 denote the fibres of f, h and g respectively and note that F_1 is the fibre of $F_2 \twoheadrightarrow F_3$. Let $\gamma \in \pi_1(X)$. Then there is a commutative square:

$$\begin{tikzcd} F_2 \arrow[r] \arrow[d, "\gamma_2"'] & F_3 \arrow[d, "\gamma_3"] \\ F_2 \arrow[r] & F_3 \end{tikzcd}$$

such that γ_2, γ_3 represent the actions of γ and $f_*(\gamma)$ respectively, and the induced map, $\gamma_1 : F_1 \to F_1$, between fibres represents the action of γ.

Proof. Let γ also denote a loop in X representing γ. We first define γ_3 in the usual way by constructing a homotopy $H : F_3 \times I \to Y$. We then define γ_2 by constructing a homotopy G as in the following lifting problem:

$$\begin{tikzcd}
F_2 \times \{0\} \cup * \times I \arrow[r, "\iota \cup \gamma"] \arrow[d] & X \arrow[d, "f"] \\
F_2 \times I \arrow[r, "H \circ (k \times 1)"'] \arrow[ur, dashed, "G"] & Y
\end{tikzcd}$$

It can then be checked that the composite $F_1 \times I \to F_2 \times I \to X$, which defines γ_1, satisfies the required properties to show that γ_1 represents γ. $\square$

To figure out what this means for the actions of fundamental groups on homotopy fibres of an arbitrary composition of maps, we consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
A \arrow[r, "i"] \arrow[u] \arrow[d] & B \arrow[r, "j"] \arrow[u] \arrow[d] & C \arrow[u] \arrow[d] \\
A \times C^{I_+} \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
A \times B^{I_+} \times C^{I_+} \arrow[r, two heads] & B \times C^{I_+} \arrow[r, two heads] & C
\end{tikzcd}$$

where all vertical maps are weak equivalence, the map between the top two rows is a Reedy cofibrant approximation and the remaining maps between rows are the canonical ones. The bottom row gives a composite of fibrations which satisfies the conditions of the Lemma 4.2.7. Using the diagram, we can conclude that $\pi_1(Y)$ acts nilpotently on $H_n(Ff)$ iff $\pi_1(B \times C^{I_+})$ acts nilpotently on $H_n(F_1)$. Similar conclusions hold for Fg and Fh, where $h = gf$. We also have, with notation as in Lemma 4.2.7, that $\pi_1(F_2)$ acts trivially on $\pi_n(F_1)$ iff the induced map $Fh \to Fg$ induces a trivial action on its homotopy fibre, which is equivalent to Ff.

4.2.3 Proof of the main theorem

The cocellular construction of the completion of a nilpotent space X can be modified in the following way. First, when replacing X by a Postnikov tower, we can modify the construction to ensure that the coattaching maps are cofibrations, as in Theorem 2.3.12. Then, to construct the completion, we inductively use commutative squares of the form:

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & \widehat{X}_i \arrow[d] \\ K(A,n) \arrow[r] & \widehat{K(A,n)} \end{tikzcd}$$

and the map $X_{i+1} \to \widehat{X}_{i+1}$ is then defined as the canonical map between homotopy fibres. Our strategy of proof will use the refinement of the Zeeman comparison theorem due to Hilton and Roitberg, [HR76], and to use this we will need:

Lemma 4.2.8: *If $f : X \to Y$ is a map of connected spaces such that Ff is connected and $\pi_1(X)$ acts nilpotently on $\pi_*(Ff)$, then $\pi_1(Y)$ acts nilpotently on $H_*(Ff)$.*

Proof. This is [Hil76, Corollary 2.2], but we will also give a direct proof using the tools we already have available. Firstly, we can assume that X and Y are well-pointed and have the homotopy type of a CW complex. Using Theorem 2.3.11, we can replace f by a relative Postnikov tower for f. Since Ff is connected, inspection of the proof shows that we can assume all cocells have coattaching maps of the form, $X_i \to K(A,n)$ with $n \geq 2$. We will inductively prove that $\pi_1(Y)$ acts nilpotently on the homology of the fibres, F_i, of the maps $d : X_i \to X_0 = Y$. This can be shown by applying Lemma 4.2.7, and subsequent discussion, to the composites:

$$\begin{tikzcd} X_{i+1} \arrow[d] \arrow[dr] & \\ X_i \arrow[r] & Y \end{tikzcd}$$

Here, we assume that $\pi_1(Y)$ acts nilpotently on $H_*(F_i)$, and we know that $\pi_1(F_i)$ acts trivially on the homology of the fibre of $F_{i+1} \twoheadrightarrow F_i$, since this is a principal fibration, and $\pi_1(F_{i+1}) \to \pi_1(F_i)$ is surjective. Therefore, an application of the Serre spectral sequence shows that $\pi_1(Y)$ acts nilpotently on $H_*(F_{i+1})$. $\square$

In particular, the lemma holds if $f : X \to Y$ is a map of nilpotent spaces inducing a surjection on fundamental groups.

We can now begin the proof of our main theorem, starting with the following special case:

Theorem 4.2.9: *Let $f : X \to A$ and $g : Y \to A$ be maps between connected nilpotent spaces such that $N(f,g)$ is connected. If we have a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & A \arrow[d] & Y \arrow[l, "g"'] \arrow[d] \\ \widehat{X}_T \arrow[r, "\widehat{f}_T"'] & \widehat{A}_T & \widehat{Y}_T \arrow[l, "\widehat{g}_T"] \end{tikzcd}$$

such that the vertical maps are T-completions, then the induced map $N(f,g) \to N(\widehat{f}_T, \widehat{g}_T)$ is a T-completion.

Proof. Via a straightforward diagram chase, we can reduce to the case where A is the limit of a Postnikov tower with coattaching maps, $A_i \to K(B,m)$ $(m \geq 2)$, which are cofibrations, and the T-completion $A \to \widehat{A}_T$ is as described at the beginning of Subsection 4.2.3. We define a filtration of $P := N(f,g)$ by pullbacks P_i:

$$\begin{tikzcd} P_i \arrow[r] \arrow[d] & (A \times A) \times_{A_i \times A_i} A_i^{I_+} \arrow[d] \\ X \times Y \arrow[r] & A \times A \end{tikzcd}$$

and let Q_i denote the pullbacks in the corresponding filtration of $Q := N(\widehat{f}_T, \widehat{g}_T)$. Observe that we have maps $P_{i+1} \to P_i$ which fit into a larger diagram of pullbacks as shown below:

$$\begin{tikzcd} P_{i+1} \arrow[r] \arrow[d] & A^2 \times_{A_{i+1}^2} A_{i+1}^{I_+} \arrow[r] \arrow[d] & A_{i+1}^{I_+} \arrow[r] \arrow[d] & PK(B,m)^{I_+} \arrow[d] \\ P_i \arrow[r] \arrow[d] & A^2 \times_{A_i^2} A_i^{I_+} \arrow[r] \arrow[d] & A_{i+1}^2 \times_{A_i^2} A_i^{I_+} \arrow[r] & (PK(B,m))^2 \times_{K(B,m)^2} K(B,m)^{I_+} \\ X \times Y \arrow[r] & A^2 & & \end{tikzcd}$$

Here, the inclusion $I \wedge \{0,1\}_+ \cup \{0\}_+ \wedge I_+ \to I \wedge I_+$, where I has basepoint 1, is homeomorphic to the inclusion $S^1 \to D^2 \cong I \wedge S^1$, and so the right hand vertical map is homeomorphic to $P\Omega K(B,m) \to \Omega K(B,m)$.

We have maps $P_i \to Q_i$ and we start by inductively showing that they are $\mathbb{F}_T$-equivalences. Since $P_0 = X \times Y$, the base case holds by assumption. Assume that $P_i \to Q_i$ is T-completion. Comparing the diagram above with the corresponding diagram for Q, we have a commutative square:

$$\begin{tikzcd} P_i \arrow[r] \arrow[d] & Q_i \arrow[d] \\ \Omega K(B,m) \arrow[r] & \Omega \widehat{K(B,m)}_T \end{tikzcd}$$

and the induced map between homotopy fibres is the map $P_{i+1} \to Q_{i+1}$.

The Zeeman comparison theorem, Theorem 2.5.1, or more specifically a refinement due to Hilton and Roitberg [HR76], can now be used either because $m > 2$ and so the base space is simply connected, or because $m = 2$ and Lemma 4.2.8 implies that B and $\mathbb{E}_T B$ act nilpotently on the homology of the respective fibres. For the left hand fibration, this follows since P connected implies that P_i is connected and $\pi_1(P_i) \to B$ is surjective. This is because if P_i were disconnected, then consideration of the fibre sequence $P_{i+1} \to P_i \to \Omega K(B, m)$ shows that P_{i+1} would also be disconnected, and similarly for $P_{i+2}, ..., P$. For the right hand fibration, we are assuming that $P_i \to Q_i$ is T-completion. Therefore, $\pi_1(Q_i) \to \mathbb{E}_T B$ is surjective, since $\mathbb{E}_T$ is right exact.

Therefore, we can inductively show that $P \to Q$ is an $\mathbb{F}_T$-equivalence. We can also use this filtration of Q to show that the homotopy groups of Q are T-complete, using Lemma 4.2.3, as well as closure of T-complete nilpotent groups under extensions, [MP12, Corollary 10.4.5], and so Q is T-complete, by [MP12, Theorem 11.1.1], and $P \to Q$ is T-completion as desired. □

4.2.4 Fracture Theorem as a Consequence

In order to deduce the fracture theorem as a corollary of our results, we first need to show that the homotopy pullback in question is connected. This is the content of the following lemma:

Lemma 4.2.10: *If G is a nilpotent group, then the function $\varphi : \mathbb{E}_T G \times G_0 \to (\mathbb{E}_T G)_0$, defined via composition with $(g, h) \to gh^{-1}$, is surjective.*

Proof. We first assume that G is abelian. Let J denote the image of φ, and P denote the kernel of φ, which is just the evident pullback. We will first show that J is rational. Since J is a subgroup of $(\mathbb{E}_T G)_0$, we have $\mathbb{H}_T J = 0$. Therefore, we have a short exact sequence:

$$0 \to \mathbb{E}_T P \to \mathbb{E}_T G \to \mathbb{E}_T J \to 0$$

The universal property of the pullback implies that the first map is split surjective. It follows that J is T-local and $\mathbb{E}_T J = \mathbb{H}_T J = 0$. Therefore, J is rational by [MP12, Proposition 10.4.7 iii) and Proposition 10.1.11]. Now φ_0 is surjective and factors through J, so $J = (\mathbb{E}_T G)_0$ and φ is surjective.

The result for general nilpotent groups G can now be proven inductively on the nilpotency class of G, using [MP12, Lemma 7.6.1]. A key point is that the image of $\mathbb{E}_T Z(G)$ in $\mathbb{E}_T G$ is a central subgroup – this can be seen from the Postnikov tower construction of completion applied to the upper central series of G. □

Finally, we give the proof of the fracture theorem that we have been building toward:

Theorem 4.2.11: *Let X be a T-local nilpotent space. Then any commutative square:*

$$\begin{tikzcd} X \arrow[r, "\widehat{\phi}"] \arrow[d, "\psi"'] & \widehat{X}_T \arrow[d, "\phi"] \\ X_0 \arrow[r, "\widehat{\phi}_0"'] & (\widehat{X}_T)_0 \end{tikzcd}$$

with $\widehat{\phi}$ a T-completion and ψ, ϕ rationalisations, is a homotopy fibre square.

Proof. We can assume that ϕ is a fibration. Then, by Lemma 4.2.10, the pullback P is connected and we have a comparison map $f : X \to P$. By Theorem 4.2.9, applying functorial rationalisation and completion shows that f_0 and $\widehat{f}_T$ are weak equivalences, respectively. It follows that $\tilde{H}_*(Cf;\mathbb{Q}) = \tilde{H}_*(Cf;\bigoplus_{p\in T}\mathbb{F}_p) = 0$. Therefore, $\tilde{H}_*(Cf)$ is local away from T and has trivial rationalisation. Moreover, $\tilde{H}_*(Cf)$ is T-local, since X and P are. Therefore, $\tilde{H}_*(Cf) = 0$ and f is a homology isomorphism between connected nilpotent spaces, so must be a weak equivalence. $\square$

Chapter 5

Localisations and completions of nilpotent G-spaces

Abstract: We develop the theory of nilpotent G-spaces and their localisations, for G a compact Lie group, via reduction to the non-equivariant case using Bousfield localisation. One point of interest in the equivariant setting is that we can choose to localise or complete at different sets of primes at different fixed point spaces – and the theory works out just as well provided that you invert more primes at $K \leq G$ than at $H \leq G$, whenever K is subconjugate to H in G. We also develop the theory in an unbased context, allowing us to extend the theory to G-spaces which are not G-connected.

5.1 Introduction

The purpose of this chapter is to develop the theory of localisations and completions of nilpotent G-spaces at sets of primes, where G is a compact Lie group. The main reference for the equivariant theory is [M$^+$96, Ch. II], which itself is a summary of the older papers [MMT82] and [May82], where it was explained how the foundations of the theory could be developed using the same arguments as in the non-equivariant setting, with some additional complications when G is compact Lie rather than just finite. Our approach is slightly different, in that we use the theory of Bousfield localisation to deduce the foundations of the theory from the non-equivariant case. This approach leads to fewer difficulties in the compact Lie case, and allows us to use a more general definition of a nilpotent G-space than in [M$^+$96], see Definition 5.3.1. For example, we prove that a nilpotent G-space X is p-complete iff all homotopy groups of the form $\pi_i(X^H)$ are p-complete. This fact was proved in [May82, Theorem 2], but only under the assumption that, for fixed i, the nilpotency classes of $\pi_i(X^H)$, as H varies, have a common bound.

Another contribution of this chapter is that we allow the set of primes we are localising or completing at to vary over the orbit category of G, and we show that this provides no extra difficulties provided that you 'invert more primes' at $K \leq G$ than at $H \leq G$, whenever K is subconjugate to H in G - we call this property the *poset condition*. For example, we could localise at p at one subgroup and complete at p at another, where, loosely speaking, completing at p 'inverts more primes' than localising at p. One might ask, why consider these localisations? In the non-equivariant setting, Bousfield proved in [Bou74, Theorem 1.1] that all localisations at connective homology theories are equivalent to localisations with respect to either $H(-;\mathbb{Z}_T)$ or $H(-;\oplus_{p\in T}\mathbb{F}_p)$ for some set of primes T. Therefore, in this chapter we are considering localisations at pointwise connective homology theories, where pointwise means we choose a connective homology theory for every closed subgroup H of G, and the localisations which satisfy the poset condition are precisely those with the property that a G-space is local iff it is pointwise local.

We develop the theory in both a based and unbased context - with different parts of the theory working better in each setting. For example, we derive some new fracture theorems for nilpotent G-spaces in Theorems 5.3.7 and 5.3.12, relate localisations of nilpotent G-spaces to equivariant Postnikov towers, and show that our homological approach to the theory is equivalent to the classical cohomological appraoch of [MMT82] and [May82], all in the based context. We use the unbased theory to extend our results on nilpotent G-spaces to G-spaces whose fixed point spaces are disjoint unions of nilpotent spaces. This is especially pertinent in the equivariant setting, since there are many examples of G-spaces which are non-equivariantly connected, but which have disconnected fixed point spaces, or no possible choice of a G-fixed basepoint at all.

5.1.1 Notations and Prerequisites

We will work with the model categories of G-spaces and based G-spaces, where G is a compact Lie group, basepoints are G-fixed, and the model structures are the Quillen or q-model structures, [MM02, Theorem 1.8]. All subgroups of G are assumed to be closed. Unless otherwise stated, we build G-CW complexes out of the maps $(\frac{G}{H})_+ \wedge S^n_+ \to (\frac{G}{H})_+ \wedge D^n_+$ in the based context, rather than using based maps out of $(\frac{G}{H})_+ \wedge S^n$. The notation $[A, B]$ denotes homotopy classes of maps, which may be based/unbased/equivariant depending on the context.

This chapter should be accessible to any reader who is familiar with the non-equivariant theory of nilpotent spaces and their localisations, as well as the basics of equivariant homotopy theory.

5.2 Localisation systems

5.2.1 Bousfield localisation at the T-equivalences

In this subsection, we define localisation systems, $\mathbf{T}$, as well as the notion of a $\mathbf{T}$-equivalence between based G-spaces. We develop the basic properties of the $\mathbf{T}$-equivalences, and then use the Bousfield cardinality argument to show that there exists a model structure on the category of based G-spaces, where a map is a weak equivalence iff it is a $\mathbf{T}$-equivalence. We develop the basic properties of this model structure, including Theorem 5.2.15 below, which is the key to deducing the equivariant theory of nilpotent G-spaces from the non-equivariant theory.

Let $\mathbf{P}$ denote the poset of subsets of the set of prime numbers partially ordered by inclusion, and let $\mathcal{O}$ denote the orbit category of a compact Lie group G. Let $\mathbb{Z}_T$ denote the integers localised at T, where recall that localising at T inverts the primes not in T. We begin with the following four important definitions, which are followed by a more informal discussion of the definitions in the case $G = C_2$, in Example 5.2.5.

Definition 5.2.1: *A localisation system is a functor $\mathbf{T} : \mathcal{O}^{op} \to \mathbf{P}^{op} \times \mathbf{1}$, where we denote by $\mathbf{1}$ the category with objects 0 and 1 and a single arrow from 0 to 1.*

We think of $\mathbf{T}([{}^{G}/{}_{H}])$ as a set of primes with coefficient, where the coefficient is either 0 or 1. If we drop the bold font on the $\mathbf{T}$, then $T([{}^{G}/{}_{H}])$ denotes only the underlying set of primes of $\mathbf{T}([{}^{G}/{}_{H}])$. Recall that a map of spaces is called a $\mathbb{Z}_T$-equivalence if it induces an isomorphism on homology with coefficients in $\mathbb{Z}_T$. Similarly, a map is called an $\mathbb{F}_T$-equivalence if it induces an isomorphism on homology with coefficients in $\mathbb{F}_p$, for every $p \in T$. When the basepoints are nondegenerate, it is equivalent to define these equivalences using the respective reduced homology theories instead.

Definition 5.2.2: *Let $\mathbf{T}$ be a set of primes with coefficient and $f : X \to Y$ a map of spaces. If the coefficient is 0, then we call f a $\mathbf{T}$-equivalence if it is a $\mathbb{Z}_T$-equivalence. If the coefficient is 1, then we call f a $\mathbf{T}$-equivalence if it is an $\mathbb{F}_T$-equivalence.*

Intuitively, a coefficient of 0 means we are localising at T, and a coefficient of 1 means we are completing at T. In a similar vein, we have:

Definition 5.2.3: *Let $\mathbf{T}$ be a set of primes with coefficient and let X be a space. If the coefficient is 0, we say that X is $\mathbf{T}$-local if it is T-local after forgetting the coefficient. If the coefficient is 1, we say that X is $\mathbf{T}$-local if it is T-complete after forgetting the coefficient.*

We can now make the following definition:

Definition 5.2.4: *Let $\mathbf{T}$ be a localisation system and $f : X \to Y$ be a map of based G-spaces. We say that f is a $\mathbf{T}$-equivalence if for all $H \leq G$, $f^H : X^H \to Y^H$ is a $\mathbf{T}([{}^{G}/{}_{H}])$-equivalence.*

Example 5.2.5: In this example, we let $G = C_2$, and make some additional comments that might shed further light on the above definitions. For any G-space, X, we have a map $X^G \to X^e$, and the definition of a localisation system is chosen so that localisation at the $\mathbf{T}$-equivalences 'inverts more primes' at $[{}^{G}/{}_{e}]$, which corresponds to X^e, than at $[{}^{G}/{}_{G}]$, which corresponds to X^G. Roughly speaking, one effect of this is that if X is $\mathbf{T}$-local and nilpotent, then the induced map on homotopy groups $\pi_i(X^G) \to \pi_i(X^e)$ will, generally, be a map from a less local group to a more local group, which is the usual direction of map. The reverse direction is often degenerate, for example the only map from $\mathbb{Z}[p^{-1}, q^{-1}] \to \mathbb{Z}[p^{-1}]$ is the zero map, and the only map from $\widehat{\mathbb{Z}}_p \to \mathbb{Z}_p$ is the zero map, where recall that we view p-completion as 'inverting more primes' than p-localisation. In total, we have the following three types of localisation system:

$\mathbf{T}([G/G])$	Effect of $\mathbf{T}$-localisation on X^G	$\mathbf{T}([G/e])$	Effect of $\mathbf{T}$-localisation on X^e
$(T,0)$	T-localisation	$(S,0), S \subset T$	S-localisation
$(T,0)$	T-localisation	$(S,1), S \subset T$	S-completion
$(T,1)$	T-completion	$(S,1), S \subset T$	S-completion

Our next task is to Bousfield localise the category of G-spaces with respect to the class of $\mathbf{T}$-equivalences, where $\mathbf{T}$ is a localisation system. However, to aid clarity at certain points of this section, we will consider, at first, the more general situation of Bousfield localisation with respect to the class of $\mathbf{A}$-equivalences, as defined below, where $\mathbf{A}$ is any function from the set of objects, $[G/H]$, of the orbit category to abelian groups.

Definition 5.2.6: *Let $\mathbf{A}$ be a function from the set of objects of the orbit category of G, or equivalently from the set of subgroups of G, to abelian groups. A map of G-spaces, $f : X \to Y$, is said to be an $\mathbf{A}$-equivalence if, for all subgroups H of G, f^H induces an isomorphism on homology with coefficients in $\mathbf{A}([G/H])$.*

Note that, if $\mathbf{T}$ is a localisation system, then a $\mathbf{T}$-equivalence is equivalent to an $\mathbf{A}$-equivalence, for some function $\mathbf{A}$ taking values in abelian groups of the form $\mathbb{Z}_T$ or $\oplus_{p\in T}\mathbb{F}_p$.

We will need the following minimal list of properties of the $\mathbf{A}$-equivalences, where a property is *pointwise* if it holds for all fixed point spaces:

Lemma 5.2.7: *i) The class of $\mathbf{A}$-equivalences is closed under retracts, satisfies 2-out-of-3, and every weak equivalence is an $\mathbf{A}$-equivalence,*
ii) the pushout of an $\mathbf{A}$-equivalence that is a pointwise h-cofibration is an $\mathbf{A}$-equivalence,
iii) the colimit of a transfinite sequence of $\mathbf{A}$-equivalences which are closed inclusions is an $\mathbf{A}$-equivalence.

Proof. i) is easy. For ii), since taking fixed points preserves pushouts along closed inclusions, we can work pointwise and replace the spaces with nondegenerately based ones. The result then follows from consideration of cofibre sequences. For iii), taking fixed points preserves transfinite colimits of closed inclusions, [MM02, Lemma 1.6], and so the result follows from the fact that homology preserves these colimits. □

We can use the Bousfield-Smith cardinality argument on the $\mathbf{A}$-equivalences. The argument is essentially the same as the classical case of localising spaces with respect to homology theories, which is treated in [MP12, Section 19.3]. The key lemma is as follows, where all cell complexes are G-cell complexes:

Lemma 5.2.8: *There exists a cardinal κ with the following property: if $i : A \to B$ is the inclusion of a subcomplex into a cell complex B which is also an $\mathbf{A}$-equivalence, then, for any cell e of B, there is a subcomplex C of size $< \kappa$ containing e such that $A \cap C \to C$ is an $\mathbf{A}$-equivalence.*

Proof. Choose a regular cardinal, $\kappa > \aleph_0$, with the following properties:

i) every cell of any cell complex is contained within a subcomplex of size $< \kappa$,
ii) if Z is a cell complex of size $< \kappa$, then $\bigoplus_{*,H} H_*(Z^H; \mathbf{A}([G/H]))$ has cardinality $< \kappa$,
iii) if W is any cell complex, then for any $*$ and H:

$$H_*(W^H; \mathbf{A}([G/H])) = \mathrm{colim}_{<\kappa} H_*(Z^H; \mathbf{A}([G/H]))$$

where the colimit is over all subcomplexes of W of size $< \kappa$.

To start the proof, choose a subcomplex C_0 of B of size $< \kappa$ which contains e, so we have a map $C_0 \cap A \to C_0$. By some $c \in H_*(C_0; \mathbf{A})$, we mean an element of $H_n(C_0^H; \mathbf{A}([G/H]))$ for some H and n. For each $c \in H_*(C_0; \mathbf{A})$, its image in $H_*(B; \mathbf{A})$ is the image of an element, a, in the homology of a $< \kappa$ dimensional subcomplex of A, D. Moreover, there is a $< \kappa$ dimensional subcomplex E of B, containing C_0 and D, such that the images of a and c in $H_*(E; \mathbf{A})$ are equal. Define C_0^1 by adding such a subcomplex E to C_0 for every $c \in H_*(C_0; \mathbf{A})$ - the conditions i) - iii) above ensure that C_0^1 has size $< \kappa$. So every element of $H_*(C_0^1; \mathbf{A})$ which is in the image of $H_*(C_0; \mathbf{A})$ is also in the image of $H_*(C_0^1 \cap A; \mathbf{A})$. Now if $k \in H_*(C_0 \cap A; \mathbf{A})$ is sent to 0 in $H_*(C_0; \mathbf{A})$, it is also sent to 0 in $H_*(B; \mathbf{A})$ and $H_*(A; \mathbf{A})$, so there is a $< \kappa$ dimensional subcomplex of A, L, containing $C_0 \cap A$, such that the image of k in $H_*(L; \mathbf{A})$ is 0. Define C_1 by adding such a subcomplex L to C_0^1, for every $k \in H_*(C_0 \cap A; \mathbf{A})$ which is sent to 0 in $H_*(C_0; \mathbf{A})$. It follows that if $k \in H_*(C_0 \cap A; \mathbf{A})$ is sent to 0 in $H_*(C_0; \mathbf{A})$, it is also sent to 0 in $H_*(C_1 \cap A; \mathbf{A})$. Moreover, every element of $H_*(C_1; \mathbf{A})$ which is in the image of $H_*(C_0; \mathbf{A})$ is also in the image of $H_*(C_1 \cap A; \mathbf{A})$. We can repeat this process to form $e \in C_0 \subset C_1 \subset C_2 \subset ...$, and we let C be the union of the C_i which still has size $< \kappa$. Since homology preserves these sequential colimits, it follows that $H_*(C \cap A; \mathbf{A}) \to H_*(C; \mathbf{A})$ is an isomorphism, as desired. □

We now deduce the standard consequences of Lemma 5.2.8. Firstly, using transfinite induction, we have:

Corollary 5.2.9: *A map has the RLP with respect to all inclusions of cell complexes which are* $\mathbf{A}$*-equivalences iff it has the RLP with respect to all inclusions of cell complexes of dimension* $< \kappa$ *which are* $\mathbf{A}$*-equivalences.*

Proof. See [Hir03, Proposition 4.5.6]. □

Any map with the RLP with respect to inclusions of cell complexes that are $\mathbf{A}$-equivalences is a q-fibration, since the generating acyclic cofibrations $(\frac{G}{H})_+ \wedge (D^n)_+ \to (\frac{G}{H})_+ \wedge (D^n \times I)_+$ are inclusions of subcomplexes. Therefore, using left properness we have:

Lemma 5.2.10: *A map has the RLP with respect to all q-cofibrations which are* $\mathbf{A}$*-equivalences iff it has the RLP property with respect to all inclusions of cell complexes that are* $\mathbf{A}$*-equivalences.*

Proof. See [Hir03, Proposition 13.2.1]. □

If we call such a map an **A**-fibration, then we see that an **A**-fibration that is an **A**-equivalence is a q-acyclic q-fibration by the retract argument. Using this and the small object argument we can now conclude:

Theorem 5.2.11: *There is a left proper model structure on the category of based G-spaces where the weak equivalences are the* **A***-equivalences, the cofibrations are the q-cofibrations and the fibrations are the* **A***-fibrations.*

This model structure is monoidal:

Lemma 5.2.12: *If $i : A \to B$ and $j : C \to D$ are cofibrations, then $i \square j : A \wedge D \cup B \wedge C \to B \wedge D$ is a cofibration which is an* **A***-equivalence if either i or j is an* **A***-equivalence.*

Proof. The fact that $i \square j$ is a cofibration is classical and is a consequence of the fact that $\frac{G}{H} \times \frac{G}{K}$ is G-homeomorphic to a G-CW complex. Similarly, since $(\frac{G}{H})^K$ is homeomorphic to a CW-complex, by [Ill83, Corollary 7.2] and [Bre72, Ch. VI, Corollary 2.5], we have that a cofibration is a pointwise cofibration. Therefore, for the remaining statement concerning **A**-equivalences we can assume that G is the trivial group. Note also that the cofibre of $i \square j$ is homotopy equivalent to $\frac{B}{A} \wedge \frac{D}{C}$. Suppose that j is an **A**-equivalence and p is an **A**-fibration. Then $i \square j$ has the left lifting property with respect to p iff i has the left lifting property with respect to $p^{\square j}$. Therefore, it suffices to show that $i \square j$ is an **A**-equivalence in the case where $i : (\frac{G}{H})_+ \wedge (S^{n-1})_+ \to (\frac{G}{H})_+ \wedge (D^n)_+$ and j is an inclusion of a subcomplex which is an **A**-equivalence. Since we are assuming that G is trivial, from the third sentence of this proof, the cofibre of $i \square j$ is homotopy equivalent to $S^n \wedge \frac{D}{C}$, which has vanishing reduced homology with the required coefficients as desired. $\square$

We have the following characterisation of the fibrant objects:

Lemma 5.2.13: *A based G-space Z is* **A***-local (that is fibrant in the model structure of Theorem 5.2.11) iff for all* **A***-equivalences $f : A \to B$ between cofibrant objects, the map $f^* : [B, Z] \to [A, Z]$ is a bijection.*

Proof. If for all **A**-equivalences $f : A \to B$ between cofibrant objects the map $[B, Z] \to [A, Z]$ is a bijection, then it is easy to show that $Z \to *$ has the right lifting property with respect to any inclusion of cell complexes that is an **A**-equivalence, using the fact that inclusions of cell complexes are h-cofibrations, and so Z is **A**-local by Lemma 5.2.10. On the other hand, if Z is **A**-local, we can assume that f is a cofibration. Considering lifts of $Z \to *$ with respect to f shows that f^* is surjective, and considering lifts with respect to $f \square i$, where i is the inclusion $\{0, 1\}_+ \to I_+$, shows that f^* is injective. $\square$

From now on, we return to the context of Bousfield localisation at the **T**-equivalences, where **T** is a localisation system. In this context, by a **T**-fibration etc., we mean an **A**-fibration, as above, for the function **A** defined by **T**. The next lemma is the key to deducing our results on nilpotent G-spaces from the non-equivariant theory, and is also the first to make use of the definition of a localisation system:

Lemma 5.2.14: *Let $\mathbf{T}$ be a localisation system. If a based G-space Z is $\mathbf{T}$-local, then Z^H is $\mathbf{T}([{}^{G}/_{H}])$-local for every $H \leq G$.*

Proof. Let $f : A \to B$ be a $\mathbf{T}([{}^{G}/_{H}])$-equivalence between cofibrant spaces. Note that A and B are just spaces, not G-spaces. Let $g = 1 \wedge f : ({}^{G}/_{H})_+ \wedge A \to ({}^{G}/_{H})_+ \wedge B$. We have $({}^{G}/_{H})_+^K = \mathcal{O}([{}^{G}/_{K}], [{}^{G}/_{H}])_+$, and it follows that g is a $\mathbf{T}$-equivalence by Lemma 5.2.12, the fact that $\mathbf{T}$ is a localisation system and the following observations:

i) if $S \subset T$, then a $\mathbb{Z}_T$-equivalence is a $\mathbb{Z}_S$-equivalence,
ii) if $S \subset T$, then an $\mathbb{F}_T$-equivalence is an $\mathbb{F}_S$-equivalence,
iii) a $\mathbb{Z}_T$-equivalence is an $\mathbb{F}_T$-equivalence.

It follows that $g^* : [({}^{G}/_{H})_+ \wedge B, Z] \to [({}^{G}/_{H})_+ \wedge A, Z]$ is a bijection. This is equivalent to $[B, Z^H] \to [A, Z^H]$ being a bijection, and it follows that Z^H is $\mathbf{T}([{}^{G}/_{H}])$-local. □

Using Lemma 5.2.14, we can deduce:

Theorem 5.2.15: *i) A based G-space Z is $\mathbf{T}$-local iff Z^H is $\mathbf{T}([{}^{G}/_{H}])$-local for every $H \leq G$,*
ii) A map of based G-spaces $X \to Y$ is a $\mathbf{T}$-localisation iff $X^H \to Y^H$ is a $\mathbf{T}([{}^{G}/_{H}])$-localisation for every $H \leq G$.

Proof. i) For the direction we haven't already proved, let Z be a based G-space such that Z^H is $\mathbf{T}([{}^{G}/_{H}])$-local for every $H \leq G$. Consider a $\mathbf{T}$-localisation $Z \to W$. Then, each map $Z^H \to W^H$ is a $\mathbf{T}([{}^{G}/_{H}])$-equivalence between $\mathbf{T}([{}^{G}/_{H}])$-local objects, and so a weak equivalence as desired.
ii) This follows from i). □

To end this subsection, we quickly give a counterexample to indicate what can happen if $\mathbf{T}$ is not a localisation system. Let $G = C_2$, and define a $\mathbf{T}$-equivalence to be a map of based G-spaces, $f : X \to Y$, such that $H_*(f^e; \mathbb{Z}[p^{-1}])$ and $H_*(f^G; \mathbb{Z}[p^{-1}, q^{-1}])$ are isomorphisms, where p and q are distinct primes. If $\mathbf{T}$-local G-spaces were always pointwise local, then the analogue of Theorem 5.2.15 would also have to hold. Let $\underline{\mathbb{Z}[p^{-1}]}$ denote the constant coefficient system and consider the map $K(\underline{\mathbb{Z}[p^{-1}]}, 1) \to K(\underline{\mathbb{Z}[p^{-1}]}, 1)_{\mathbf{T}}$. The induced map between systems of homotopy groups would result in a commutative triangle:

$$\begin{tikzcd}
\mathbb{Z}[p^{-1}] \arrow[d, "\phi"'] \arrow[dr, "1"] & \\
\mathbb{Z}[p^{-1}, q^{-1}] \arrow[r] & \mathbb{Z}[p^{-1}]
\end{tikzcd}$$

where ϕ denotes localisation, which is a contradiction since the bottom map has to be the zero map. Therefore, a $\mathbf{T}$-local space is not necessarily pointwise local.

5.2.2 Unbased T-localisations

The theory described in Subsection 5.2.1 goes through essentially unchanged in the unbased context. Recall that we are using the same notation for based and unbased homotopy classes of maps. We have:

Theorem 5.2.16: *Let $\mathbf{T}$ be a localisation system. There is a left proper, monoidal model structure on the category of G-spaces where the weak equivalences are the $\mathbf{T}$-equivalences, the cofibrations are the q-cofibrations and the fibrations are the $\mathbf{T}$-fibrations (which are defined as in Lemma 5.2.10). A G-space Z is $\mathbf{T}$-local (that is fibrant in this model structure) iff for all $\mathbf{T}$-equivalences $f : A \to B$ between cofibrant objects, the map $f^* : [B, Z] \to [A, Z]$ is a bijection.*

Proof. The existence of the left proper model structure follows from the Bousfield cardinality argument, as in Subsection 5.2.1. If $i : A \to B$ is a cofibration and $f : X \to Y$ is a cofibration which is a $\mathbf{T}$-equivalence, then $A \times Y \cup B \times X \to B \times Y$ is a cofibration as in Lemma 5.2.12 and it will be a $\mathbf{T}$-equivalence if $(A \times Y \cup B \times X)_+ \to (B \times Y)_+$ is a $\mathbf{T}$-equivalence. The latter map can be identified with $i_+ \Box f_+$, which is a $\mathbf{T}$-equivalence by Lemma 5.2.12. The characterisation of the fibrant objects now follows as in Lemma 5.2.13. $\Box$

Since $\mathbf{T}$ is a localisation system, the arguments of Lemma 5.2.14 and Theorem 5.2.15 show:

Theorem 5.2.17: *i) A G-space Z is $\mathbf{T}$-local iff Z^H is $\mathbf{T}([{}^{G}/{}_{H}])$-local for every $H \leq G$,*
ii) A map of G-spaces $X \to Y$ is a $\mathbf{T}$-localisation iff $X^H \to Y^H$ is a $\mathbf{T}([{}^{G}/{}_{H}])$-localisation for every $H \leq G$.

At this point, it is helpful to compare based and unbased localisations in the non-equivariant setting. In this setting, a localisation system, $\mathbf{T}$, is equivalent to a set of primes with coefficient. We have:

Lemma 5.2.18: *Let Z be an unbased space. Then:*
i) Z is $\mathbf{T}$-local iff $f^* : [B, Z] \to [A, Z]$ is a bijection for all $\mathbf{T}$-equivalences, $f : A \to B$, between connected cofibrant spaces,
ii) if $Z = \sqcup_{i \in I} Z_i$, then Z is $\mathbf{T}$-local iff Z_i is $\mathbf{T}$-local for every i. In particular, a map of spaces which induces a bijection on connected components is a $\mathbf{T}$-localisation iff each component is a $\mathbf{T}$-localisation.

Proof. If $f : A \to B$ is a $\mathbf{T}$-equivalence between cofibrant spaces, then f induces a bijection between the connected components of A and B, so f is a disjoint union of $\mathbf{T}$-equivalences $A_i \to B_i$, for i in the set of connected components of A. Now, $[\sqcup A_i, Z] = \prod_i [A_i, Z]$, and $i)$ follows. For $ii)$, if A is connected we have $[A, \sqcup_i Z_i] = \sqcup_i [A, Z_i]$, and so $ii)$ follows from $i)$. $\Box$

Lemma 5.2.19: *Let $f : X \to Y$ be a map of unbased spaces, with X non-empty. Then the following are equivalent:*

i) f is an unbased $\mathbf{T}$-localisation,
ii) f is a based $\mathbf{T}$-localisation for some $x \in X$,

iii) f is a based $\mathbf{T}$-localisation for all $x \in X$,

iv) f_+ is a based $\mathbf{T}$-localisation, with respect to the adjoined basepoint $+$.

Proof. The key point is that if Z is a $\mathbf{T}$-local based space, then it is also $\mathbf{T}$-local as an unbased space. This is a consequence of the fact that unbased homotopy classes $[A, Z]$ are equivalent to based homotopy classes $[A_+, Z]$, and the observation that if $A \to B$ is a $\mathbf{T}$-equivalence between cofibrant unbased spaces, then $A_+ \to B_+$ is a $\mathbf{T}$-equivalence between cofibrant based spaces. Now, $iii) \implies ii)$ is trivial, and $ii) \implies i)$ follows from the above. For $i) \implies iii)$, let $x \in X$. Since $\mathbf{T}$-localisations are preserved by composing with weak equivalences, we can assume that X is a CW-complex and f is a cofibration. Let $f_{\mathbf{T}} : X \to X_{\mathbf{T}}$ be a based $\mathbf{T}$-localisation. Then $f_{\mathbf{T}}$ is also an unbased $\mathbf{T}$-localisation, since $ii) \implies i)$. Therefore, there is a weak equivalence, g, such that $gf = f_{\mathbf{T}}$, and so f is also a based $\mathbf{T}$-localisation, as desired. The fact that $iv) \implies i)$ follows from $ii) \implies i)$ and Lemma 5.2.18ii), and $i) \implies iv)$ follows from Lemma 5.2.18ii) and $i) \implies iii)$. □

Returning to the equivariant setting, we have the following consequence:

Theorem 5.2.20: *Let $\mathbf{T}$ be a localisation system.*

i) if $f : X \to Y$ is a based $\mathbf{T}$-localisation, then it is also an unbased $\mathbf{T}$-localisation,

ii) if $f : X \to Y$ is a map of unbased G-spaces, then f is a $\mathbf{T}$-localisation iff f_+ is a based $\mathbf{T}$-localisation. Moreover, if X^G is non-empty, then f is a $\mathbf{T}$-localisation iff f is a based $\mathbf{T}$-localisation with respect to any G-fixed basepoint iff f is a based $\mathbf{T}$-localisation with respect to all G-fixed basepoints.

5.2.3 An algebraic analogue

Before moving on to the theory of nilpotent G-spaces, we record the following result, which can be viewed as an algebraic analogue of the above theory. Recall that coefficient systems are functors $h\mathcal{O}^{op} \to \mathbf{Ab}$, and there are free coefficient systems defined by:

Definition 5.2.21: *The free coefficient system associated to the object $[{}^G/_H]$ is defined by $\mathbf{F}_{[{}^G/_H]}([{}^G/_K]) = \bigoplus_{h\mathcal{O}([{}^G/_K],[{}^G/_H])} \mathbb{Z}$ along with the evident definition on morphisms.*

The free coefficient systems have the property that $Hom_{[h\mathcal{O}^{op},\mathbf{Ab}]}(A \otimes \mathbf{F}_{[{}^G/_H]}, \mathbf{L}) \cong Hom_{\mathbf{Ab}}(A, \mathbf{L}([{}^G/_H]))$, where A is any abelian group.

Lemma 5.2.22: *Let $\mathbf{T}$ be a localisation system and let $\mathbf{A}$ and $\mathbf{B}$ be coefficient systems such that:*

i) if the coefficient of $\mathbf{T}([{}^G/_H])$ is 0, then $\mathbf{A}([{}^G/_H]) \otimes \mathbb{Z}_{T([{}^G/_H])} = 0$ and $\mathbf{B}([{}^G/_H])$ is $T([{}^G/_H])$-local,

ii) if the coefficient of $\mathbf{T}([{}^G/_H])$ is 1, then $\mathbf{A}([{}^G/_H])$ is a $\mathbb{Z}[T([{}^G/_H])^{-1}]$-module and $\mathbf{B}([{}^G/_H])$ is $T([{}^G/_H])$-complete.

Then $Ext^i_{[h\mathcal{O}^{op},\mathbf{Ab}]}(\mathbf{A}, \mathbf{B}) = 0$ for all $i \geq 0$.

Proof. We first claim that if $\mathbf{T}([{}^{G}/_{H}])$ has coefficient 0, and n is a product of primes not in $T([{}^{G}/_{H}])$, then $Ext^i_{[h\mathcal{O}^{op},\mathbf{Ab}]}(\mathbf{F}_{[G/H]} \otimes {}^{\mathbb{Z}}/_{n\mathbb{Z}}, \mathbf{B}) = 0$ for all $i \geq 0$. The category $[h\mathcal{O}^{op}, \mathbf{Ab}]$ has enough injectives, [Wei94, Exercise 2.3.7], so we can calculate this by taking an injective resolution $\{\mathbf{Q}_i\}$ of $\mathbf{B}$. Such a resolution is, in particular, an objectwise injective resolution of $\mathbf{B}([{}^{G}/_{H}])$, and $Hom_{[h\mathcal{O}^{op},\mathbf{Ab}]}(\mathbf{F}_{[G/H]} \otimes {}^{\mathbb{Z}}/_{n\mathbb{Z}}, \mathbf{Q}_i) = Hom_{\mathbf{Ab}}({}^{\mathbb{Z}}/_{n\mathbb{Z}}, \mathbf{Q}_i([{}^{G}/_{H}]))$, so taking homology calculates $Ext^i({}^{\mathbb{Z}}/_{n\mathbb{Z}}, \mathbf{B}([{}^{G}/_{H}]))$, which vanishes by the non-equivariant case. Similarly, if $\mathbf{T}([{}^{G}/_{H}])$ has coefficient 1, then $Ext^i_{[h\mathcal{O}^{op},\mathbf{Ab}]}(\mathbf{F}_{[G/H]} \otimes \mathbb{Z}[\mathbf{T}([{}^{G}/_{H}])^{-1}], \mathbf{B}) = 0$ by [MP12, 10.1.22].

We will use this to define a $Hom_{[h\mathcal{O}^{op},\mathbf{Ab}]}(-, \mathbf{B})$-acyclic resolution, $\{\mathbf{P}_i\}$, of $\mathbf{A}$. If $\mathbf{T}([{}^{G}/_{H}])$ has coefficient 0, there is a coproduct, $\mathbf{K}_{[G/H]}$, of functors of the form $\mathbf{F}_{[G/H]} \otimes {}^{\mathbb{Z}}/_{n\mathbb{Z}}$, with n being a product of primes not in $T([{}^{G}/_{H}])$, such that there is a natural transformation $\mathbf{K}_{[G/H]} \to \mathbf{A}$ which is a surjection at $[{}^{G}/_{H}]$. If $\mathbf{T}([{}^{G}/_{H}])$ has coefficient 1, then there is a coproduct, $\mathbf{K}_{[G/H]}$, of functors of the form $\mathbf{F}_{[G/H]} \otimes \mathbb{Z}[\mathbf{T}([{}^{G}/_{H}])^{-1}]$, such that there is a natural transformation $\mathbf{K}_{[G/H]} \to \mathbf{A}$ which is a surjection at $[{}^{G}/_{H}]$. We define $\mathbf{P}_0 := \bigoplus_{[G/H]} \mathbf{K}_{[G/H]}$, so we have a surjection $\mathbf{P}_0 \to \mathbf{A}$, and $\mathbf{P}_0$ is $Hom_{[h\mathcal{O}^{op},\mathbf{Ab}]}(-, \mathbf{B})$-acyclic by the previous paragraph.

The key point now is that the functor $\mathbf{P}_0$ satisfies the conditions in i) and ii) that $\mathbf{A}$ does, and this follows from the fact that $\mathbf{T}$ is a localisation system. In more detail, $\mathbf{F}_{[G/H]}([{}^{G}/_{K}])$ is only non-zero when there is a map $[{}^{G}/_{K}] \to [{}^{G}/_{H}]$ in $\mathcal{O}$, and then we have the following observations:

i) if $S \subset T$, then a torsion group with no T-torsion is also a torsion group with no S-torsion,

ii) if $S \subset T$, then a $\mathbb{Z}[T^{-1}]$-module is a $\mathbb{Z}[S^{-1}]$-module,

iii) a torsion group with no T-torsion is a $\mathbb{Z}[T^{-1}]$-module.

Therefore, we can inductively construct a $Hom_{[h\mathcal{O}^{op},\mathbf{Ab}]}(-, \mathbf{B})$-acyclic resolution $\{\mathbf{P}_i\}$ of $\mathbf{A}$, since the kernel of $\mathbf{P}_0 \to \mathbf{A}$ also satisfies i) and ii) in the statement of the lemma. Using the first paragraph of the proof, we can use this acyclic resolution to compute $Ext^i_{[h\mathcal{O}^{op},\mathbf{Ab}]}(\mathbf{A}, \mathbf{B}) = 0$ for all $i \geq 0$, as desired. □

5.3 Nilpotent G-spaces

5.3.1 The main theorems

We now move on to the theory of nilpotent G-spaces and we begin with the definition of a nilpotent G-space. This differs from the definition given in [M$^+$96, Ch. II] in that we do not require a common bound on the nilpotency classes at each fixed point space. To understand this, we will show in Subsection 5.3.4 that any nilpotent G-space can be approximated by a weak Postnikov tower, but if we assume a common bound on the nilpotency classes, then a nilpotent G-space can be approximated by a (strict) Postnikov tower, a distinction which becomes important when using co-HELP to deduce theorems about nilpotent spaces, as in [MP12, Section 3.3].

Definition 5.3.1: *A based G-space X is said to be nilpotent if X^H is a nilpotent space for all subgroups H*

of G.

In the unbased context, we have the following definition:

Definition 5.3.2: *An unbased G-space X is said to be componentwise nilpotent if for every subgroup H of G, every component of X^H is a nilpotent space.*

In general, if we speak about componentwise nilpotent G-spaces we are working in an unbased context, and if we speak about nilpotent G-spaces we are working in a based context.

By reduction to fixed point spaces, we can immediately deduce one of the most important properties of localisations of componentwise nilpotent G-spaces:

Theorem 5.3.3: *Let $\mathbf{T}$ be a localisation system where all the coefficients are 0. Let $f : X \to Y$ be a map from a componentwise nilpotent G-space X to a $\mathbf{T}$-local unbased G-space Y, such that for every $H \leq G$, f^H induces a bijection on connected components. Then, the following are equivalent:*

i) f is a $\mathbf{T}$-localisation,
ii) for all $H \leq G$, $* \geq 1$, and $b \in X^H$, $f^H_* : \pi_*(X^H, b) \to \pi_*(Y^H, f^H(b))$ is a $\mathbf{T}([{}^G/_H])$-localisation of nilpotent groups,
iii) for all $H \leq G$ and $* \geq 1$, $f^H_* : H_*(X^H) \to H_*(Y^H)$ is a direct sum of $\mathbf{T}([{}^G/_H])$-localisations, where the sum ranges over the connected components of X^H.

Proof. This follows from [MP12, Theorem 6.1.2], as well as Lemma 5.2.18ii). □

Recall that if T is a set of primes and A is an abelian group, then $\mathbb{E}_T A$ and $\mathbb{H}_T A$ denote the zeroth and first derived functors of T-completion applied to A, respectively. These functors can be extended to take nilpotent groups as input by using the homotopy groups of completions of Eilenberg-MacLane spaces. In the current context, we use the above definition of $\mathbb{E}_{\mathbf{T}}G$ and $\mathbb{H}_{\mathbf{T}}G$ for sets of primes with coefficient 1. If, instead, $\mathbf{T}$ is a set of primes with coefficient 0, and G is a nilpotent group, we define $\mathbb{E}_{\mathbf{T}}G = G_T$ and $\mathbb{H}_{\mathbf{T}}G = 0$. This corresponds to using the homotopy groups of localisations of Eilenberg-MacLane spaces. A system of nilpotent groups, $\underline{G}$, is a continuous functor from $\mathcal{O}^{op}$ to the category of nilpotent groups, and we call such a system $\mathbf{T}$-local if it is pointwise $\mathbf{T}([{}^G/_H])$-local. The $\mathbf{T}$-localisation $K(\underline{G}, 1) \to K(\underline{G}, 1)_{\mathbf{T}}$ specifies a homomorphism $\underline{G} \to \mathbb{E}_{\mathbf{T}}(\underline{G})$ and the, up to homotopy, universal property of $\mathbf{T}$-localisation implies the following universal property:

Lemma 5.3.4: *Let $\underline{G}$ and $\underline{H}$ be systems of nilpotent groups, with $\underline{H}$ $\mathbf{T}$-local. Then any homomorphism $f : \underline{G} \to \underline{H}$ factors uniquely through the $\mathbf{T}$-localisation $\underline{G} \to \mathbb{E}_{\mathbf{T}}\underline{G}$.*

Proof. This follows from the fact that $[K(\underline{G}, 1)_{\mathbf{T}}, K(\underline{H}, 1)] \cong [K(\underline{G}, 1), K(\underline{H}, 1)]$. □

If X is a nilpotent G-space, then this universal property defines a map from $\mathbb{E}_{\mathbf{T}}\underline{\pi}_i(X) \to \underline{\pi}_i(X_{\mathbf{T}})$ and we have the following theorem:

Theorem 5.3.5: *If X is a nilpotent G-space, then there is a natural short exact sequence:*

$$1 \to \mathbb{E}_{\mathbf{T}}\underline{\pi}_i(X) \to \underline{\pi}_i(X_{\mathbf{T}}) \to \mathbb{H}_{\mathbf{T}}\underline{\pi}_{i-1}(X) \to 1$$

If $f : X \to Y$ is a map between componentwise nilpotent G-spaces such that each f^H induces a bijection on connected components, and $\mathbb{H}_{\mathbf{T}([G/H])}(\pi_i(X^H, x)) = 0$ for all $H \leq G, i \geq 1$ and $x \in X^H$, then the following are equivalent:

i) f is a $\mathbf{T}$-localisation,

ii) for all $i \geq 1$, $H \leq G$ and $x \in X^H$, $\pi_i(X^H, x) \to \pi_i(Y^H, f^H(x))$ is a $\mathbf{T}([G/H])$-localisation.

For example, the hypothesis holds if, for all H, $X^H_{\mathbf{T}([G/H])}$ is a disjoint union of f$\mathbb{Z}_{T([G/H])}$-nilpotent spaces.

Proof. This follows from [MP12, Theorem 11.1.2, Proposition 10.1.23], as well as Lemma 5.2.18. □

Non-equivariantly, the fact that $Ext(\mathbb{H}_T B, \mathbb{E}_T A) = 0$, [MP12, Corollary 10.4.9], implies that the short exact sequence of Theorem 5.3.5 splits, however, equivariantly the sequence does not necessarily split as the following example shows. Take $G = C_2$. Then, consideration of Elmendorf's theorem, [Elm83, Theorem 1], shows that to find a counterexample to the splitting, we can use the following counterexample to the naturality of the splitting in the non-equivariant case. For this, we let $X = K(\frac{\mathbb{Z}[p^{-1}]}{\mathbb{Z}}, 1)$, so that $\widehat{X}_p = K(\widehat{\mathbb{Z}}_p, 2)$, and a map $X \to K(\widehat{\mathbb{Z}}_p, 2)$ is equivalent to a homomorphism $\widehat{\mathbb{Z}}_p \to \widehat{\mathbb{Z}}_p$. Then, any non-zero homomorphism, such as the identity, suffices to show that the splitting cannot be natural.

5.3.2 T-localisation and fibre squares

From now on, we work in a based context. In this subsection, we discuss how $\mathbf{T}$-localisation interacts with fibre sequences and homotopy pullbacks. Let $N(f, g)$ denote the double mapping path space of f and g. We have:

Theorem 5.3.6: *Let $f : X \to Z$ and $g : Y \to Z$ be maps of nilpotent G-spaces such that $N(f, g)$ is G-connected. If we have a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \arrow[d] & Y \arrow[l, "g"'] \arrow[d] \\ X' \arrow[r, "f'"'] & Z' & Y' \arrow[l, "g'"] \end{tikzcd}$$

such that the vertical maps are $\mathbf{T}$-localisations, then the induced map $N(f, g) \to N(f', g')$ is a $\mathbf{T}$-localisation.

Proof. This follows from Theorem 4.2.9. □

It follows that any functorial **T**-localisation, such as one obtained via the small object argument applied to the model structure of Theorem 5.2.11, preserves homotopy fibre squares of nilpotent G-spaces. The special case where $Y = Y' = *$ results in the connected fibre lemma.

5.3.3 Fracture Theorems

In this subsection, we move on to derive fracture squares associated to nilpotent G-spaces, a topic which was not discussed in the classical sources for the equivariant theory. In the non-equivariant setting, we have fracture squares relating to localisation and completion, [MP12, Theorem 8.1.3, Theorem 13.1.4], and we would like to generalise these results to the equivariant setting, perhaps localising and completing at different sets of primes at each fixed point space. For example, the following two squares are homotopy fibre squares associated to a T-local nilpotent space X, where T is a set of primes containing 7:

$$\begin{array}{ccc} X & \longrightarrow & \widehat{X}_T \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & (\widehat{X}_T)_0 \end{array} \tag{5.1}$$

$$\begin{array}{ccc} X & \longrightarrow & (\prod_{p\in T\setminus\{7\}} X_p) \times \widehat{X}_7 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & (\prod_{p\in T\setminus\{7\}} X_p)_0 \times (\widehat{X}_7)_0 \end{array} \tag{5.2}$$

In 5.2, we complete at 7 to illustrate to point that there are an abundance of fracture squares that we can ask for, especially in the equivariant case. With this in mind, the following theorem subsumes all of the examples that we are aware of:

Theorem 5.3.7: *Consider a commutative square of nilpotent G-spaces:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ A & \xrightarrow[g]{} & B \end{array}$$

such that, for each subgroup H of G, there are sets of primes T_H, S_H satisfying:

i) X^H, Y^H are T_H-local and A^H, B^H are S_H-local,

ii) f^H is an $\mathbb{F}_{T_H}$-equivalence and g^H is an $\mathbb{F}_{S_H}$-equivalence, *iii)* ϕ^H, ψ^H *are* $\mathbb{Q}$*-equivalences.*

Then the square is a homotopy fibre square.

Proof. Since taking fixed points detects homotopy fibre squares, we can reduce the theorem to a pointwise statement with sets of primes T and S. In this case, the theorem follows from (5.1) and successive applications of the pasting lemma for homotopy pullbacks, [Hir03, Proposition 13.3.15]. □

When $\mathbf{T}$ is a constant localisation system, we can derive, as in [MP12, Theorem 13.1.1], a fracture square for homotopy classes $[K, X]$, under certain finiteness hypotheses on K and X. In particular, K will always be a finite *based* G-CW complex, by which we mean that K is built by starting with a G-fixed basepoint and attaching finitely many cells along based maps out of G-spaces of the form $(\frac{G}{H})_+ \wedge S^n$, with $n \geq 0$. In order to give what we feel is the cleanest exposition of our main result, and the corresponding counterexample when $\mathbf{T}$ is not a constant localisation system, we begin by recalling some preliminaries on homotopy pullbacks.

Definition 5.3.8: *Let $f : K \to X$ be a map of G-spaces. We define $[K \wedge (I^n)_+, f]$ to be the set of homotopy classes of maps $K \wedge (I^n)_+ \to X$ relative to the boundary $K \wedge (\partial I^n)_+$, where at each point on the boundary ∂I^n, the induced map is equal to f.*

Lemma 5.3.9: *Let $A \xrightarrow{i} B \xrightarrow{j} Ci$ be a cofibre sequence of G-spaces, and $f : Ci \to X$ a map of G-spaces. Then there is a natural long exact sequence of groups:*

$$... \to [B \wedge (I^2)_+, fj] \to [\Sigma^2 A, X] \to [Ci \wedge I_+, f] \to [B \wedge I_+, fj] \to [\Sigma A, X]$$

Moreover, the image of $[\Sigma^2 A, X]$ in $[Ci \wedge I_+, f]$ is central.

Proof. Modify f so that it is radially constant in a neighbourhood of the boundary of the cone. Consider the sequence of based maps:

$$... \to \Omega Map(A, X) \xrightarrow{\partial} Map(Ci, X) \xrightarrow{j^*} Map(B, X) \xrightarrow{i^*} Map(A, X)$$

where the spaces are given basepoints f and the constant loop to f. The fact that f is radially constant in a neighbourhood of the boundary of the cone allows us to define a based map, which is also a weak equivalence, $Fi^* \to Map(Ci, X)$. The map ∂ is then induced by a comparison of the fibre sequences associated to j^* and $Fi^* \to Map(B, X)$. It follows that $[S^1, -]$ takes the above sequence of maps to an exact sequence of groups, since it does so for the homotopy fibre sequence induced by i^*. The fact that the image of $\pi_1(\partial)$ is central follows from [MP12, Lemma 1.4.7 v).]. □

Let $N(f,g)$ denote the double mapping path space associated to maps $f : X \to A$ and $g : Y \to A$. We will make use of the following result on homotopy classes of maps into a homotopy pullback:

Lemma 5.3.10: *Let K be a based G-CW complex. Then the natural map of pointed sets:*

$$[K, N(f,g)] \to [K,X] \times_{[K,A]} [K,Y]$$

is a surjection. Suppose that f is a fibration, so that every element of $[K,X] \times_{[K,A]} [K,Y]$ can be represented by a pair of maps $u : K \to X, v : K \to Y$ such that $fu = gv := w$. Then the preimage of (u,v) is isomorphic to the set of orbits of $[K \wedge I_+, w]$ under the right action of the group $[K \wedge I_+, u] \times [K \wedge I_+, v]$. In particular, the map is injective iff each of the functions $[K \wedge I_+, u] \times [K \wedge I_+, v] \to [K \wedge I_+, w]$ is surjective.

Proof. This follows from the same arguments as in [MP12, Proposition 2.2.2], where the result is proved in the special case when u and v are nullhomotopic. □

Next, we seek to understand how the groups $[K \wedge I_+, f]$ behave with respect to $\mathbf{T}$-localisation.

Lemma 5.3.11: *Let $\mathbf{T}$ be a constant localisation system. Let K be a finite based G-CW complex, let X be a nilpotent G-space, and let $f : K \to X$ be a map. Then:*

i) $[K \wedge I_+, f]$ is a nilpotent group, which is finitely T-generated (see [MP12, Definition 5.6.3]) if, for every $i \geq 2$ and H, $\pi_i(X^H)$ is finitely T-generated,
ii) if the coefficient of $\mathbf{T}$ is 0, then $[K \wedge I_+, f] \to [K \wedge I_+, \phi_{\mathbf{T}} f]$ is T-localisation, where $\phi_{\mathbf{T}}$ is a $\mathbf{T}$-localisation of X,
iii) if the coefficient of $\mathbf{T}$ is 1, $\mathbb{H}_{\mathbf{T}}\underline{\pi}_1(X) = 0$, and, for every $i \geq 2$ and H, $\pi_i(X^H)$ is finitely T-generated, then $[K \wedge I_+, f] \to [K \wedge I_+, \phi_{\mathbf{T}} f]$ is T-completion.

Proof. This follows by induction up the CW structure on K, using Lemma 5.3.9. In more detail, part i) follows from [MP12, Lemma 3.1.3] and the fact that a nilpotent group G is finitely T-generated iff G_T is $f\mathbb{Z}_T$-nilpotent. Part ii) follows from [MP12, Corollary 5.4.11]. Part iii) follows from [MP12, Corollary 10.4.5], and the condition that $\mathbb{H}_{\mathbf{T}}\underline{\pi}_1(X) = 0$ ensures that $\underline{\pi}_2(X) \to \underline{\pi}_2(X_{\mathbf{T}})$ is $\mathbf{T}$-localisation, by Theorem 5.3.5. □

We can now state our main fracture theorem for homotopy classes of maps:

Theorem 5.3.12: *Let $\mathbf{T}, \mathbf{S}$, and, for each i in some indexing set I, $\mathbf{T_i}$ be constant localisation systems such that $\mathbf{T}$ and $\mathbf{S}$ have coefficient 0, $T = \cup_i T_i$ and $T_i \cap T_j = S$, for all $i \neq j$. Let K be a finite based G-CW complex and let X be a T-local nilpotent G-space such that, if $\mathbf{T_i}$ has coefficient 1, then for every subgroup H of G, $\mathbb{H}_{T_i}\pi_1(X^H) = 0$, and for each $i \geq 2$, $\pi_i(X^H)$ is finitely T_i-generated. Then the following diagram is a pullback of sets:*

$$\begin{array}{ccc} [K,X] & \longrightarrow & [K,\prod_i X_{\mathbf{T_i}}] \\ \downarrow & & \downarrow \\ [K,X_{\mathbf{S}}] & \longrightarrow & [K,(\prod_i X_{\mathbf{T_i}})_{\mathbf{S}}] \end{array}$$

Proof. The map from $[K,X]$ to the pullback is surjective by Theorem 5.3.7 and Lemma 5.3.10. The map is injective by Lemma 5.3.10 and Lemma 5.3.11. In more detail, to see that the surjectivity hypothesis in the final sentence of Lemma 5.3.10 is satisfied, surjectivity tells us that we can find a map $\mu : K \to X$ projecting onto a representative (u,v) of any element of the pullback. Then we can apply Theorem 5.3.7 to give a fracture square for $K([K \wedge I_+, \mu], 1)$, and the fact that $K([K \wedge I_+, \mu], 1)$ is connected tells us that the required map is surjective, via use of Lemma 5.3.11. □

Note that the fracture theorems for nilpotent groups given in [MP12, Theorem 7.2.1 ii), Theorem 12.3.2], are both consequences of Theorem 5.3.12.

To finish this subsection, we give an example to show that the square:

$$\begin{array}{ccc} [K,X] & \longrightarrow & [K,X_{\mathbf{T}}] \\ \downarrow & & \downarrow \\ [K,X_0] & \longrightarrow & [K,(X_{\mathbf{T}})_0] \end{array}$$

need not be a pullback of sets if $\mathbf{T}$ is not a constant localisation system, where K is a finite based G-CW complex and, for every H, X^H is $f\mathbb{Z}_{T([G/H])}$-nilpotent. Of course, the comparison map $[K,X] \to [K,X_{\mathbf{T}}] \times_{[K,(X_{\mathbf{T}})_0]} [K,X_0]$ is always a surjection by Theorem 5.3.7 and Lemma 5.3.10. We let $G = C_2$, and let $\mathbf{T}([G/G]) = (\{p,q\},1)$, and $\mathbf{T}([G/e]) = (\{p\},1)$. We let $X = K(\underline{\mathbb{Z}}, n+2)_T$, where $\underline{\mathbb{Z}}$ is the constant coefficient system to $\mathbb{Z}$, and we let T denote the localisation system induced by $\mathbf{T}$ taking values with coefficient 0, so that $X^G = K(\mathbb{Z}_{\{p,q\}}, n+2)$ and $X^e = K(\mathbb{Z}_{\{p\}}, n+2)$. We let K be the cofibre:

$$\Sigma S^n \wedge (G_+) \to \Sigma S^n \wedge (e_+) \to K$$

where the first map is induced by the constant map $G \to e$. The cofibre sequence implies that there is a map of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [\Sigma K, X_{\mathbf{T}}] \oplus [\Sigma K, X_0] & \longrightarrow & \widehat{\mathbb{Z}}_{\{p,q\}} \oplus \mathbb{Q} & \longrightarrow & \widehat{\mathbb{Z}}_p \oplus \mathbb{Q} & \longrightarrow & 0 \\ & & \downarrow & & \twoheaddownarrow & & \twoheaddownarrow & & \\ 0 & \longrightarrow & [\Sigma K, (X_{\mathbf{T}})_0] & \longrightarrow & \widehat{\mathbb{Q}}_{\{p,q\}} & \longrightarrow & \widehat{\mathbb{Q}}_p & \longrightarrow & 0 \end{array}$$

It follows that the map $[\Sigma K, X_{\mathbf{T}}] \oplus [\Sigma K, X_0] \to [\Sigma K, (X_{\mathbf{T}})_0]$ can be identified with the rationalisation $\widehat{\mathbb{Z}}_q \to \widehat{\mathbb{Q}}_q$ which is not surjective. It follows that the square above is not a pullback of sets. In particular, there exist maps $f, g : K \to X$, such that $f_{\mathbf{T}} \simeq g_{\mathbf{T}}$ and $f_0 \simeq g_0$, but f and g are not homotopic.

5.3.4 Nilpotent G-spaces and Postnikov towers

We now discuss the relationship between **T**-localisations of nilpotent G-spaces and towers of principal fibrations. The arguments of this section are similar to those of [May82], except that we derive the results appropriate to non-constant localisation systems, and make a distinction between nilpotent G-spaces and bounded nilpotent G-spaces, and the types of Postnikov tower they are equivalent to. First, we will define what it means for a $\underline{\pi}$-group to be $\mathcal{B}$-nilpotent, where $\mathcal{B}$ is a class of coefficient systems. Then we will define the analogue of Postnikov towers in the equivariant setting, and we will show that a G-space, X, is equivalent to a *weak* Postnikov $\mathcal{B}$-tower iff its homotopy groups are $\mathcal{B}$-nilpotent $\underline{\pi}$-groups, where $\underline{\pi} = \underline{\pi}_1(X)$. Finally, we will show that **T**-local nilpotent G-spaces are equivalent to weak Postnikov $\mathcal{B}_{\mathbf{T}}$-towers, where $\mathcal{B}_{\mathbf{T}}$ is the class of **T**-local coefficient systems.

Definition 5.3.13: *Let $\mathcal{B}$ be a class of coefficient systems of abelian groups. Let $\underline{\pi}$ be a coefficient system of groups and let $\mathbf{G}$ be a coefficient system of groups admitting an action of $\underline{\pi}$ by automorphisms. We say that $\mathbf{G}$ is a $\mathcal{B}$-nilpotent $\underline{\pi}$-group if there is a descending sequence of normal $\underline{\pi}$-subgroups:*

$$\mathbf{G} = \mathbf{G_0} \supseteq \mathbf{G_1} \supseteq \mathbf{G_2} \supseteq ...$$

such that:

i) $\underline{\pi}$ acts trivially on $\frac{\mathbf{G_{i-1}}}{\mathbf{G_i}}$,
ii) $\frac{\mathbf{G_{i-1}}}{\mathbf{G_i}} \in \mathcal{B}$,
iii) for every H, $\frac{\mathbf{G_{i-1}}}{\mathbf{G_i}}([G/H]) \to \frac{\mathbf{G}}{\mathbf{G_i}}([G/H])$ has central image,
iv) for every H, $\frac{\mathbf{G_{i-1}}}{\mathbf{G_i}}([G/H]) = 0$ for all but finitely many i.

Definition 5.3.14: *Call a $\mathcal{B}$-nilpotent $\underline{\pi}$-group bounded if the filtration in Definition 5.3.13 can be replaced by a finite filtration terminating at* 1.

Definition 5.3.15: *We call a G-space X $\mathcal{B}$-nilpotent if it is G-connected and, for all $i \geq 1$, $\underline{\pi}_i(X)$ is a $\mathcal{B}$-nilpotent $\underline{\pi}_1(X)$-group. We say that a $\mathcal{B}$-nilpotent G-space, X, is bounded if the homotopy groups $\underline{\pi}_i(X)$ are all bounded $\mathcal{B}$-nilpotent $\underline{\pi}_1(X)$-groups.*

Note that a G-space is nilpotent iff it is $\mathcal{A}$-nilpotent, where $\mathcal{A}$ is the class of all coefficient systems of abelian groups. This follows from the fact that if X is a nilpotent space, then there are functorial filtrations of $\pi_i(X)$ satisying the conditions of the previous definition - the lower central series when $i = 1$, and the filtration induced by the augmentation ideal, $\{I^n \pi_i(X)\}$, for $i \geq 2$.

Definition 5.3.16: *A map of G-spaces, $f : X \to Y$, is called a principal $K(\underline{A}, n)$-fibration if it is the pullback* *of the path-space fibration along a map $k : Y \to K(\underline{A}, n+1)$. In particular, f is a fibration with fibre $K(\underline{A}, n)$.*

Definition 5.3.17: *Let $\mathbf{Q}$ be the totally ordered set consisting of pairs of natural numbers ordered by $(m, n) \leq (p, q)$ iff $m < p$ or $m = p$ and $n \leq q$. A weak Postnikov $\mathcal{B}$-tower is a functor $\mathbf{Q} \to G\text{–}Sp$, where $G\text{–}Sp$ is the category of G-spaces, satisfying:*

i) $X_{1,1} = *$,
ii) $X_{n+1,1} \to \lim_i X_{n,i}$ is a weak equivalence,
iii) The map $X_{n,i+1} \to X_{n,i}$ is a principal $K(\underline{B}_{n,i}, n)$-fibration for some $\underline{B}_{n,i} \in \mathcal{B}$,
iv) for every n and H, $X^H_{n,i+1} \to X^H_{n,i}$ is a weak equivalence for all but finitely many i.

Definition 5.3.18: *A Postnikov $\mathcal{B}$-tower is a weak Postnikov $\mathcal{B}$-tower such that the maps $X_{n+1,1} \to \lim_i X_{n,i}$ of condition ii) above are G-homeomorphisms.*

We have the principal fibration lemma:

Lemma 5.3.19: *Let $f : X \to Y$ be a map of well-pointed G-connected G-spaces with the homotopy type of a G-CW complex, such that $Ff \simeq K(\underline{A}, n)$ for some coefficient system $\underline{A}$ and $n \geq 1$. Then $\underline{\pi}_1(X)$ acts trivially on $\underline{\pi}_*(Ff)$ iff there is a weak equivalence $X \to Fk$ over Y, for some cofibration $k : Y \to K(\underline{A}, n+1)$.*

Proof. See Lemma 2.3.10. □

Lemma 5.3.20: *A G-space is $\mathcal{B}$-nilpotent iff it is weakly equivalent to a weak Postnikov $\mathcal{B}$-tower.*

Proof. To see that weak Postnikov $\mathcal{B}$-towers are $\mathcal{B}$-nilpotent, it suffices to show that $\underline{\pi}_n(X)$ is a $\mathcal{B}$-nilpotent $\underline{\pi}_1(X)$-group. Let $\mathbf{G}_i$ be the kernel of $\underline{\pi}_n(X) \to \underline{\pi}_n(X_{n,i+1})$. The quotients $\frac{\mathbf{G}_{i-1}}{\mathbf{G}_i}$ correspond to the coefficient systems $\underline{B}_{n,i}$ appearing in the tower, and so these are in $\mathcal{B}$ by assumption. Now, $\underline{\pi}_1(X)$ acts trivially on $\frac{\mathbf{G}_{i-1}}{\mathbf{G}_i}$ by Lemma 5.3.19, and the required inclusions are central for the same reason as in the non-equivariant case, namely [MP12, Lemma 1.4.7 v)]. Finally, $\frac{\mathbf{G}_{i-1}}{\mathbf{G}_i}([G/H]) = 0$ for all but finitely many i, since $X^H_{n,i+1} \to X^H_{n,i}$ is a weak equivalence for all but finitely many i.

Next assume that a G-CW complex, X, is $\mathcal{B}$-nilpotent. So, for each n, we have a filtration of $\underline{\pi}_n(X)$, $\{\mathbf{G}^n_i\}$, satisfying the conditions of Definition 5.3.13. We define $X^0_{n,i+1}$ by first attaching cells to X along all possible maps $(\frac{G}{H})_+ \wedge S^n \to X$ representing an element of $\mathbf{G}^n_i([G/H])$, for some $H \leq G$. Then, inductively define $X^j_{n,i+1}$, for each $j \geq 1$, by attaching a cell to $X^{j-1}_{n,i+1}$ along every possible map $(\frac{G}{H})_+ \wedge S^{n+j} \to X^j_{n,i+1}$, for any $H \leq G$. Define $X_{n,i+1}$ as the union of the $X^j_{n,i+1}$. Then:

i) $\underline{\pi}_j(X_{n,i+1}) = \underline{\pi}_j(X)$ for $j < n$,
ii) $\underline{\pi}_n(X_{n,i+1}) = \frac{\underline{\pi}_n(X)}{\mathbf{G}_i}$,
iii) $\underline{\pi}_j(X_{n,i+1}) = 0$ for $j > n$.

Moreover, we have an inclusion $X_{n,i} \to X_{m,j}$, whenever $(m, j) \leq (n, i)$ in $\mathbf{Q}$. Since the action of $\underline{\pi}_1(X)$ is trivial on each $\frac{\mathbf{G}_{i-1}}{\mathbf{G}_i}$, each of the maps $X_{n,i+1} \to X_{n,i}$ is equivalent to a $K(\underline{B}_{n,i}, n)$-principal fibration, with

$\underline{B}_{n,i} \in \mathcal{B}$. To define the weak Postnikov $\mathcal{B}$-tower, we keep each $X_{n,1}$ fixed, and inductively replace each $X_{n,i}$, for $i \geq 2$, using Lemma 5.3.19. Results of Waner, [Wan80, Corollary 4.14], imply that by doing this we never leave the category of well-pointed G-spaces with the homotopy type of a G-CW complex. Unfortunately, we could leave this category by taking inverse limits, which is why we leave each $X_{n,1}$ fixed and only require a weak equivalence in Definition 5.3.17 ii). □

If we restrict attention to bounded $\mathcal{B}$-nilpotent G-spaces, such as pointwise simply connected G-spaces, then the same proof shows:

Lemma 5.3.21: *A bounded $\mathcal{B}$-nilpotent G-space is weakly equivalent to a Postnikov $\mathcal{B}$-tower.*

As promised, we have the following definition and characterisation of $\mathbf{T}$-local nilpotent spaces:

Definition 5.3.22: *A coefficient system, $\underline{B}$, is said to be $\mathbf{T}$-local if $K(\underline{B}, 1)$ is a $\mathbf{T}$-local G-space.*

Lemma 5.3.23: *A nilpotent G-space is $\mathbf{T}$-local iff it is $\mathcal{B}_{\mathbf{T}}$-nilpotent, where $\mathcal{B}_{\mathbf{T}}$ is the class of $\mathbf{T}$-local coefficient systems. A G-space is a $\mathbf{T}$-local bounded $\mathcal{A}$-nilpotent G-space iff it is bounded $\mathcal{B}_{\mathbf{T}}$-nilpotent.*

Proof. If X is $\mathcal{B}_{\mathbf{T}}$-nilpotent it is easily verified that all of the homotopy groups of X^H are $\mathbf{T}([G/H])$-local, which implies that X is $\mathbf{T}$-local. Suppose that X is $\mathbf{T}$-local. Recall that we have a central $\underline{\pi}_1(X)$-series for $\underline{\pi}_i(X)$ induced by the functorial lower central series when $i = 1$, or the functorial augmentation ideal series when $i \geq 2$. Localising these series at $\mathbf{T}$, and considering their images in $\underline{\pi}_i(X)$, expresses each $\underline{\pi}_i(X)$ as a $\mathcal{B}_{\mathbf{T}}$-nilpotent $\underline{\pi}_1(X)$-series, and so X is $\mathcal{B}_{\mathbf{T}}$-nilpotent. If X is bounded $\mathcal{A}$-nilpotent, then the lower central series terminates after finitely many stages and so X is bounded $\mathcal{B}_{\mathbf{T}}$-nilpotent. □

5.3.5 Localisation at equivariant cohomology theories

We end this chapter by tying up the following loose end. Namely, in [M+96, Ch. II], localisations of nilpotent G-spaces were defined relative to equivariant cohomology theories, and we would like to compare this to our localisations relative to $\mathbf{T}$-equivalences.

Definition 5.3.24: *A map of G-spaces, $f : X \to Y$, is a cohomology $\mathbf{T}$-equivalence if for all $\mathbf{T}$-local coefficient systems $\underline{A}$, $f^* : H^*(Y; \underline{A}) \to H^*(X; \underline{A})$ is an isomorphism.*

We have:

Theorem 5.3.25: *A map of G-spaces, $f : X \to Y$, is a $\mathbf{T}$-equivalence iff it is a cohomology $\mathbf{T}$-equivalence.*

Proof. If f is a $\mathbf{T}$-equivalence, then, since each $K(\underline{A}, n)$ is $\mathbf{T}$-local for every $\mathbf{T}$-local coefficient system $\underline{A}$, f is a cohomology $\mathbf{T}$-equivalence.

If f is a cohomology $\mathbf{T}$-equivalence, then we can assume that X, Y are well-pointed. Now, $\Sigma^2 f : \Sigma^2 X \to \Sigma^2 Y$ is a cohomology $\mathbf{T}$-equivalence between pointwise simply connected G-spaces. By the first part, $(\Sigma^2 f)_{\mathbf{T}}$ is also a cohomology $\mathbf{T}$-equivalence between pointwise simply connected $\mathbf{T}$-local G-spaces. Simply connected $\mathbf{T}$-local G-spaces are weakly equivalent to strict Postnikov $\mathcal{B}_{\mathbf{T}}$-towers by the results of Subsection 5.3.4, so the equivariant analogue of co-HELP, [MP12, Theorem 3.3.7], implies that $(\Sigma^2 f)_{\mathbf{T}}$ is a weak equivalence and, therefore, that $\Sigma^2 f$ is a $\mathbf{T}$-equivalence. It follows that f is a $\mathbf{T}$-equivalence. $\square$

Corollary 5.3.26: *Localisation with respect to* $\mathbf{T}$*-equivalences is equivalent to localisation with respect to cohomology* $\mathbf{T}$*-equivalences.*

Chapter 6

Model Structures and Fibrations

In this chapter, which has the flavour of an appendix, we discuss results related to the closure properties of the category of well-pointed spaces with the homotopy type of a CW-complex. We begin, in Section 6.1, by reviewing the q,h, and m-model structures on the category of G-spaces, where the m-model structure is the mixed model structure of Cole, [Col06]. In Section 6.2, we record some point-set topological lemmas that will be useful in later sections. In Section 6.3, we derive the closure properties of the category of well-pointed spaces of the homotopy type of a CW-complex that were used in earlier chapters. For these spaces cohomology is represented by homotopy classes of maps into Eilenberg-MacLane spaces, which played a role in the inductive construction of Postnikov towers, Theorem 2.3.11, albeit indirectly via Lemma 2.3.10. In this section, we also showcase another short proof of Stasheff's theorem on spaces with the homotopy type of a CW-complex in fibre sequences, the original short proof being [Sch77], one direction making use of the classification of the m-cofibrant objects as precisely the spaces with the homotopy type of a CW complex. In Section 6.4, we consider the equivariant generalisation of Stasheff's thoerem, one direction of which was proved by Waner in [Wan80]. In this section, we show that our proof from Section 6.3, which makes use of the m-model structure, generalises to the equivariant setting to give a proof of the converse of Waner's theorem. On the other hand, our proof of the non-equivariant version of Waner's theorem does not generalise readily to the equivariant setting, and so we refer the reader to [Wan80] for the proof of the results on G-spaces with the G-homotopy type of a G-CW complex that we needed for our inductive construction of equivariant Postnikov towers. The issue with generalising our proof is that, non-equivariantly, the simplicial construction of a CW-approximation functor which preserves h-fibrations and finite limits depends on the theory of minimal fibrations of simplicial sets, which does not readily generalise to the equivariant setting. We end the thesis in Section 6.5 by proving Theorem 6.5.9 which, roughly speaking, states that the geometric realisation of a locally trivial map of simplicial spaces is an h-fibration. Since this theorem generalises the proof that the realisation of a minimal fibration is an h-fibration, this section can, perhaps, be viewed as a very first step toward generalising our non-equivariant proof from Section 6.3.

6.1 The q,h and m-model structures

In this section, we'll review the q,h, and m-model structures on the category of G-spaces, where G is a compact Lie group. The m-model structure, [Col06], is of interest to us, since we will use it to study spaces of the homotopy type of a G-CW complex in Sections 6.3 and 6.4 - these are precisely the spaces which are m-cofibrant. The q-model structure needs no introduction, and we'll just state it as a theorem:

Theorem 6.1.1: *There is a proper model structure on G-spaces where:*

i) the q-weak equivalences are the maps, f, such that every f^H is a weak homotopy equivalence,

ii) the q-cofibrations are the retracts of relative G-cell complexes,

iii) the q-fibrations are the maps with the RLP with respect to the q-acyclic q-cofibrations.

Proof. See [MM02, Theorem 1.8]. □

We now move on the h-model structure, where we take all homotopies, fibrations and cofibrations to be equivariant, unless otherwise stated:

Theorem 6.1.2: *There is a proper model structure on G-spaces where:*

i) the h-weak equivalences are the homotopy equivalences,

ii) the h-cofibrations are the Hurewicz cofibrations,

iii) the h-fibrations are the Hurewicz fibrations.

Here, Hurewicz cofibrations satisfy the equivariant homotopy extension property and Hurewicz fibrations satisfy the equivariant covering homotopy property, [May99, Ch. 6 and 7]. As in the non-equivariant case, the proof hinges on the fact that h-cofibrations are equivalent to G-NDR pairs, where a G-NDR pair is simply an NDR-pair (X, A), defined by some (H, λ) with both H and λ equivariant. The proofs of their basic properties are identical to the non-equivariant case, see eg [Lüc89, Lemma 1.10].

It is clear that the classes above are closed under composition and retracts, and that homotopy equivalences satisfy 2-out-of-3. The lifting axiom follows from:

Lemma 6.1.3: *Suppose that we have a commutative square, where i is an h-cofibration and p is an h-fibration:*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "f"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

Then a lift exists in the above diagram if either i or p is a G-homotopy equivalence.

Proof. See [MP12, Proposition 17.1.4]. □

It remains to prove the factorisation axiom, and we take the opportunity to showcase our favourite proof from the non-equivariant setting. Firstly, recall that we can decompose any map into an h-cofibration followed by a homotopy equivalence using the mapping cylinder construction. Similarly, we can decompose any map into a homotopy equivalence followed by a h-fibration using the mapping path space construction. We will modify the second of these constructions to show that we can decompose any map into an h-acyclic h-cofibration followed by an h-fibration. The remaining half of the factorisation axiom then follows immediately from the mapping cylinder construction.

Lemma 6.1.4: *If $f : X \to Y$ is a map between topological spaces, then f is of the form pi where i is an h-acyclic h-cofibration and p is an h-fibration.*

Proof. Define Qf to be the subspace of $X \times Y^I \times I$ consisting of triples (x, γ, t) such that $\gamma(0) = f(x)$ and $\gamma(s) = \gamma(t)$ for all $s \geq t$. Define a map $p : Qf \to Y$ by $p(x, \gamma, t) = \gamma(1)$. Define a map $i : X \to Qf$ by $i(x) = (x, c_{f(x)}, 0)$. Then, $f = pi$ so it is enough to show that i is an h-acyclic cofibration and p is a fibration.

In the case of i, consider the homotopy $H : Qf \times I \to Qf$ defined by:

$$H((x, \gamma, s), t) = (x, \gamma_{1-t}, \min(s, 1-t)),$$

where γ_{1-t} denotes the restriction of γ to $[0, 1-t]$ extended to a path from the unit interval by $\gamma_{1-t}(s) = \gamma(1-t)$ for $s \geq 1-t$. Then H defines a strong deformation retract of Qf onto $i(X)$ and, since $i(X) = u^{-1}(0)$, where $u(x, \gamma, t) = t$, it follows that i is an h-acyclic cofibration.

To see that p is a fibration, we will construct a path-lifting function $\lambda : Qf \times_p Y^I \to Qf^I$ by defining its adjoint $\tilde{\lambda} : Qf \times_p Y^I \times I \to Qf$ by:

$$\tilde{\lambda}((x, \gamma, t), \tau, s) = (x, \mu_{s,t}, \min(t+s, 1))$$

where, if $t + s \leq 1$:

$$\mu_{s,t}(a) = \begin{cases} \gamma(a) & \text{if } a \leq t \\ \tau(a-t) & \text{if } t \leq a \leq t+s \\ \tau(s) & \text{if } a \geq t+s \end{cases}$$

and, if $t + s \geq 1$:

$$\mu_{s,t}(a) = \begin{cases} \gamma(a(t+s)) & \text{if } a \leq \frac{t}{t+s} \\ \tau((a-1)t + as) & \text{if } \frac{t}{t+s} \leq a \leq 1 \end{cases}$$

□

For another proof of Lemma 6.1.4, see Strøm's original proof in [Str72, Proposition 2]. This completes our derivation of the h-model structure. Observe that it is proper since all objects are bifibrant.

Finally, we will demonstrate how the q and h-model structures can be combined to create a mixed model structure on G-spaces, as described in the following theorem:

Theorem 6.1.5: *There is a proper model structure on G-spaces where:*

i) the m-weak equivalences are the q-equivalences,

ii) the m-cofibrations are the h-cofibrations of the form fi where i is a q-cofibration and f is an h-equivalence,

iii) the m-fibrations are the h-fibrations.

Following [Col06] and [MP12, Section 17.3], we will work in a more general context. Suppose that M is a category with two model structures $(\mathcal{W}_q, \mathcal{C}_q, \mathcal{F}_q)$ and $(\mathcal{W}_h, \mathcal{C}_h, \mathcal{F}_h)$ such that $\mathcal{W}_h \subset \mathcal{W}_q$ and $\mathcal{F}_h \subset \mathcal{F}_q$ (and, so, $\mathcal{C}_q \subset \mathcal{C}_h$). The first thing we will prove is that the mixed model structure exists:

Lemma 6.1.6: *There is a model structure on M where:*

i) the weak equivalences are the q-equivalences,

ii) the fibrations are the h-fibrations.

Proof. Define $\mathcal{C}_m$ to be the class of maps with the left lifting property with respect to all q-acyclic h-fibrations. Clearly, these classes are closed under composition and retracts and the q-equivalences satisfy 2-out-of-3. One of the lifting axioms is a definition.

For the first factorisation axiom, first factor $f = pi$ where p is an h-fibration and i is an h-acyclic h-cofibration. Then p is an m-fibration and i has the LLP with respect to $\mathcal{F}_h$ and, hence, with respect to $\mathcal{F}_h \cap \mathcal{W}_q$. It is also a q-equivalence since it is an h-equivalence.

For the other factorisation axiom, first factor $f = gi$ where i is a q-cofibration and g is a q-equivalence. Note that i is an m-cofibration. Next factor g as $g = pj$ where j is an m-acyclic m-cofibration and p is an m-fibration. Then, by the two out of three property, p is an m-acyclic m-fibration and so $p(ji)$ is our desired factorisation.

Finally, for the remaining lifting axiom, it is enough to show that an m-acyclic m-cofibration, i, is an h-acyclic h-cofibration. Factor i as pj where p is an h-fibration and j is an h-acyclic h-cofibration. Since i is m-acyclic, p is m-acyclic. Since i is an m-cofibration it has the LLP with respect to p. Hence, the retract argument tells us that i is a retract of j and so an h-acyclic h-cofibration itself. □

With the model structure demonstrated, Ken Brown's lemma, [Hir03, Lemma 7.7.1], has the following consequence:

Lemma 6.1.7: *Suppose that i and j are m-cofibrations in the following diagram:*

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

i) if f is a q-equivalence, then it is an h-equivalence,

ii) if f is an h-cofibration, then it is an m-cofibration.

Proof. i) Consider the under-category $(A \downarrow M)$. Since f is an m-equivalence between m-cofibrant objects we can factor f as pi where i is an m-acyclic m-cofibration and p has a right inverse which is an m-acyclic m-cofibration. Now m-acyclic m-cofibrations are equivalent to h-acyclic h-cofibrations and, so, f is an h-equivalence.

ii) Factor f as pi where i is an m-cofibration and p is an m-acyclic m-fibration, By i), p is an h-acyclic h-fibration and so has the RLP with respect to f. Hence, the retract argument applies to show that f is an m-cofibration. □

We can use Lemma 6.1.7 to characterise the m-cofibrations:

Lemma 6.1.8: *A map $j : A \to X$ is an m-cofibration iff j is an h-cofibration which can be factored as fi, where i is a q-cofibration and f is an h-equivalence.*

Proof. Suppose that j is an m-cofibration. Clearly, it is also an h-cofibration. Factor j as pi where i is a q-cofibration and p is a q-acyclic q-fibration. Then, since all q-cofibrations are m-cofibrations, Lemma 6.1.7 applies to show that p is an h-equivalence, so we're done.

On the other hand, suppose that j is an h-cofibration which can be factored as fi where i is a q-cofibration and f is an h-equivalence. Factor f as pk, where p is an h-acyclic h-fibration and k is an h-acyclic h-cofibration. Note that i and k are both m-cofibrations and p has the RLP with respect to j. Therefore, the retract argument shows that j is an m-cofibration. □

Before considering properness, it is useful to record the following definition and reformulation of the retract argument:

Definition 6.1.9: *Let $i : A \to X$ and $j : A \to Y$ be maps. We say that i is a retract of j relative to A if there is a commutative diagram:*

$$\begin{array}{ccccc} A & \xrightarrow{1} & A & \xrightarrow{1} & A \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{s} & Y & \xrightarrow{r} & X \end{array}$$

such that $rs = 1$.

The retract argument can be stated as:

Lemma 6.1.10: *Let $i = pj$ and suppose that $i : A \to B$ has the LLP with respect to p. Then i is a retract of j relative to A.*

Note that if i is a retract of j relative to A, and $f : A \to X$ is a map, then the pushout of f along i is a retract of the pushout of f along j. We can now prove:

Lemma 6.1.11: *If M is right q-proper, then it is right m-proper. M is left m-proper iff it is left q-proper.*

Proof. The first sentence is obvious, as is the statement that left m-properness implies left q-properness. Therefore, suppose that M is left q-proper and that we have a pushout square where f is an m-equivalence and i is an m-cofibration:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \xrightarrow[g]{} & Y \end{array}$$

Note that i is a retract, rel A, of the composite of a q-cofibration followed by an m-acyclic m-cofibration. Since M is left q-proper, the pushout of a q-equivalence along either of these is a q-equivalence. Therefore, g is the retract of a q-equivalence. □

This completes the proof of Theorem 6.1.5. Specialising to the m-model structure on G-spaces, we record the following corollary of Lemma 6.1.8:

Corollary 6.1.12: *A G-space is m-cofibrant iff it has the homotopy type of a G-CW complex.*

Proof. Lemma 6.1.8 implies that a G-space is m-cofibrant iff it has the homotopy type of a q-cofibrant G-space. Then G-CW approximation, [M$^+$96, Theorem 3.6], implies that any q-cofibrant G-space has the homotopy type of a G-CW complex. □

6.2 Point set topological lemmas

In this section, we record some point set topological lemmas about the category of CGWH spaces that will be used at various points in the remaining sections. Since these are not the main focus of our work, we will refer the reader to Strickland's notes on CGWH spaces, [Str09], for some of the proofs. We begin with a useful criterion for recognising when a continuous bijection of CGWH spaces is a homeomorphism:

Lemma 6.2.1: *If $f : X \to Y$ is a proper (the preimages of compact subspaces are compact) continuous bjiection between CGWH spaces, then it is a homeomorphism.*

Proof. This is [Str09, Proposition 3.17]. □

The following lemma is useful in various situations, as we will see in the proof of Lemma 6.2.3 below:

Lemma 6.2.2: *In the category of CGWH spaces, the pullback of a quotient map is a quotient map.*

Proof. This is [Str09, Proposition 2.36]. □

The next result is related, but not identical, to Mather's second cube theorem, [Mat76, Theorem 25]:

Lemma 6.2.3: *If we have a commutative diagram of spaces:*

$$\begin{tikzcd}
 & A_1 \arrow[rr] \arrow[dl, "j"'] \arrow[dd, dotted] & & B_1 \arrow[dl] \arrow[dd] \\
X_1 \arrow[rr] \arrow[dd] & & Y_1 \arrow[dd] & \\
 & A_0 \arrow[rr, dotted] \arrow[dl, dotted, "i"'] & & B_0 \arrow[dl] \\
X_0 \arrow[rr] & & Y_0 &
\end{tikzcd}$$

such that the bottom square is a pushout, the vertical squares are pullbacks, and the map $A_0 \to X_0$ is a closed inclusion, then the top square is a pushout.

Proof. Since i is a closed inclusion, the categorical pushout of i agrees with the usual quotient of $X_0 \sqcup B_0$, [May99, pg.40], and, since closed inclusions are preserved by pullbacks, [Str09, Proposition 2.33], the pushout of j is also given by the usual quotient. Therefore, we have a pullback:

$$\begin{array}{ccc} X_1 \sqcup B_1 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ X_0 \sqcup B_0 & \longrightarrow & Y_0 \end{array}$$

where the bottom map is a quotient. Therefore, by Lemma 6.2.2, the top arrow is also a quotient map. There are two equivalence relations on $X_1 \sqcup B_1$ that we care about. The first identifies $u \sim_1 v$ if $u, v \in X_1 \sqcup B_1$ have the same image in Y_1. The second is the smallest equivalence relation $\sim_2$ generated by the relations $x_1 \sim_2 b_1$ whenever there exists some $a_1 \in A_1$ with images $x_1 \in X_1$ and $b_1 \in B_1$. We want to show both of these equivalence relations are equal. Since $x_1 \sim_1 b_1$ whenever there is such an a_1, we have that $u \sim_2 v \implies u \sim_1 v$. If $u \sim_1 v$, write u_0 and v_0 for the images of u and v in $X_0 \sqcup B_0$. Since their images in Y_0 are equal, there is a sequence $u_0 = e_0, e_1, ..., e_n = v_0$, with $n \geq 0$, of elements of $X_0 \sqcup B_0$ such that for every $i < n$, there is some $a_i \in A_0$ such that e_i and e_{i+1} are images of a_i. It follows that each e_i has the same image in Y_0, which corresponds to the image of y, where y is the image of u and v in Y_1. So (a_i, y) is a well-defined element of A_1, and both $(e_i, y) \in X_1 \sqcup B_1$ and $(e_{i+1}, y) \in X_1 \sqcup B_1$ are images of (a_i, y). So $u \sim_2 v$. □

6.3 Well-pointed spaces of the homotopy type of a CW complex

In this section, we work non-equivariantly, and derive the closure properties of the category of well-pointed spaces of the homotopy type of a CW complex that we used in our previous work. For example, we used these results in our inductive construction of a Postnikov tower associated to a nilpotent space, since we implicitly required that cohomology is represented by homotopy classes of maps into Eilenberg-MacLane spaces. Of course, this is guaranteed to be true if the space has the based homotopy type of a CW complex.

We first consider well-pointed spaces, and the results we need are a consequence of the characterisation of an h-cofibration as an NDR-pair. We begin with:

Lemma 6.3.1: *The pullback of an h-cofibration along an h-fibration is an h-cofibration.*

Proof. See [MP12, Lemma 1.3.1]. □

The next two results are [Str72, Lemmas 5 and 6].

Lemma 6.3.2: *If $i : A \to B$ and $j : B \to C$ are maps and both j and $k = ji$ are h-cofibrations, then i is an h-cofibration.*

Proof. Firstly, i is an inclusion by verification of the universal property. Let (H, λ) and (K, μ) represent (C, B) and (C, A) as NDR-pairs. Define $\mu^{'}(c) = \mu(c) + \sup_{t \in I} \lambda(K(c, t))$. Define:

$$L(b,t) = \begin{cases} H(K(b,t),1) & \mu^{'}(b) \leq \frac{1}{2} \\ H(K(b,2(1-\mu^{'}(b))t),1) & \mu^{'}(b) \geq \frac{1}{2} \end{cases}$$

Then $(L, 2\mu^{'})$ represents (B, A) as an NDR-pair. □

Lemma 6.3.3: *If E, B, X are well-pointed and p is an h-fibration in the following pullback square:*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & E \arrow[d, two heads, "p"] \\ X \arrow[r] & B \end{tikzcd}$$

Then P is well-pointed.

Proof. This is a corollary of Lemma 6.3.1 and Lemma 6.3.2. □

We now turn our attention to spaces with the unbased homotopy type of a CW complex, or equivalently m-cofibrant spaces. We will give the shortest proof we know of the following well-known, [Sch77], theorem:

Theorem 6.3.4: *Let $p : E \to B$ be an h-fibration with fibre F and connected base B. If B is m-cofibrant, then E is m-cofibrant iff F is m-cofibrant.*

Paired with the results about well-pointed spaces, this explains why we are free to assume that all spaces are well-pointed with the homotopy type of a CW complex in our inductive construction of Postnikov towers. Since the h-model structure is proper, we can and will assume that B is already a CW-complex. The m-model structure will play a simplifying role in the proof of Lemma 6.3.6 below. However, first, we use the fact that geometric realisation takes Kan fibrations to h-fibrations, to prove:

Lemma 6.3.5: *If E is m-cofibrant, then so is F.*

Proof. Consider the square:

$$\begin{tikzcd} {|Sing(E)|} \arrow[r, "\simeq"] \arrow[d, two heads, "{|Sing(p)|}"'] & E \arrow[d, two heads, "p"] \\ {|Sing(B)|} \arrow[r, "\simeq"'] & B \end{tikzcd}$$

Since p is an h-fibration it is a q-fibration, so $Sing(p)$ is a Kan fibration and $|Sing(p)|$ is an h-fibration, by [MP12, Theorem 17.5.7] or [FP90, Theorem 4.5.25]. Since the h-model structure is proper, there is an induced homotopy equivalence between F and $|Sing(F)|$, which is a CW complex. □

Lemma 6.3.6: *If F is m-cofibrant, then so is E.*

Proof. Suppose that the n-skeleton of B is defined by attaching maps $\alpha_i : \partial\Delta^n \to B_{(n-1)}$. Let e_i denote the inclusion of the cell into $B_{(n)}$, $e_i : \Delta^n \to B_{(n)}$. Define $P_i^{\partial\Delta^n}$ to be the pullback of α_i along p, and $P_i^{\Delta^n}$ to be the pullback of e_i along p. Then $P_i^{\partial\Delta^n} \simeq \partial\Delta^n \times F$ and $P_i^{\Delta^n} \simeq \Delta^n \times F$ and so both are m-cofibrant. Using Lemma 6.2.3, we have a pushout:

$$\begin{tikzcd}
\sqcup_i P_i^{\partial\Delta^n} \arrow[r] \arrow[d] & p^{-1}(B_{(n-1)}) \arrow[d] \\
\sqcup_i P_i^{\Delta^n} \arrow[r] & p^{-1}(B_{(n)})
\end{tikzcd}$$

Since the pullback of an h-cofibration along an h-fibration is an h-cofibration, $P_i^{\partial\Delta^n} \to P_i^{\Delta^n}$ is an h-cofibration. Therefore, the LHS map is an h-cofibration between m-cofibrant objects and, hence, an m-cofibration, by Lemma 6.1.7. Therefore, the RHS is also an m-cofibration, and so, inductively, $p^{-1}(B) = E$ is m-cofibrant. □

6.4 Well-pointed spaces of the homotopy type of a G-CW complex

The results about well-pointed spaces of Section 6.3 generalise effortlessly to the equivariant context, using the fact that h-cofibrations are equivalent to G-NDR pairs. Therefore, the main purpose of this section is to use the m-model structure to prove one direction of the following theorem:

Theorem 6.4.1: *Let $p : E \to B$ be an h-fibration, and suppose that B has the G-homotopy type of a G-CW complex. Then E has the G-homotopy type of a G-CW complex iff for every $b \in B$, $p^{-1}(b)$ has the H_b-homotopy type of an H_b-CW complex, where H_b is the isotropy group of b.*

The more difficult direction, and the direction needed to construct equivariant Postnikov towers, was proved by Waner in [Wan80, Corollary 4.14] - namely, that if E and B have the G-homotopy type of a G-CW complex, then each fibre has the H_b-homotopy type of an H_b-CW complex. In order to generalise the proof we gave in the non-equivariant setting in Lemma 6.3.5, we would need an m-cofibrant replacement functor that preserves m-fibrations and finite limits. However, non-equivariantly the construction of such a functor used the theory of minimal fibrations of simplicial sets, a tool which we do not evidently have available in the equivariant context.

For the converse direction, we begin with a few routine lemmas:

Lemma 6.4.2: *If $p : E \to B \times I$ is an h-fibration, then there is a G-homotopy equivalence over B, $E_0 \simeq E_1$, where E_t is the preimage of $B \times \{t\}$.*

Proof. The proof is verbatim to the non-equivariant case, [May75, Lemma 2.4]. □

Lemma 6.4.3: *If $p : E \to \frac{G}{H} \times \Delta^n$ is an h-fibration, then E is G-homotopy equivalent over $\frac{G}{H} \times \Delta^n$ to $(G \times_H F) \times \Delta^n$, where $F = p^{-1}(1, *)$ for some $* \in \Delta^n$.*

Proof. By Lemma 6.4.2, E is G-homotopy equivalent to $p^{-1}(\frac{G}{H} \times *) \times \Delta^n$ over $\frac{G}{H} \times \Delta^n$. Since the preimage of a slice is a slice, $p^{-1}(\frac{G}{H} \times *) = G \times_H p^{-1}(1, *)$. □

Since $G \times_H \frac{H}{K} = G \times_H (H \times_K *) = (G \times_H H) \times_K * = G \times_K * = \frac{G}{K}$, it is easy to see that:

Lemma 6.4.4: *The functor $G \times_H -$ takes H-CW complexes to G-CW complexes, and satisfies $G \times_H (X \times I) = (G \times_H X) \times I$.*

If H is a subgroup of G, then $\frac{G}{K}$ is an H-manifold, and so has the H-homotopy type of an H-CW complex, by [Ill83, Corollary 7.2]. Therefore, we also have:

Lemma 6.4.5: *If H is a subgroup of G, and X is a G-space with the G-homotopy type of a G-CW complex, then X has the H-homotopy type of an H-CW complex.*

We can now prove:

Theorem 6.4.6: *Let $p : E \to B$ be an h-fibration such that B has the G-homotopy type of a G-CW complex, and for every H and $b \in B$ with isotropy group H, $p^{-1}(b)$ has the H-homotopy type of an H-CW complex. Then, E has the G-homotopy type of a G-CW complex.*

Proof. Firstly, we'll show that we can reduce to the case where B is a G-CW complex. Let $i : \tilde{B} \to B$ be a G-CW approximation and consider the pullback:

$$\begin{tikzcd} \tilde{E} \arrow[r, "j"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ \tilde{B} \arrow[r, "i"'] & B \end{tikzcd}$$

Since the h-model structure is proper, both i and j are G-homotopy equivalences. If $\tilde{b} \in \tilde{B}$, $b = i(\tilde{b})$ and $\tilde{b}$ has isotropy group $\tilde{H}$, then b has isotropy group $H \supseteq \tilde{H}$. Therefore, by assumption, $q^{-1}(\tilde{b}) = p^{-1}(b)$ has the H-homotopy type of an H-CW complex, and so, in particular, has the $\tilde{H}$-homotopy type of an $\tilde{H}$-CW complex. So we will assume from now on that B is a G-CW complex.

Suppose that the n-skeleton, $B_{(n)}$, of B is defined by attaching maps $\alpha_i : \frac{G}{H} \times \partial\Delta^n \to B_{(n-1)}$. Let e_i denote the inclusion of the cell into $B_{(n)}$, $e_i : \frac{G}{H} \times \Delta^n \to B_{(n)}$. Define $P_i^{\partial\Delta^n}$ to be the pullback of α_i along p, and $P_i^{\Delta^n}$ to be the pullback of e_i along p. Then, using Lemma 6.4.3, $P_i^{\partial\Delta^n} \simeq (G \times_H F) \times \partial\Delta^n$ and $P_i^{\Delta^n} \simeq (G \times_H F) \times \Delta^n$, where $F = p^{-1}(b)$ for some $b \in B$ with isotropy group H - and so both are m-cofibrant. Using Lemma 6.2.3, we have a pushout:

$$\begin{tikzcd} \sqcup_i P_i^{\partial\Delta^n} \arrow[r] \arrow[d] & p^{-1}(B_{(n-1)}) \arrow[d] \\ \sqcup_i P_i^{\Delta^n} \arrow[r] & p^{-1}(B_{(n)}) \end{tikzcd}$$

Since the pullback of an h-cofibration along an h-fibration is an h-cofibration, $P_i^{\partial\Delta^n} \to P_i^{\Delta^n}$ is an h-cofibration. Therefore, the LHS map is an h-cofibration between m-cofibrant objects and, hence, an m-cofibration, by Lemma 6.1.7. Therefore, the RHS is also an m-cofibration, and so, inductively, $p^{-1}(B) = E$ is m-cofibrant. □

Finally, we record the following analogue for principal $(\Pi;\Gamma)$-bundles, [ML86], which can be proven using similar methods. This is a useful result if one wishes to view the universal principal $(\Pi;\Gamma)$-bundle, $E(\Pi;\Gamma)$, as an equivariant Eilenberg MacLane space, $K(\underline{M}, 0)$, and use the universal property of such Eilenberg-MacLane spaces to prove a classification theorem for bundles. See [M+96, pg.55,72] and [ML86, pg. 270] for further discussion of this perspective.

Lemma 6.4.7: *If $p : E \to B$ is a numerable principal $(\Pi;\Gamma)$-bundle such that B has the G-homotopy type of a G-CW complex, then E has the Γ-homotopy type of a Γ-CW complex.*

Proof. Firstly, we can consider the pullback of the principal $(\Pi;\Gamma)$-bundle along a G-CW complex $\tilde{B} \to B$.

$$\begin{tikzcd} P \arrow[r] \arrow[d] & E \arrow[d] \\ \tilde{B} \arrow[r] & B \end{tikzcd}$$

Since numerable bundles over a base of the form $B \times I$ are of the form $E_0 \times I$, [ML86, Theorem 6], it follows that the map $P \to E$ is a Γ-homotopy equivalence. Therefore, we assume from now on that B is a G-CW complex.

Suppose that the n-skeleton, $B_{(n)}$, of B is defined by attaching maps $\alpha_i : \frac{G}{H} \times \partial\Delta^n \to B_{(n-1)}$. Let e_i denote the inclusion of the cell into $B_{(n)}$, $e_i : \frac{G}{H} \times \Delta^n \to B_{(n)}$. Define $P_i^{\partial\Delta^n}$ to be the pullback of α_i along p, and $P_i^{\Delta^n}$ to be the pullback of e_i along p. Then, using [ML86, Lemma 3 and Theorem 6], we have that

$P_i^{\partial\Delta^n} \cong \frac{\Gamma}{\Lambda} \times \partial\Delta^n$, and $P_i^{\Delta^n} \cong \frac{\Gamma}{\Lambda} \times \Delta^n$, where $\Lambda \cap \Pi = 1$ and $q(\Lambda) = H$. Using Lemma 6.2.3, we have a pushout:

$$\begin{tikzcd}
\sqcup_i P_i^{\partial\Delta^n} \arrow[r] \arrow[d] & p^{-1}(B_{(n-1)}) \arrow[d] \\
\sqcup_i P_i^{\Delta^n} \arrow[r] & p^{-1}(B_{(n)})
\end{tikzcd}$$

which exhibits E as a Γ-CW complex. □

6.5 Simplicial Spaces and Fibrations

Since geometric realisation of simplicial spaces preserves finite limits, it would not be too surprising if geometric realisation takes 'locally trivial' maps of simplicial spaces to locally trivial Hurewicz fibrations. The purpose of this section is to prove such a theorem, which reads as follows:

Theorem 6.5.1: *If $p : X \to Y$ is a locally F-trivial map of simplicial spaces, and Y is a proper simplicial space, then $|p|$ is a Hurewicz fibration.*

For the precise definition of a locally F-trivial map of simplicial spaces see Definition 6.5.8. The proof we give is inspired by the proof of Goerss and Jardine that geometric realisation takes minimal fibrations to Hurewicz fibrations, [GJ09, Ch. I. Theorem 10.9]. In that context, Theorem 10.9 is a key component for showing that geometric realisation takes Kan fibrations to Hurewicz fibrations. As an application of Theorem 6.5.1, we recover the theorem:

Theorem 6.5.2: *If G is a topological group with a nondegenerate basepoint, then, for any left G-space X and right G-space Y, $B(Y, G, X) \to B(Y, G, *)$ is a Hurewicz fibration.*

For a more direct proof, see [May75, Theorem 8.2] and corollaries. Taking $Y = *$ and $X = G$, we can deduce that the orbit map $EG \to BG$ is a Hurewicz fibration, whenever G is a topological group with a nondegenerate basepoint.

We begin with a discussion of locally trivial maps of spaces, before moving on to simplicial spaces:

Definition 6.5.3: *Let F be a space. We call a map $f : X \to Y$ locally F-trivial over a subspace A of Y if there is a homeomorphism $\phi_A : f^{-1}(A) \to A \times F$ over A:*

$$\begin{tikzcd} f^{-1}(A) \arrow[rr, "\phi_A", "\cong"'] \arrow[dr, "f"'] & & A \times F \arrow[dl, "\pi_A"] \\ & A & \end{tikzcd}$$

Clearly, we have:

Lemma 6.5.4: *If Y has a numerable, locally finite open cover, $\{U_i\}$, such that f is locally F-trivial over each U_i, then f is a Hurewicz fibration.*

Proof. See [May99, pg. 51]. □

The next lemma is central to our main theorem:

Lemma 6.5.5: *Let F be a space, let $f : X \to Y$ be a map, and let A, B be subspaces of Y such that $A \subset B$, A is a retract of B, and f is locally F-trivial over A and B via homeomorphisms ϕ_A and ϕ_B respectively. Then f is locally F-trivial over B via a homeomorphism $\phi_B^{'}$ which agrees with ϕ_A on $f^{-1}(A)$.*

Proof. Form the diagram:

$$\begin{tikzcd} f^{-1}(A) \arrow[r, "\phi_B"'] \arrow[d] \arrow[rr, bend left, "\phi_A"] & A \times F \arrow[r, "\sigma"'] \arrow[d] & A \times F \arrow[d] \\ f^{-1}(B) \arrow[r, "\phi_B"'] \arrow[d] & B \times F \arrow[r, "\epsilon"'] \arrow[d] & B \times F \arrow[d] \\ B \arrow[r, "1"'] & B \arrow[r, "1"'] & B \end{tikzcd}$$

where $\sigma = \phi_A \phi_B^{-1}$ and $\epsilon(b, f) = (b, \pi_F \sigma(r(b), f))$. Then all horizontal arrows are homeomorphisms, and we can define $\phi_B^{'} = \epsilon \phi_B$. □

Corollary 6.5.6: *Let F be a space, $f : X \to Y$ be a map, and A, B closed subspaces of Y such that f is locally F-trivial over A and B. If $A \cap B$ is a retract of B, then f is locally F-trivial over $A \cup B$ via a homeomorphism $\phi_{A \cup B}$ which agrees with ϕ_A on $f^{-1}(A)$.*

As a quick application we have:

Lemma 6.5.7: *If Y is a CW complex, and $f : X \to Y$ is a map which is locally F-trivial over the images of every cell, $\epsilon(D^n)$, then f is a Hurewicz fibration.*

Proof. We can express Y as the transfinite composite of maps $Y_\lambda \to Y_{\lambda+1}$ where each $Y_{\lambda+1}$ is obtained from Y_λ by attaching a single cell. Given a pair of subspaces (A, U) of some Y_λ such that $U \subset A$, A is closed, U is open and numerable, and p is locally F-trivial over A, we can extend (A, U) to such a pair, $(\tilde{A}, \tilde{U})$, on Y by using open/closed collars around U and A respectively corresponding to half the radius of each attached cell, D^n. It follows by induction that $\tilde{A}$ is locally F-trivial, using Corollary 6.5.6, and that $\tilde{U}$ is numerable. If we single out the pairs (A, U) of the form $(\bar{B}(0, \frac{1}{2}), B(0, \frac{1}{2}))$ corresponding to interior balls of half the radius in each cell, then the corresponding cover of Y by the induced $\tilde{U}$ is a locally finite numerable open cover of Y such that p is locally F-trivial over each $\tilde{U}$. The result now follows from Lemma 6.5.4. □

Our next task is to define a locally F-trivial map of simplicial spaces:

Definition 6.5.8: *Let F be a simplicial space and $f : X \to Y$ a map of simplicial spaces. Define P_n to be the pullback:*

$$\begin{tikzcd} P_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ \underline{Y_n} \times \Delta^n \arrow[r] & Y \end{tikzcd}$$

where an underline denotes the constant simplicial space on the underlined space. We say that f is locally F-trivial if, for every $n \geq 0$, there is an homeomorphism of simplicial spaces $\underline{Y_n} \times \Delta^n \times F \to P_n$ over $\underline{Y_n} \times \Delta^n$:

$$\begin{tikzcd} \underline{Y_n} \times \Delta^n \times F \arrow[rr, "\cong"] \arrow[dr] & & P_n \arrow[dl] \\ & \underline{Y_n} \times \Delta^n & \end{tikzcd}$$

From this point on, the reader would benefit from being familiar with the basic properties of the geometric realisation of a simplicial space, X, as developed in [May72, Chapter 11], particularly with regard to the topology on the geometric realisation, $|X|$, being equivalent to the topology induced by the filtration $F_n X$. If $p : X \to Y$ is any map of simplicial spaces, then upon passage to geometric realisations, and using the notations of [May72, Definition 11.1], we have a diagram of spaces:

$$\begin{tikzcd}
& P_\partial \arrow[rr] \arrow[dl] \arrow[dd, dotted, "q"] & & p^{-1}(F_{n-1}Y) \arrow[dl] \arrow[dd, "p"] \\
P_\Delta \arrow[rr] \arrow[dd, "q"] & & p^{-1}(F_nY) \arrow[dd, "p"] & \\
& s(Y_{n-1}) \times \Delta^n \cup Y_n \times \partial\Delta^n \arrow[rr, dotted, "\alpha"] \arrow[dl, dotted] & & F_{n-1}Y \arrow[dl] \\
Y_n \times \Delta^n \arrow[rr, "\epsilon"] & & F_nY &
\end{tikzcd}$$

where the bottom square is a pushout and the vertical squares are pullbacks. By Lemma 6.2.3, the top square is also a pushout, since $s(Y_{n-1}) \to Y_n$ is a closed inclusion. If p is locally F-trivial, then we have $P_\Delta \cong Y_n \times \Delta^n \times F$ over $Y_n \times \Delta^n$, and $P_\partial \cong (s(Y_{n-1}) \times \Delta^n \cup Y_n \times \partial\Delta^n) \times F$ over $s(Y_{n-1}) \times \Delta^n \cup Y_n \times \partial\Delta^n$. Recall that a simplicial space is proper if, for every n, $s(Y_{n-1}) \to Y_n$ is a Hurewicz cofibration. In this case, let (H, λ) denote $(Y_n \times \Delta^n, s(Y_{n-1}) \times \Delta^n \cup Y_n \times \partial\Delta^n)$ as an NDR-pair. Then, $H(-,1)$ defines a retraction $r : \bar{\lambda}^{-1}([0,1)) \to s(Y_{n-1}) \times \Delta^n \cup Y_n \times \partial\Delta^n$, where $\bar{\lambda}^{-1}([0,1))$ denotes the closure of $\lambda^{-1}([0,1))$. Our main theorem now states:

Theorem 6.5.9: *If $p : X \to Y$ is a locally F-trivial map of simplicial spaces, and Y is a proper simplicial space, then $|p|$ is a Hurewicz fibration.*

Proof. Suppose inductively that we have a locally finite numerable open cover $\{U_i\}_{i \in I}$ of $F_{n-1}Y$ such that p is locally F-trivial over each U_i. For each $i \in I$, let $V_i^{'} = r^{-1}\alpha^{-1}(U_i) \cap \lambda^{-1}((0,1))$, where r, λ and α are as defined above. Let $W_i = \epsilon(V_i^{'}) \cup U_i$, which can be viewed as a collar around U_i. Then W_i is open in F_nY, since U_i is open in $F_{n-1}Y$. We'll show that p is locally F-trivial over W_i. By assumption, we have a trivialisation:

$$\begin{tikzcd}
p^{-1}(U_i) \arrow[rr, "\phi_{U_i}", "\cong"'] \arrow[dr] & & U_i \times F \arrow[dl] \\
& U_i &
\end{tikzcd}$$

Letting $U_i^{'} = \alpha^{-1}(U_i)$, pulling back along α induces a trivialisation:

$$\begin{tikzcd} q^{-1}(U_i^{'}) \arrow[rr, "\phi_{U_i'}", "\cong"'] \arrow[dr] & & U_i^{'} \times F \arrow[dl] \\ & U_i^{'} & \end{tikzcd}$$

Since q is locally F-trivial over $V_i^{'}$, Lemma 6.5.6 implies that there is a trivialisation of q over $V_i^{'}$ which agrees with $\phi_{U_i'}$ on $q^{-1}(U_i^{'})$, call it $\phi_{V_i'}$:

$$\begin{tikzcd} q^{-1}(V_i^{'}) \arrow[rr, "\phi_{V_i'}", "\cong"'] \arrow[dr] & & V_i^{'} \times F \arrow[dl] \\ & V_i^{'} & \end{tikzcd}$$

By Lemma 6.2.3, $p^{-1}(W_i)$ is the pushout of the maps $q^{-1}(U_i^{'}) \to q^{-1}(V_i^{'})$ and $q^{-1}(U_i^{'}) \to p^{-1}(U_i)$. Similarly, since left adjoints preserves colimits, $W_i \times F$ is the pushout of the maps $U_i^{'} \times F \to V_i^{'} \times F$ and $U_i^{'} \times F \to U_i \times F$. Therefore, $\phi_{V_i'}$ and ϕ_{U_i} induce a homeomorphism ϕ_{W_i}:

$$\begin{tikzcd} p^{-1}(W_i) \arrow[rr, "\phi_{W_i}", "\cong"'] \arrow[dr] & & W_i \times F \arrow[dl] \\ & W_i & \end{tikzcd}$$

which shows that p is locally F-trivial over W_i, as desired. We now explain how to complete the inductive proof of the theorem. We define $W^{'} = \lambda^{-1}((\frac{1}{2}, 1]) \subset Y_n \times \Delta^n$ and let $W = \epsilon(W^{'})$. Then, W is an open subspace of F_nY and p is locally F-trivial over W, since $W^{'} \to W$ is a homeomorphism, via Lemma 6.2.3. It is clear that W is a numerable open subspace of F_nY. We also need to check that each W_i is a numerable subspace of F_nY. If $U_i = \mu_i^{-1}((0, 1])$, for some $\mu_i : F_{n-1}Y \to I$, we can define $\nu_i : F_nY \to I$ using the map μ_i, and the map κ on $Y_n \times \Delta^n$ defined by:

$$\kappa(y,t) = \begin{cases} (1 - \lambda(y,t))\mu_i \alpha r(y,t), & \text{if } \ (y,t) \in \bar{\lambda}([0,1)) \\ 0, \ \text{ if } \ \lambda(y,t) = 1 \end{cases}$$

So $\kappa^{-1}((0, 1]) = V_i^{'}$ and $\nu_i^{-1}((0, 1]) = W_i$. Now observe that $\{W_i\}_{i \in I}$ along with W is a locally finite numerable cover of F_nY, and if $y \in F_{n-1}Y$ has an open neighbourhood P in $F_{n-1}Y$ intersecting U_j only if $j \in J \subset I$, then there exists an open neighbourhood Q of y in F_nY which intersects W_j only if $j \in J$ and doesn't intersect W. Moreover, we can take $Q \cap F_{n-1}Y = P$. Note also that $W_i \cap F_{n-1}Y = U_i$, and ϕ_{W_i} agrees with ϕ_{U_i} on $p^{-1}(U_i)$. Therefore, we can iterate this procedure along the sequential colimit of the maps $F_iY \to F_{i+1}Y$, and we will end up with a numerable locally finite open cover of Y, such that p is locally F-trivial over each open set in the cover. It follows that p is a Hurewicz fibration by Lemma 6.5.4. □

We will now apply Theorem 6.5.9 to prove that the orbit map $EG \to BG$ is a Hurewicz fibration, whenever G is a topological group with a nondegenerate identity element, which we define to be the basepoint. It is straightforward to show that this condition ensures that $B(Y, G, X)$ is a proper simplicial space, [May75, Proposition 7.1]. We now prove:

Theorem 6.5.10: *If G is a topological group with a nondegenerate basepoint, then, for any spaces X and Y, $B(Y, G, X) \to B(Y, G, *)$ is a Hurewicz fibration.*

Proof. We will show that the corresponding map of simplicial spaces is X-locally trivial. We have a commutative square:

$$\begin{array}{ccc} \underline{B(Y,G,X)}_n \times \Delta^n & \longrightarrow & B(Y,G,X) \\ \downarrow & & \downarrow \\ \underline{B(Y,G,*)}_n \times \Delta^n & \longrightarrow & B(Y,G,*) \end{array}$$

and, therefore, we have an induced map of simplicial spaces $\phi : \underline{B(Y,G,X)}_n \times \Delta^n \cong \underline{B(Y,G,*)}_n \times X \times \Delta^n \to P_n$ over $\underline{B(Y,G,*)}_n \times \Delta^n$. It suffices to show that ϕ is a homeomorphism. A generic element of $\underline{B(Y,G,*)}_n \times X \times \Delta^n_m$ is of the form $\alpha = (\tilde{y}, \tilde{g}_1, ..., \tilde{g}_n, \tilde{x}, \tilde{\partial})$, where $\tilde{\partial}$ is a morphism from $\mathbf{m} \to \mathbf{n}$ in Δ. A generic element of $(P_n)_m$ is of the form $\beta = ((y, g_1, ..., g_n, \partial), (y', g_1', ..., g_m', x'))$, where ∂ is a morphism $\mathbf{m} \to \mathbf{n}$ in Δ and $\partial(y, g_1, ..., g_n) = (y', g_1', ..., g_m')$ in $B(Y, G, *)$. Now $\beta = \phi(\alpha)$ iff $\tilde{y} = y, \tilde{g_1} = g_1, ..., \tilde{g_n} = g_n, \tilde{\partial} = \partial$ and $\tilde{x} = h^{-1}x'$ where h is a product of the g_i which depends on ∂. It follows that for every β there is a unique α such that $\phi(\alpha) = \beta$, so ϕ is a continuous bijection. We will show that ϕ is proper, and then we will be done by Lemma 6.2.1. Using the existence of inverses, and the fact that Δ^n_m is a discrete space, we have that $\underline{B(Y,G,X)}_n \times \Delta^n \to B(Y,G,X) \times \underline{B(Y,G,*)}_n \times \Delta^n$ is the inclusion of a retract. It follows that if A is compact in $(P_n)_m$, then the closed subspace $\phi^{-1}(A)$ is contained within a compact subspace, and therefore is compact itself. □

When G is a topological monoid with a nondegenerate basepoint, it is straightforward to check that $B(Y, G, X) \to B(Y, G, *)$ is not necessarily a Hurewicz fibration. However, as explained in [May90], it is possible to use an inductive argument of a similar nature to our proof of Theorem 6.5.9 to prove:

Theorem 6.5.11: *If G is a grouplike topological monoid with a nondegenerate basepoint, then $B(Y, G, X) \to B(Y, G, *)$ is a quasifibration.*

Proof. See [May90]. $\square$

Chapter 7

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