A double coset formula for the genus of a nilpotent group

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Abstract

We derive double coset formulae for the genus and extended genus of a finitely generated nilpotent group G, using the notions of bounded and bounded above automorphisms of $\prod G_S$, which are defined relative to a fixed fracture square for G.

1 Introduction

Let T, S and, for each i in some indexing set I, T_i be sets of primes such that $T = \cup_i T_i$ and $T_i \cap T_j = S$ for all $i \neq j$. Suppose also that $T \neq S$. Throughout, we let G be an $f\mathbb{Z}_T$-nilpotent group and consider a fixed reference diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{(\psi_i)} & \prod G_{T_i} \\
\sigma \downarrow & & \downarrow \phi \quad \searrow^{\prod \phi_i} \\
G_S & \xrightarrow{\omega} & (\prod G_{T_i})_S \xrightarrow{\tilde{\pi}} \prod G_S
\end{array}
$$

where each ψ_i is a localisation at T_i, ϕ is a localisation at S, σ is a localisation at S, ϕ_i is the unique localisation at S such that $\phi_i\psi_i = \sigma$, ω is the localisation of (ψ_i) and $\tilde{\pi}$ is the unique map making the triangle on the right commute. It follows from these definitions that $\tilde{\pi}\omega = \Delta$.

The purpose of this paper is to derive double coset formulae for the genus and extended genus of G, and we begin by recalling the relevant definitions from [1]:

Definition 1.1: *i) the genus of G is the set of isomorphism classes of $f\mathbb{Z}_T$-nilpotent groups H such that for every $i \in I$, $H_{T_i} \cong G_{T_i}$,*

ii) the extended genus of G is the set of isomorphism classes of T-local nilpotent groups H such that for every $i \in I$, $H_{T_i} \cong G_{T_i}$.

We remark that these definitions depend on G being $f\mathbb{Z}_T$-nilpotent, and the sets of primes T_i. The fact that the extended genus is a set is a consequence of the fracture theorem, [1, Theorem 7.2.1 ii)], for T-local nilpotent groups.

In [1, Section 7.5], a map was defined which sends an automorphism $\alpha \in \prod Aut(G_S)$ to the pullback of $\alpha \circ \prod \phi_i$ along Δ, and it was claimed that this map was a surjection onto the extended genus of G. However, it is not necessarily true that the image of this map is contained within the extended genus of G. To see this consider the following fracture square for $\mathbb{Z}$, where the product is indexed over the natural numbers, p_i is the ith prime number, and each of the undefined maps is the inclusion sending 1 to 1:

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{(\psi_i)} & \prod \mathbb{Z}_{\{p_i\}} \\
\sigma \downarrow & & \downarrow \prod \phi_i \\
\mathbb{Q} & \xrightarrow{\Delta} & \prod \mathbb{Q}
\end{array}
$$

Consider the automorphism $\alpha = \prod p_i$ of $\prod \mathbb{Q}$, where p_i also denotes multiplication p_i. Then the image of $\alpha \circ \prod \phi_i$ consists of elements $(q_i) \in \prod \mathbb{Q}$ such that if $q_i = \frac{a_i}{b_i}$ with a_i, b_i coprime, then a_i is divisible by p_i. In particular, the image of $\alpha \circ \prod \phi_i$ intersects the image of Δ only at 0. Therefore, the pullback group of $\alpha \circ \prod \phi_i$ along Δ is 0, which does not localise to $\mathbb{Z}_{\{p_i\}}$ for any i.

Nevertheless, this example turns out to be instructive. Suppose, instead, that $\alpha = \prod(\frac{u_i}{v_i})$ with u_i and v_i coprime non-zero integers. Suppose that α is 'bounded' in the sense that there are only finitely many primes which divide some u_i or v_i. Then the induced pullback is isomorphic to $\mathbb{Z}$, which is the unique abelian group in the genus of $\mathbb{Z}$. If, instead, α is only 'bounded above' in the sense that there are only finitely many primes which divide some u_i, then the pullback turns out to be in the extended genus of $\mathbb{Z}$ and the maps ψ_i are localisations at T_i. In fact, we will see that the pullback group is not finitely generated unless α is 'bounded'. Note that in the counterexample we formulated, the map α was neither 'bounded' nor 'bounded above'.

With this in mind, the purpose of this paper will be to prove the following pair of double coset results, relating to the genus and extended genus of G respectively:

Theorem 1.2: *The genus of G is in 1-1 correspondence with the double coset:*

$$Aut(G_S) \backslash Aut_b(\textstyle\prod_i G_S) / \textstyle\prod_i Aut(G_{T_i})$$

where $Aut_b(\prod_i G_S)$ is the subgroup of automorphisms of the form $\prod_i \alpha_i$ which are S-bounded, see Definition 4.1.

Theorem 1.3: *The extended genus of G is in 1-1 correspondence with the double coset:*

$$Aut(G_S) \backslash Aut_{b.a.}(\textstyle\prod_i G_S) / \textstyle\prod_i Aut(G_{T_i})$$

where $Aut_{b.a.}(\prod_i G_S)$ is the monoid of automorphisms of the form $\prod_i \alpha_i$ which are S-bounded above, see Definition 3.1.

We will begin in Section 2, with a review of the results about localisations of nilpotent groups that we will use in this paper. Then, in Sections 3 and 4, we will derive our double coset formulae for the extended genus and genus, respectively. We then conclude the paper in Section 5, by relating our results to the formal fracture square, and deriving a double coset formula in that context.

Finally, for this introductory section, we will review some other notions of genus, and corresponding double coset formulae, that can be found in the literature. Firstly, we adopt the following definition from [1, Definition 12.4.6]:

Definition 1.4: *Let G be an $f\mathbb{Z}_T$-nilpotent group. The adelic genus of G is defined to be the set of isomorphism classes of $f\mathbb{Z}_T$-nilpotent groups, H, such that $H_0 = G_0$ and $\hat{H}_p = \hat{G}_p$ for every $p \in T$.*

We will restrict attention to the case where T is the set of all primes. In this case, Pickel has shown that the adelic genus of a torsion free finitely generated nilpotent group is in 1-1 correspondence with a subset of the double coset $\mathcal{G}_A^\infty \backslash \mathcal{G}_A / \mathcal{G}_Q$, in [2, Proposition 3.2]. Our proofs of Theorems 1.2 and 1.3 will make heavy use of the universal property of localisations; however, in the case of the adelic genus, proofs of double coset formulae do not seem to hinge on the universal property of completion, but, instead, on the universal property of extensions of scalars. To justify this, note that if A is abelian, then $(\hat{A}_p)_0$ is not p-complete, but is a $\mathbb{Q} \otimes \hat{\mathbb{Z}}_p$-module. In the nilpotent setting, the notion of an R-module is replaced by the notion of a nilpotent R-group, [3, Definition 10.4], and Warfield has shown that, if R is a binomial domain (eg $\mathbb{Q}, \hat{\mathbb{Q}}_p, \hat{\mathbb{Z}}_p$), then we can define the tensor product of a nilpotent group with R, with the universal property that a group homomorphism from G to a nilpotent R-group H factors uniquely through $G \to G \otimes R$, via an R-map $G \otimes R \to H$ ([3, Theorem 10.14]).

Moving on to spaces, there is an entirely analogous definition of the genus and adelic genus of an $f\mathbb{Z}_T$-nilpotent space, and Wilkerson has derived a double coset formula for the adelic genus of a simply connected CW-complex of finite type, in [4, Theorem 3.8]. Here, Sullivan's formal completion, [5, page 76], takes the place of the extension of scalars functor $- \otimes \hat{\mathbb{Z}}_p$. In order to generalise Wilkerson's double coset formula to nilpotent spaces, it would be interesting if a homotopical adjoint functor theorem, such as those of [6] or [7], could be used to construct the tensor product of a nilpotent space with a ring, with an appropriate universal property. In theory, we want that, if X is an $f\mathbb{Z}$-nilpotent space, then $X \to X \otimes \hat{\mathbb{Z}}_p$ would be a p-completion of X, and $X \to X \otimes \mathbb{Q}$ would be a rationalisation.

We are not aware of a double coset formula for the extended adelic genus, or fully general double coset formulae for the genus or extended genus of an $f\mathbb{Z}_T$-nilpotent space. However, the special case where I is a finite indexing set is worth mentioning. In this case, the genus and extended genus of an $f\mathbb{Z}_T$-nilpotent space are the same, and the formal arguments of [1, Proposition 7.5.2] go through to yield a double coset formula for both. Moreover, in [8], this double coset formula is derived as an application of a general formula for calculating conjugates in an ∞-category. Finally, we remark that some other double coset formulae for the

(adelic) genus of a nilpotent group/space are claimed in [4, Theorem 1.2] and [1, Proposition 8.5.10, Remark 12.4.8, Theorem 13.6.6], but the proofs are incorrect, or missing in detail.

2   Review of nilpotent groups and their localisations

In this short, introductory section, we recall some definitions and results about nilpotent groups and their localisations which will help us on our way. The following result, which is an easy generalisation of a theorem of Warfield, [3, Theorem 3.25], is used repeatedly throughout this paper:

Lemma 2.1: *Let G be a nilpotent group of nilpotency class c, H a subgroup of G, and A a set of elements of G such that there exists an $s \in \mathbb{N}$ such that $a \in A \implies a^s \in H$. Then, if $g \in G$ is in the subgroup generated by elements of A and H, $g^{s^d} \in H$ where $d = \frac{1}{2}c(c+1)$.*

Proof. Let K be the subgroup of G generated by elements of A and H. Then K has nilpotency class $e \leq c$. Let:

$$1 = \Gamma^e K \subset ... \subset \Gamma^1 K \subset \Gamma^0 K = K$$

be the lower central series of K. Suppose that $k \in K$ is of the form xh where $x \in \Gamma^i K$ and $h \in H$. Recall that $\frac{\Gamma^i K}{\Gamma^{i+1} K}$ is an abelian group generated by commutators of the form $[z_0, ..., z_i]$ where each $z_i \in A \cup H$, [9, Corollary 2.10]. Since the commutators are bilinear and $\frac{\Gamma^i K}{\Gamma^{i+1} K}$ is a central subgroup of $\frac{K}{\Gamma^{i+1} K}$, it follows that $k^{s^{i+1}} = yh'$ for some $y \in \Gamma^{i+1} K$, $h' \in H$. $\qquad\square$

Moving on to localisations, recall that a nilpotent group is T-local iff it is uniquely p-divisible for all $p \in T$. Recall, also, the following definitions from [1]:

Definition 2.2: *If R is a set of primes, then an R-number is a natural number which is a product of primes not in R.*

Definition 2.3: *Let $f : G \to H$ be a homomorphism between nilpotent groups. Then, we call f an:*

i) R-monomorphism if $f(g) = 1 \implies$ there is an R-number, r, such that $g^r = 1$,

ii) R-epimorphism if, for all $h \in H$, there exists an R-number, r, such that $h^r \in im(f)$,

iii) R-isomorphism if it is both an R-monomorphism and an R-epimorphism.

Unsurprisingly, we have:

Lemma 2.4: *A homomorphism between nilpotent groups, f, is an R-monomorphism/R-epimorphism/R-isomorphism iff f_R is a monomorphism/epimorphism/isomorphism, respectively.*

Proof. This is [1, Proposition 5.5.4]. $\qquad\square$

We will also use:

Lemma 2.5: *R-localisation preserves pullbacks.*

Proof. This is [1, Lemma 5.5.7]. □

Recall that our reference group, G, is $f\mathbb{Z}_T$-nilpotent. The next few results record some consequences of this, starting with the observation that G is finitely T-generated in the following sense:

Definition 2.6: *A nilpotent group G is said to be finitely T-generated if there exists a finite subset A of G such that, for every $g \in G$, there exists a T-number, t, such that g^t is in the subgroup generated by A.*

Lemma 2.7: *A nilpotent group G is finitely T-generated iff G_T is $f\mathbb{Z}_T$-nilpotent.*

Proof. Firstly, if G is finitely T-generated, then the images of a finite T-generating set for G give a finite T-generating set for G_T. Conversely, if G_T is finitely T-generated, we can assume that the finite T-generating set is contained in the image of G, by Lemma 2.1. Then, we can form a finite T-generating set for G by picking an element in the preimage of each element of the finite T-generating set for G. The fact that this is a finite T-generating set for G again follows from Lemma 2.1. So we can assume that G is T-local, and this case is already proved in [1, Proposition 5.6.5]. To sketch how the argument goes, if G is $f\mathbb{Z}_T$-nilpotent, it is straightforward use a central series to inductively show that G is finitely T-generated, using Lemma 2.1. Conversely, if G is finitely T-generated, then it is clear that $Ab(G) = \frac{G}{[G,G]}$ is an $f\mathbb{Z}_T$-module, and so we can use the epimorphisms $Ab(G) \otimes ... \otimes Ab(G) \to \frac{\Gamma^i G}{\Gamma^{i+1} G}$ onto the quotients of the lower central series, [9, Corollary 2.10], to conclude that the lower central series expresses G as an $f\mathbb{Z}_T$-nilpotent group. □

Lemma 2.8: *Let G be an $f\mathbb{Z}_T$-nilpotent group with reference diagram as in the introduction. Then:*
i) G is T-Noetherian; that is G satisfies the ascending chain condition for T-local subgroups,
ii) $\tilde{\pi}$ is a monomorphism,
iii) G_{T_i} has no $(T_i - S)$-torsion for all but finitely many i. Equivalently, ϕ_i is a monomorphism for all but finitely many i.

Proof. i) This follows in the abelian case from the fact that $\mathbb{Z}_T$ is Noetherian, and the general nilpotent case follows via induction up a central series.

ii) It suffices to prove that $\prod \phi_i : \prod G_{T_i} \to \prod G_S$ is an S-monomorphism. This will follow from iii) and the fact that each ϕ_i is an S-monomorphism,

iii) Let $P = \{p_1, ..., p_k\}$ be a finite set of prime numbers and define:

$$G^P = \{g \in G \mid g^p = 1 \text{ for some product p of primes in P}\}$$

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Then G^P is a T-local subgroup of G, by Lemma 2.1. Since G is T-Noetherian it follows that there is a finite set of primes Q such that if $g^n = 1$ for some $n \in \mathbb{N}$, then $g^q = 1$ for some product of primes in Q. Now suppose that T_i does not contain any primes in Q. If $a \in G_{T_i}$ is such that $a^s = 1$ for some product of primes in $(T_i - S)$, let t_1 be a T_i-number such that $a^{t_1} = \psi_i(g)$ for some $g \in G$. We have that $\psi_i(g^s) = 1$ and so there is a T_i-number t_2 such that $g^{st_2} = 1$. Since s is coprime to each of the primes in Q, it follows that $g^{t_2} = 1$ and, therefore, that $a^{t_1 t_2} = 1$. Since G_{T_i} is T_i-local, it follows that $a = 1$, as desired. $\qquad\square$

3 A double coset formula for the extended genus

The aim of this section is to show that if $\alpha = \prod \alpha_i$ is an automorphism of $\prod G_S$, then in the pullback diagram below:

$$
\begin{array}{ccc}
H & \xrightarrow{(\varphi_i)} & \prod G_{T_i} \\
\mu \downarrow & & \downarrow \prod \alpha_i \phi_i \\
G_S & \xrightarrow{\Delta} & \prod G_S
\end{array}
$$

φ_i is a T_i-localisation for all i iff α is S-bounded above in the following sense:

Definition 3.1: *An automorphism $\alpha = \prod \alpha_i \in \prod Aut(G_S)$ is said to be S-bounded above if there exists an S-number s such that for all i and for all $g_i \in G_{T_i}$, $\alpha_i^{-1} \phi_i(g_i^s) \in im(\phi_i)$.*

From this, the double coset formula for the extended genus will follow in the expected manner. We start with:

Lemma 3.2: *If φ_i is a T_i-localisation for all i, then α is S-bounded above.*

Proof. Let A be a finite T-generating set for G. Since the φ_i are T_i-localisations, μ is an S-localisation. It follows that there exists an S-number s such that for all $a \in A$, $\sigma(a^s) \in im(\mu) \subset im(\alpha_i \phi_i)$ for all i. Since $\psi_i(A)$ is a finite T_i-generating set for G_{T_i}, it follows from Lemma 2.1 that if $g_i \in G_{T_i}$, then $\phi_i(g_i^{s^d}) \in im(\alpha_i \phi_i)$, where $d = \frac{1}{2} c(c + 1)$, for c the nilpotency class of G. $\qquad\square$

For the reverse direction, we start with the following observation which does not require α to be S-bounded above:

Lemma 3.3: *μ is an S-monomorphism.*

Proof. By Lemma 2.8, $\prod \phi_i$ is an S-monomorphism, and, therefore, so is $\prod \alpha_i \phi_i$. The result follows since the pullback of an S-monomorphism is an S-monomorphism, by Lemmas 2.4 and 2.5. $\qquad\square$

Lemma 3.4: *If α is S-bounded above, then μ is an S-epimorphism, hence an S-localisation.*

Proof. If $x \in G_S$, then since σ is an S-localisation, there exists an S-number r such that $x^r \in im(\sigma) \subset im(\phi_i)$ for all i. Since α is S-bounded, there exists an S-number s such that $x^{rs} \in im(\alpha_i \phi_i)$ for all i. It follows that x^{rs} is in the image of μ by the definition of a pullback. $\qquad\square$

We now have:

Lemma 3.5: *If α is S-bounded above, then φ_i is a T_i-localisation for all i.*

Proof. If $h \in H$ and $\varphi_i(h) = 1$, then $\mu(h) = 1$ and so there exists an S-number s such that $h^s = 1$. Write s as a product of a T_i-number t and a product of primes in T_i, r. Then, if $j \neq i$, $\varphi_j(h^t) = 1$, since G_{T_j} is T_j-local and $T_i \cap T_j = S$. Clearly $\varphi_i(h^t) = 1$, so it follows that φ_i is a T_i-monomorphism.

Now suppose that $g_i \in G_{T_i}$ and let $x = \alpha_i \phi_i(g_i)$. Since μ is an S-localisation, there exists an S-number s and $h \in H$ such that $x^s = \mu(h)$. Write s as a product of a T_i-number t and a product of primes in T_i, r. If $j \neq i$, then the image of $\alpha_j \phi_j$ is a T_j-local subgroup of G_S and so $x^t \in im(\alpha_j \phi_j)$ for all $j \neq i$. Since x^t is also in $im(\alpha_i \phi_i)$, it follows that g_i^t is in the image of φ_i by the definition of a pullback. $\qquad\square$

In order to state a double coset formula for the extended genus, we need to show that $\prod \alpha$ is S-bounded above, for $\alpha \in Aut(G_S)$. In fact, we will prove the stronger result that $\prod \alpha$ is S-bounded, and the reader is invited to skip ahead and read the definition of an S-bounded automorphism in Definition 4.1.

Lemma 3.6: *If $\alpha \in Aut(G_S)$, then $\prod \alpha$ is S-bounded.*

Proof. Let A be a finite set of T-generators for G. Since σ is an S-localisation, there exists an S-number s such that for all $a \in A$, $\alpha\sigma(a^s), \alpha^{-1}\sigma(a^s) \in im(\sigma)$. Since, for all i, $\sigma = \phi_i \psi_i$ and $\psi_i(A)$ is a finite set of T_i generators for G_{T_i}, this implies, by Lemma 2.1, that for all $g_i \in G_{T_i}$, $\alpha\phi_i(g_i^{s^d})$ and $\alpha^{-1}\phi_i(g_i^{s^d}) \in im(\phi_i)$, where $d = \frac{1}{2}c(c+1)$ is independent of i. It follows that $\prod \alpha$ is S-bounded. $\qquad\square$

It is clear than an automorphism of the form $\prod_i \beta_i \in \prod_i Aut(G_{T_i})$ also induces an S-bounded automorphism of $\prod_i G_S$, and we can now prove:

Theorem 3.7: *The extended genus of G is in 1-1 correspondence with the double coset:*

$$Aut(G_S) \backslash Aut_{b.a.}(\textstyle\prod_i G_S) / \textstyle\prod_i Aut(G_{T_i})$$

where $Aut_{b.a.}(\prod_i G_S)$ is the monoid of automorphisms of the form $\prod_i \alpha_i$ which are S-bounded above. The correspondence sends an S-bounded above automorphism α to the pullback group of $\alpha \circ (\prod \phi_i)$ along Δ.

Proof. The fact that the map factors through the double coset follows from the commutative diagram, in which all vertical maps are isomorphisms:

$$G_S \xrightarrow{\Delta} \prod G_S \xleftarrow{\prod \alpha_i \phi_i} \prod G_{T_i}$$

$$\alpha \downarrow \qquad \downarrow \prod \alpha \qquad \downarrow \prod \beta_i$$

$$G_S \xrightarrow{\Delta} \prod G_S \xleftarrow[\prod \alpha\alpha_i(\beta_i)_S^{-1}\phi_i]{} \prod G_{T_i}$$

where $\prod \alpha_i \in Aut_{b.a.}(\prod_i G_S)$, $\alpha \in Aut(G_S)$ and, for every i, $\beta_i \in Aut(G_{T_i})$.

For surjectivity, if H is in the extended genus of G, then we can form a diagram:

$$H \xrightarrow{(\epsilon_i)} \prod G_{T_i}$$

$$\mu \downarrow \qquad \downarrow \phi \qquad \searrow^{\prod \phi_i}$$

$$G_S \xrightarrow{\omega_H} (\prod G_{T_i})_S \xrightarrow{\tilde{\pi}} \prod G_S$$

Here, each ϵ_i is a T_i-localisation, μ is a S-localisation, and ω_H is then defined as the localisation of (ϵ_i). By [1, Theorem 7.2.1ii)], the left hand square is a pullback, and, since $\tilde{\pi}$ is a monomorphism, by Lemma 2.8ii), so is the larger square with base $\tilde{\pi}\omega_H$. Now $\tilde{\pi}\omega_H \neq \Delta$, in general. Instead, it is the product of localisations of each ϵ_i - that is, $\tilde{\pi}\omega_H = (\alpha_i)$, where each α_i is an automorphism of G_S. Let $\alpha := \prod \alpha_i$. Rearranging the pullback, we see that H is isomorphic to the pullback of $\alpha^{-1} \circ \prod \phi_i$ along Δ, and α^{-1} is S-bounded above by Lemma 3.2.

For injectivity, suppose that $\alpha = \prod \alpha_i, \beta = \prod \beta_i$ are S-bounded above automorphisms and we have pullbacks:

$$P \xrightarrow{(\varphi_i)} \prod G_{T_i} \qquad\qquad P \xrightarrow{(\bar{\varphi}_i)} \prod G_{T_i}$$

$$\mu \downarrow \qquad \downarrow \prod \alpha_i \phi_i \qquad\quad \bar{\mu} \downarrow \qquad \downarrow \prod \beta_i \phi_i$$

$$G_S \xrightarrow{\Delta} \prod G_S \qquad\qquad G_S \xrightarrow{\Delta} \prod G_S$$

By uniqueness of localisations, there is an automorphism $\gamma = \prod \gamma_i$ of $\prod G_{T_i}$ such that $\gamma(\varphi_i) = (\bar{\varphi}_i)$. Therefore, since we only care about equivalence classes in the double coset we may assume that $\varphi_i = \bar{\varphi}_i$ for all i. Similarly, there is an automorphism γ' of G_S such that $\gamma'\mu = \bar{\mu}$, and so we can reduce to the case $\mu = \bar{\mu}$. Now, for all i, $\alpha_i\phi_i$ and $\beta_i\phi_i$ are both the unique factorisation of μ through φ_i. By uniqueness of factorisation through ϕ_i, we must have $\alpha_i = \beta_i$, as desired. $\square$

4 A double coset formula for the genus of G

The purpose of this section is to prove that if we restrict the map of Theorem 3.7 to the S-bounded automorphisms, defined as follows, then its image is precisely the genus of G.

Definition 4.1: *An automorphism* $\alpha = \prod \alpha_i \in \prod Aut(G_S)$ *is said to be S-bounded if there exists an S-*

number s such that for all $(g_i) \in \prod G_{T_i}$, $\alpha \circ (\prod \phi_i)(g_i^s) \in im(\prod(\phi_i))$ and $\alpha^{-1} \circ (\prod \phi_i)(g_i^s) \in im(\prod(\phi_i))$.

We begin with the following observation, which is also [1, Proposition 7.4.3]:

Lemma 4.2: *If I is a finite indexing set and H is a T-local nilpotent group such that H_{T_i} is finitely T_i-generated for all $i \in I$, then H is finitely T-generated.*

Proof. Let $H_0 \subset H_1 \subset ...$ be an ascending chain of T-local subgroups of H. For each i, let ψ_i denote a T_i-localisation of H and let H_j^i denote the T_i-local subgroup of H_{T_i} generated by $\psi_i(H_j)$. Choose an integer N such that $H_0^i \subset H_1^i \subset ...$ terminates at H_N^i for all i. Now let $n \geq N$; we claim that $H_n = H_N$. If $h \in H_n$, then there exists a T_i-number t_i and a $k \in H_N$ such that $\psi_i(h^{t_i}) = \psi_i(k)$. It follows that there is a T_i-number s_i such that $(h^{t_i} k^{-1})^{s_i} = 1$. Since the set of $g \in H$ such that there exists a T_i-number s such that $g^s \in H_N$ is a subgroup of G which contains $h^{t_i} k^{-1}$ and k, it follows that there is a T_i-number r_i such that $h^{r_i} \in H_N$. Now any common factor of each of the r_i lies outside of T and so there is a T-number r such that $h^r \in H_N$. Since H_N is T-local, it follows that $h \in H_N$ as desired. So H is T-Noetherian, which implies that H is finitely T-generated. $\square$

Now suppose that H is in the image of an S-bounded automorphism. We consider the finite subset, F, of I consisting of i such that ϕ_i is not a monomorphism. Then H fits into a diagram of the form:

$$
\begin{array}{ccc}
H & \longrightarrow & (\prod_{i \notin F} G_{T_i}) \times (\prod_{j \in F} G_{T_j}) \\
\downarrow & & \downarrow {\scriptstyle 1 \times (\prod(\alpha_j \phi_j))} \\
P & \longrightarrow (\prod_{i \notin F} G_{T_i}) \times (\prod_{j \in F} G_S) & \longrightarrow \prod_{i \notin F} G_{T_i} \\
\downarrow & & \downarrow {\scriptstyle (\prod(\alpha_i \phi_i)) \times 1} \qquad \downarrow \\
G_S & \xrightarrow[\Delta \times \Delta]{} (\prod_{i \notin F} G_S) \times (\prod_{j \in F} G_S) & \longrightarrow \prod_{i \notin F} G_S
\end{array}
$$

where $\alpha = \prod \alpha_i$ is S-bounded, and each of the squares is a pullback. Consider the localisation of the diagram at $T' = \cup_{i \notin F} T_i$. The groups in the bottom two rows are all T'-local. If $j \in F$, $T_j \cap T' = S$, so $\alpha_j \phi_j$ is a T'-localisation. It follows that P is a T'-localisation of H. In light of Lemma 4.2, if we want to show that H is finitely T-generated, it suffices to show that P is finitely T'-generated. Note also that P is the image of an S-bounded automorphism in the extended genus of $G_{T'}$. In this way we can reduce the next lemma to the case where ϕ_i is a monomorphism for all i.

Lemma 4.3: *If H is the image of an S-bounded automorphism, then H is finitely T-generated.*

Proof. As discussed above, we can reduce to the case where ϕ_i is a monomorphism for all i. Let $\alpha = \prod \alpha_i$ be an S-bounded automorphism such that we have a pullback square:

$$\begin{array}{ccc} H & \longrightarrow & \prod G_{T_i} \\ \downarrow & & \downarrow \prod \alpha_i \phi_i \\ G_S & \xrightarrow{\;\;\Delta\;\;} & \prod G_S \end{array}$$

Let K be the T-local subgroup of H consisting of pairs $(x, (g_i))$ with $x \in G_S, g_i \in G_{T_i}$, such that, for all i, $\alpha_i \phi_i(g_i) = x$ and $x \in im(\phi_i)$, say $x = \phi_i(a_i)$. Then there is an injective group homomorphism $K \to G$ sending $(x, (g_i))$ to $(x, (a_i))$. Since G is finitely T-generated so is K, and since α is S-bounded there exists an S-number s such that if $h \in H$, then $h^s \in K$. Consider a T-subnormal series for K:

$$K = K_0 \subset K_1 \subset ... \subset K_m = H$$

If we localise at T_i, then all of the groups in the chain become finitely T_i-generated. Moreover, $(\frac{K_{j+1}}{K_j})_{T_i}$ is a finitely T_i-generated nilpotent group such that if $k \in (\frac{K_{j+1}}{K_j})_{T_i}$, then $k^s = 1$. For all but finitely many i this implies that $(\frac{K_{j+1}}{K_j})_{T_i}$ is trivial. For the remaining i, $(\frac{K_{j+1}}{K_j})_{T_i}$ is finitely T_i-generated (in fact it is finite). Therefore, using the fracture square [1, Theorem 7.2.1ii)], we see that $\frac{K_{j+1}}{K_j}$ is finitely T-generated (in fact it is finite). Inductively, it follows that H is finitely T-generated (and K is a subgroup of finite index in H). $\qquad\square$

It remains to prove that every element of the genus is the image of an S-bounded automorphism. We start with the following observation:

Lemma 4.4: *If H is in the genus of G, then there is a finite subset F of I such that if $T^{'} = \cup_{i \notin F} T_i$, then $G_{T'} \cong H_{T'}$.*

Proof. By [10, Theorem I.3.3], since $G_S \cong H_S$, there is a finitely T-generated nilpotent group P equipped with S-isomorphisms $f : P \to G$ and $g : P \to H$. In fact, we just need to consider the pullback:

$$\begin{array}{ccc} P & \longrightarrow & G \\ \downarrow & & \downarrow \phi_S \\ H & \xrightarrow{\;\;\phi_S\;\;} & G_S \end{array}$$

to get the desired maps, where ϕ_S denotes a localisation at S. Since G, H and P are finitely T-generated, we can use Lemma 2.1 to show that there exists an S-number s such that if $p \in ker(f)$ or $p \in ker(g)$, then $p^s = 1$ and, if $g \in G, h \in H$, then $g^s \in im(f), h^s \in im(g)$. This implies that if we take $T^{'}$ to be the union of the T_i which don't contain any prime factors of s, then both f and g are $T^{'}$-isomorphisms, which implies the result. $\qquad\square$

We can now prove:

Lemma 4.5: *If H is in the genus of G, then H is the image of an S-bounded automorphism.*

Proof. Let F be a finite subset of I such that if $T' = \cup_{i \notin F} T_i$, then $H_{T'} \cong G_{T'}$. Let $\mu : H \to G_{T'}$ and $\epsilon : G \to G_{T'}$ be T'-localisations. Then there are unique factorisations of σ and ψ_i, for $i \notin F$, through ϵ; denote them by σ', ψ_i'. Note that $\phi_i \psi_i' = \sigma_i'$. Since H is finitely T-generated, we can form a global to local fracture square, [1, Theorem 7.2.1 iii)], of the form:

$$
\begin{array}{ccc}
H & \xrightarrow{(\psi_i'\mu) \times (\varphi_j)} & (\prod_{i \notin F} G_{T_i}) \times (\prod_{j \in F} G_{T_j}) \\
{\scriptstyle \sigma'\mu} \downarrow & & \downarrow {\scriptstyle (\phi_i) \times (\alpha_j \phi_j)} \\
G_S & \xrightarrow[\Delta \times \Delta]{} & (\prod_{i \notin F} G_S) \times (\prod_{j \in F} G_S)
\end{array}
$$

where φ_j is any T_j-localisation of H and $\alpha_j \in Aut(G_S)$. Since F is finite, $1 \times (\alpha_j)$ is S-bounded, which can be seen directly or via Lemma 5.2 of the next section, as desired. $\qquad\square$

We can now prove our double coset formula for the genus of G:

Theorem 4.6: *The genus of G is in 1-1 correspondence with the double coset:*

$$Aut(G_S) \backslash Aut_b(\textstyle\prod_i G_S) / \textstyle\prod_i Aut(G_{T_i})$$

where $Aut_b(\prod_i G_S)$ is the subgroup of automorphisms of the form $\prod_i \alpha_i$ which are S-bounded. The correspondence sends an S-bounded automorphism α to the pullback group of $\alpha \circ (\prod \phi_i)$ along Δ.

Proof. We have already shown that the correspondence is well-defined and surjective in Lemmas 4.3 and 4.5. It is injective by Theorem 3.7. $\qquad\square$

5 Relationship to the formal fracture square

So far we have phrased our results in terms of the fracture square [1, Theorem 7.2.1 iii)] with the diagonal map as the base. In this section, we investigate what happens if we try to define a double coset formula relative to the fracture square:

$$
\begin{array}{ccc}
G & \xrightarrow{(\psi_i)} & \prod G_{T_i} \\
{\scriptstyle \sigma} \downarrow & & \downarrow {\scriptstyle \phi} \\
G_S & \xrightarrow{\omega} & (\prod G_{T_i})_S
\end{array}
$$

It turns out that this fracture square only sees the genus of G, and not the whole of the extended genus. Recall that we have previously considered 'diagonal' automorphisms of $\prod G_S$ of the form $\prod \alpha_i$, for $\alpha_i \in Aut(G_S)$.

We first define the analogue of a diagonal automorphism in $Aut((\prod G_{T_i})_S)$:

Definition 5.1: $DAut((\prod G_{T_i})_S)$ *is the subgroup of* $Aut((\prod G_{T_i})_S)$ *consisting of automorphisms* α *such that, for every* $j \in I$, *under the identification of* $(\prod G_{T_i})_S$ *with* $G_{T_j} \times (\prod_{i \neq j} G_{T_i})_S$ *determined by* ϕ *and any* S-*localisation of* $G_{T_j} \times (\prod_{i \neq j} G_{T_i})$ *of the form* $\phi_j \times \phi'_j$, $\alpha = \alpha_j \times \beta$ *for some automorphisms* α_j *of* G_S *and* β *of* $(\prod_{i \neq j} G_{T_i})_S$.

Note that the subgroup of diagonal automorphisms, $DAut((\prod G_{T_i})_S)$, is independent of the choice of the collection $\{\phi'_j\}_{j \in I}$. Also, note that if α is a diagonal automorphism of $(\prod G_{T_i})_S$, then there is a commutative diagram:

$$
\begin{array}{ccc}
(\prod G_{T_i})_S & \xrightarrow{\tilde{\pi}} & \prod G_S \\
\alpha \downarrow & & \downarrow \prod \alpha_i \\
(\prod G_{T_i})_S & \xrightarrow{\tilde{\pi}} & \prod G_S
\end{array}
$$

Since $\tilde{\pi}$ is a monomorphism, it follows that there is an injective homomorphism $DAut((\prod G_{T_i})_S) \to \prod Aut(G_S)$. We will now show that the image of this map is the subgroup of S-bounded automorphisms of $\prod G_S$. It follows, from Lemma 3.6, that $Aut(G_S)$ defines a subgroup of $DAut((\prod G_{T_i})_S)$.

Lemma 5.2: *An automorphism* $\alpha \in \prod Aut(G_S)$ *is the image of a diagonal automorphism* β *iff* α *is* S-*bounded.*

Proof. First suppose that α is the image of a diagonal automorphism β. Let A be a finite set of T-generators for G. Since ϕ is an S-epimorphism, there exists an S-number s such that for all $a \in A$, $\beta\omega\sigma(a^s)$ and $\beta^{-1}\omega\sigma(a^s) \in im(\phi)$. Now $\psi_i(A)$ is a finite set of T_i generators for G_{T_i} and $im(\phi_i)$ is a T_i-local subgroup of G_S. It follows, by Lemma 2.1, that if $g_i \in G_{T_i}$ then $\alpha_i\phi_i(g_i^{s^d}), \alpha_i^{-1}\phi_i(g_i^{s^d}) \in im(\phi_i)$, where $d = \frac{1}{2}c(c+1)$, for c the nilpotency class of G. Since d is independent of i, it follows that α is S-bounded as desired.

Now suppose that α is S-bounded. Let $F \subset I$ be the finite subset of I such that ϕ_i is not a monomorphism for $i \in F$. If $i \notin F$, let H_i be the subgroup of G_{T_i} consisting of g_i such that $\alpha_i\phi_i(g_i) \in im(\phi_i)$. Define unique homomorphisms, f_i, such that the following square commutes:

$$
\begin{array}{ccc}
\prod_{i \notin F} H_i & \xrightarrow{\prod f_i} & \prod_{i \notin F} G_{T_i} \\
(\prod \phi_i) \circ \iota \downarrow & & \downarrow \prod \phi_i \\
\prod_{i \notin F} G_S & \xrightarrow{\prod \alpha_i} & \prod_{i \notin F} G_S
\end{array}
$$

where ι is the inclusion of $\prod H_i$ into $\prod G_{T_i}$. Since α is S-bounded, ι is an S-isomorphism. Similarly, the image of the monomorphism $f := \prod f_i$ is $\{(g_i) \mid \forall i \; \alpha_i^{-1}\phi_i(g_i) \in im(\phi_i)\}$ and so, since α is S-bounded, f

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is also an S-isomorphism. Let $\phi_{I/F}$ be an S-localisation of $\prod_{i \notin F} G_{T_i}$, so there is an induced isomorphism $(\prod G_{T_i})_S \cong_\tau \prod_{i \in F} G_S \times (\prod_{i \notin F} G_{T_i})_S$ induced by ϕ and $(\prod_{i \in F} \phi_i) \times \phi_{I/F}$. Since the vertical arrows in the diagram below are S-localisations, there is a unique map f_S making the diagram commute:

$$
\begin{array}{ccc}
\prod_{i \notin F} H_i & \xrightarrow{\;\;f\;\;} & \prod_{i \notin F} G_{T_i} \\
{\scriptstyle \phi_{I/F}\iota}\downarrow & & \downarrow{\scriptstyle \phi_{I/F}} \\
(\prod_{i \notin F} G_{T_i})_S & \xrightarrow[\;\;f_S\;\;]{} & (\prod_{i \notin F} G_{T_i})_S
\end{array}
$$

Since f is an S-isomorphism, $(\prod \alpha_i) \times f_S$ defines an automorphism of $\prod_{i \in F} G_S \times (\prod_{i \notin F} G_{T_i})_S$. Noting that the S-localisation of f_i with respect to $\phi_i \circ \iota_i$ and ϕ_i is α_i, it follows that if we define $\beta \in Aut((\prod G_{T_i})_S)$ to correspond to $(\prod \alpha_i)_{i \in F} \times f_S$ under the isomorphism, τ, given above, then β is a diagonal automorphism whose image is α. $\qquad\square$

It is now an easy matter to reformulate our double coset formula for the genus of G in terms of the formal fracture square:

Theorem 5.3: *There is a 1-1 correspondence between the genus of G and the double coset:*

$$Aut(G_S) \setminus DAut((\prod G_{T_i})_S) \;/\; \prod Aut(G_{T_i})$$

The correspondence sends a diagonal automorphism α to the pullback group of $\alpha\phi$ along ω.

Proof. By Theorem 4.6 and Lemma 5.2, it is immediate that there is a 1-1 correspondence between the double coset and the genus of G, sending α to the pullback group of $\prod \alpha_i \phi_i$ along Δ. This is equivalent to sending α to the pullback group of $\alpha\phi$ along ω, since $\tilde{\pi}$ is a monomorphism. $\qquad\square$

Our final result tells us that a nilpotent group, H, in the extended genus of G, is finitely T-generated iff the S-localisation of the map $H \to \prod G_{T_i}$ is equivalent to ω. To make this precise, we have:

Definition 5.4: *Define $Orb(G_S, (\prod G_{T_i})_S)$ to be the set of orbits of $Hom(G_S, (\prod G_{T_i})_S)$ under the action of the group $Aut(G_S) \times DAut((\prod G_{T_i})_S)$.*

If $E(G)$ denotes the extended genus of G, then we have a map $L : E(G) \to Orb(G_S, (\prod G_{T_i})_S)$ defined by sending H to the S-localisation of some product of T_i-localisations, $(\epsilon_i) : H \to \prod G_{T_i}$, with respect to some S-localisation $\mu : H \to G_S$, and ϕ. By definition, $L(H)$ is independent of the choices of μ and ϵ_i. We have:

Lemma 5.5: *The genus of G is equal to $L^{-1}(Orb(\omega))$.*

Proof. If $H \in L^{-1}(Orb(\omega))$, then the fracture square, [1, Theorem 7.2.1], exhibits H as the pullback of $\alpha\phi$ along ω, for some diagonal automorphism α. So H is in the genus of G, by Theorem 5.3. Conversely, if H is

in the genus of G, then, by Theorem 5.3, we can view H as the pullback of $\alpha\omega$ along ϕ, for some diagonal automorphism α, so $L(H) = Orb(\omega)$. $\qquad\square$

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