Simplicial Spaces and Fibrations

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1 Simplicial Spaces and Fibrations

Since geometric realisation of simplicial spaces preserves finite limits, it would not be too surprising if geometric realisation takes 'locally trivial' maps of simplicial spaces to locally trivial Hurewicz fibrations. The purpose of this article is to prove such a theorem, which reads as follows:

Theorem 1.0.1: *If $p : X \to Y$ is a locally F-trivial map of simplicial spaces, and Y is a proper simplicial space, then $|p|$ is a Hurewicz fibration.*

The proof we give is inspired by the proof of Goerss and Jardine that geometric realisation takes minimal fibrations to Hurewicz fibrations, [1, Ch. I. Theorem 10.9]. In that context, Theorem 10.9 is the key to showing that geometric realisation takes Kan fibrations to Serre fibrations, which in turn plays a crucial role in the derivation in [1] of the Quillen model structure on simplicial sets. As an application of Theorem 1.0.1, we recover the theorem:

Theorem 1.0.2: *If G is a topological group with a nondegenerate basepoint, then, for any spaces X and Y, $B(Y, G, X) \to B(Y, G, *)$ is a Hurewicz fibration.*

For a more direct proof, see [4, Theorem 8.2] and corollaries. Taking $Y = *$ and $X = G$, we can deduce that the orbit map $EG \to BG$ is a Hurewicz fibration, whenever G is a topological group with a nondegenerate basepoint.

1.1 CGWH spaces

We begin by recalling some results about the category of CGWH spaces that we will use in this article. Since these results are not the main focus of the article, we do not provide all of the proofs, instead referring to Strickland's notes on CGWH spaces, [2], for the details. Firstly, a useful criterion for recognising when a continuous bijection of CGWH spaces is a homeomorphism is:

Lemma 1.1.1: *If $f : X \to Y$ is a proper (the preimages of compact subspaces are compact) continuous bjiection between CGWH spaces, then it is a homeomorphism.*

Proof. This is [2, Proposition 3.17]. □

Lemma 1.1.2: *In the category of CGWH spaces, the pullback of a quotient map is a quotient map.*

Proof. This is [2, Proposition 2.36]. □

Lemma 1.1.3: *If we have a commutative diagram of CGWH spaces:*

$$\begin{tikzcd}
& A_1 \arrow[rr] \arrow[dl, "j"'] \arrow[dd, dotted] & & B_1 \arrow[dl] \arrow[dd] \\
X_1 \arrow[rr] \arrow[dd] & & Y_1 \arrow[dd] & \\
& A_0 \arrow[rr, dotted] \arrow[dl, dotted, "i"'] & & B_0 \arrow[dl] \\
X_0 \arrow[rr] & & Y_0 &
\end{tikzcd}$$

such that the bottom square is a pushout, the vertical squares are pullbacks, and the map $A_0 \to X_0$ is a closed inclusion, then the top square is a pushout.

Proof. Since i is a closed inclusion, the categorical pushout of i agrees with the usual quotient of $X_0 \sqcup B_0$, [3, pg.40], and, since closed inclusions are preserved by pullbacks, [2, Proposition 2.33], the pushout of j is also given by the usual quotient. Therefore, we have a pullback:

$$\begin{tikzcd}
X_1 \sqcup B_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\
X_0 \sqcup B_0 \arrow[r] & Y_0
\end{tikzcd}$$

where the bottom map is a quotient. Therefore, by Lemma 1.1.2, the top arrow is also a quotient map. There are two equivalence relations on $X_1 \sqcup B_1$ that we care about. The first identifies $u \sim_1 v$ if $u, v \in X_1 \sqcup B_1$ have the same image in Y_1. The second is the smallest equivalence relation $\sim_2$ generated by the relations $x_1 \sim_2 b_1$ whenever there exists some $a_1 \in A_1$ with images $x_1 \in X_1$ and $b_1 \in B_1$. We want to show both of these equivalence relations are equal. Since $x_1 \sim_1 b_1$ whenever there is such an a_1, we have that $u \sim_2 v \implies u \sim_1 v$. If $u \sim_1 v$, write u_0 and v_0 for the images of u and v in $X_0 \sqcup B_0$. Since their images in Y_0 are equal, there is a sequence $u_0 = e_0, e_1, ..., e_n = v_0$, with $n \geq 0$, of elements of $X_0 \sqcup B_0$ such that for every $i < n$, there is some $a_i \in A_0$ such that e_i and e_{i+1} are images of a_i. It follows that each e_i has the same image in Y_0, which corresponds to the image of y, where y is the image of u and v in Y_1. So (a_i, y) is a well-defined element of A_1, and both $(e_i, y) \in X_1 \sqcup B_1$ and $(e_{i+1}, y) \in X_1 \sqcup B_1$ are images of (a_i, y). So $u \sim_2 v$.

□

1.2 Locally trivial maps of spaces

Definition 1.2.1: *Let F be a space. We call a map $f : X \to Y$ locally F-trivial over a subspace A of Y if there is a homeomorphism $\phi_A : f^{-1}(A) \to A \times F$ over A:*

$$\begin{tikzcd}
f^{-1}(A) \arrow[rr, "\phi_A", "\cong"'] \arrow[dr, "f"'] & & A \times F \arrow[dl, "\pi_A"] \\
& A &
\end{tikzcd}$$

Clearly, we have:

Lemma 1.2.2: *If Y has a numerable, locally finite open cover, $\{U_i\}$, such that f is locally F-trivial over each U_i, then f is a Hurewicz fibration.*

Proof. See [3, pg. 51]. ☐

The next lemma is central to our main theorem:

Lemma 1.2.3: *Let F be a space, let $f : X \to Y$ be a map, and let A, B be subspaces of Y such that $A \subset B$, A is a retract of B, and f is locally F-trivial over A and B via homeomorphisms ϕ_A and ϕ_B respectively. Then f is locally F-trivial over B via a homeomorphism $\phi_B^{'}$ which agrees with ϕ_A on $f^{-1}(A)$.*

Proof. Form the diagram:

$$\begin{tikzcd}
f^{-1}(A) \arrow[r, "\phi_B"'] \arrow[d] \arrow[rr, bend left, "\phi_A"] & A \times F \arrow[r, "\sigma"'] \arrow[d] & A \times F \arrow[d] \\
f^{-1}(B) \arrow[r, "\phi_B"'] \arrow[d] & B \times F \arrow[r, "\epsilon"'] \arrow[d] & B \times F \arrow[d] \\
B \arrow[r, "1"'] & B \arrow[r, "1"'] & B
\end{tikzcd}$$

where $\sigma = \phi_A \phi_B^{-1}$ and $\epsilon(b, f) = (b, \pi_F \sigma(r(b), f))$. Then all horizontal arrows are homeomorphisms, and we can define $\phi_B^{'} = \epsilon \phi_B$. ☐

Corollary 1.2.4: *Let F be a space, $f : X \to Y$ be a map, and A, B closed subspaces of Y such that f is locally F-trivial over A and B. If $A \cap B$ is a retract of B, then f is locally F-trivial over $A \cup B$ via a homeomorphism $\phi_{A \cup B}$ which agrees with ϕ_A on $f^{-1}(A)$.*

As a quick application we have:

Lemma 1.2.5: *IF Y is a CW complex, and $f : X \to Y$ is a map which is locally F-trivial over the images of every cell, $\epsilon(D^n)$, then f is a Hurewicz fibration.*

Proof. We can express Y as the transfinite composite of maps $Y_\lambda \to Y_{\lambda+1}$ where each $Y_{\lambda+1}$ is obtained from Y_λ by attaching a single cell. Given a pair of subspaces (A, U) of some Y_λ such that $U \subset A$, A is closed, U is open and numerable, and p is locally F-trivial over A, we can extend (A, U) to such a pair, $(\tilde{A}, \tilde{U})$, on Y by using open/closed collars around U and A respectively corresponding to half the radius of each attached cell, D^n. It follows by induction that $\tilde{A}$ is locally F-trivial, using Corollary 1.2.4, and that $\tilde{U}$ is numerable. If we single out the pairs (A, U) of the form $(\bar{B}(0, \frac{1}{2}), B(0, \frac{1}{2}))$ corresponding to interior balls of half the radius in each cell, then the corresponding cover of Y by the induced $\tilde{U}$ is a locally finite numerable open cover of Y such that p is locally F-trivial over each $\tilde{U}$. The result now follows from Lemma 1.2.2. $\square$

1.3 Locally trivial maps of simplicial spaces

We now move onto the main theorem, starting with the following definition:

Definition 1.3.1: *Let F be a simplicial space and $f : X \to Y$ a map of simplicial spaces. Define P_n to be the pullback:*

$$\begin{tikzcd} P_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ \underline{Y_n} \times \Delta^n \arrow[r] & Y \end{tikzcd}$$

where an underline denotes the constant simplicial space on the underlined space. We say that f is locally F-trivial if, for every $n \geq 0$, there is an homeomorphism of simplicial spaces $\underline{Y_n} \times \Delta^n \times F \to P_n$ over $\underline{Y_n} \times \Delta^n$:

$$\begin{tikzcd} \underline{Y_n} \times \Delta^n \times F \arrow[rr, "\cong"] \arrow[dr] & & P_n \arrow[dl] \\ & \underline{Y_n} \times \Delta^n & \end{tikzcd}$$

If $p : X \to Y$ is any map of simplicial spaces, then upon passage to geometric realisations we have a diagram:

$$\begin{tikzcd}
& P_\partial \arrow[rr] \arrow[dl] \arrow[dd, dotted, "q"] & & p^{-1}(F_{n-1}Y) \arrow[dl] \arrow[dd, "p"] \\
P_\Delta \arrow[rr] \arrow[dd, "q"] & & p^{-1}(F_nY) \arrow[dd, "p"] & \\
& s(Y_{n-1}) \times \Delta^n \cup Y_n \times \partial\Delta^n \arrow[rr, dotted, "\alpha"] \arrow[dl, dotted] & & F_{n-1}Y \arrow[dl] \\
Y_n \times \Delta^n \arrow[rr, "\epsilon"] & & F_nY &
\end{tikzcd}$$

where the bottom square is a pushout and the vertical squares are pullbacks. By Lemma 1.1.3, the top square is also a pushout, since $s(Y_{n-1}) \to Y_n$ is a closed inclusion. If p is locally F-trivial, then we have $P_\Delta \cong Y_n \times \Delta^n \times F$ over $Y_n \times \Delta^n$, and $P_\partial \cong (s(Y_{n-1}) \times \Delta^n \cup Y_n \times \partial\Delta^n) \times F$ over $s(Y_{n-1}) \times \Delta^n \cup Y_n \times \partial\Delta^n$. Recall that a simplicial space is proper if, for every n, $s(Y_{n-1}) \to Y_n$ is a Hurewicz cofibration. In this case, let (H, λ) denote $(Y_n \times \Delta^n, s(Y_{n-1}) \times \Delta^n \cup Y_n \times \partial\Delta^n)$ as an NDR-pair. Then, $H(-, 1)$ defines a retraction $r : \bar{\lambda}^{-1}([0, 1)) \to s(Y_{n-1}) \times \Delta^n \cup Y_n \times \partial\Delta^n$, where $\bar{\lambda}^{-1}([0, 1))$ denotes the closure of $\lambda^{-1}([0, 1))$. Our main theorem now states:

Theorem 1.3.2: *If $p : X \to Y$ is a locally F-trivial map of simplicial spaces, and Y is a proper simplicial space, then $|p|$ is a Hurewicz fibration.*

Proof. Suppose inductively that we have a locally finite numerable open cover $\{U_i\}_{i \in I}$ of $F_{n-1}Y$ such that p is locally F-trivial over each U_i. For each $i \in I$, let $V_i' = r^{-1}\alpha^{-1}(U_i) \cap \lambda^{-1}((0, 1))$, where r, λ and α are as defined above. Let $W_i = \epsilon(V_i') \cup U_i$, which can be viewed as a collar around U_i. Then W_i is open in F_nY, since U_i is open in $F_{n-1}Y$. We'll show that p is locally F-trivial over W_i. By assumption, we have a trivialisation:

$$\begin{tikzcd}
p^{-1}(U_i) \arrow[rr, "\phi_{U_i}", "\cong"'] \arrow[dr] & & U_i \times F \arrow[dl] \\
& U_i &
\end{tikzcd}$$

Letting $U_i' = \alpha^{-1}(U_i)$, pulling back along α induces a trivialisation:

$$\begin{tikzcd}
q^{-1}(U_i') \arrow[rr, "\phi_{U_i'}", "\cong"'] \arrow[dr] & & U_i' \times F \arrow[dl] \\
& U_i' &
\end{tikzcd}$$

Since q is locally F-trivial over $V_i^{'}$, Lemma 1.2.4 implies that there is a trivialisation of q over $V_i^{'}$ which agrees with $\phi_{U_i^{'}}$ on $q^{-1}(U_i^{'})$, call it $\phi_{V_i^{'}}$:

$$\begin{tikzcd} q^{-1}(V_i^{'}) \arrow[rr, "\phi_{V_i^{'}}", "\cong"'] \arrow[dr] & & V_i^{'} \times F \arrow[dl] \\ & V_i^{'} & \end{tikzcd}$$

By Lemma 1.1.3, $p^{-1}(W_i)$ is the pushout of the maps $q^{-1}(U_i^{'}) \to q^{-1}(V_i^{'})$ and $q^{-1}(U_i^{'}) \to p^{-1}(U_i)$. Similarly, since left adjoints preserves colimits, $W_i \times F$ is the pushout of the maps $U_i^{'} \times F \to V_i^{'} \times F$ and $U_i^{'} \times F \to U_i \times F$. Therefore, $\phi_{V_i^{'}}$ and ϕ_{U_i} induce a homeomorphism ϕ_{W_i}:

$$\begin{tikzcd} p^{-1}(W_i) \arrow[rr, "\phi_{W_i}", "\cong"'] \arrow[dr] & & W_i \times F \arrow[dl] \\ & W_i & \end{tikzcd}$$

which shows that p is locally F-trivial over W_i, as desired. We now explain how to complete the inductive proof of the theorem. We define $W^{'} = \lambda^{-1}((\frac{1}{2}, 1]) \subset Y_n \times \Delta^n$ and let $W = \epsilon(W^{'})$. Then, W is an open subspace of F_nY and p is locally F-trivial over W, since $W^{'} \to W$ is a homeomorphism, via Lemma 1.1.3. It is clear that W is a numerable open subspace of F_nY. We also need to check that each W_i is a numerable subspace of F_nY. If $U_i = \mu_i^{-1}((0,1])$, for some $\mu_i : F_{n-1}Y \to I$, we can define $\nu_i : F_nY \to I$ using the map μ_i, and the map κ on $Y_n \times \Delta^n$ defined by:

$$\kappa(y,t) = \begin{cases} (1 - \lambda(y,t))\mu_i \alpha r(y,t), & \text{if} \quad (y,t) \in \bar{\lambda}([0,1)) \\ 0, \quad \text{if} \quad \lambda(y,t) = 1 \end{cases}$$

So $\kappa^{-1}((0,1]) = V_i^{'}$ and $\nu_i^{-1}((0,1]) = W_i$. Now observe that $\{W_i\}_{i \in I}$ along with W is a locally finite numerable cover of F_nY, and if $y \in F_{n-1}Y$ has an open neighbourhood P in $F_{n-1}Y$ intersecting U_j only if $j \in J \subset I$, then there exists an open neighbourhood Q of y in F_nY which intersects W_j only if $j \in J$ and doesn't intersect W. Moreover, we can take $Q \cap F_{n-1}Y = P$. Note also that $W_i \cap F_{n-1}Y = U_i$, and ϕ_{W_i} agrees with ϕ_{U_i} on $p^{-1}(U_i)$. Therefore, we can iterate this procedure along the sequential colimit of the maps $F_iY \to F_{i+1}Y$, and we will end up with a numerable locally finite open cover of Y, such that p is locally F-trivial over each open set in the cover. It follows that p is a Hurewicz fibration by Lemma 1.2.2. $\square$

Finally, we apply Theorem 1.3.2 to prove that the orbit map $EG \to BG$ is a Hurewicz fibration, whenever G is a group with a nondegenerate identity element, which we define to be the basepoint. It is straightforward to show that this condition ensures that $B(Y,G,X)$ is a proper simplicial space, [4, Proposition 7.1]. We now prove:

Theorem 1.3.3: *If G is a topological group with a nondegenerate basepoint, then, for any spaces X and Y, $B(Y,G,X) \to B(Y,G,*)$ is a Hurewicz fibration.*

Proof. We will show that the corresponding map of simplicial spaces is X-locally trivial. We have a commutative square:

$$\begin{array}{ccc} \underline{B(Y,G,X)}_n \times \Delta^n & \longrightarrow & B(Y,G,X) \\ \downarrow & & \downarrow \\ \underline{B(Y,G,*)}_n \times \Delta^n & \longrightarrow & B(Y,G,*) \end{array}$$

and, therefore, we have an induced map of simplicial spaces $\phi : \underline{B(Y,G,X)}_n \times \Delta^n \cong \underline{B(Y,G,*)}_n \times X \times \Delta^n \to P_n$ over $\underline{B(Y,G,*)}_n \times \Delta^n$. It suffices to show that ϕ is a homeomorphism. A generic element of $\underline{B(Y,G,*)}_n \times X \times \Delta^n_m$ is of the form $\alpha = (\tilde{y}, \tilde{g_1}, ..., \tilde{g}_n, \tilde{x}, \tilde{\partial})$, where $\tilde{\partial}$ is a morphism from $\mathbf{m} \to \mathbf{n}$ in Δ. A generic element of $(P_n)_m$ is of the form $\beta = ((y, g_1, ..., g_n, \partial), (y^{'}, g_1^{'}, ..., g_m^{'}, x^{'}))$, where ∂ is a morphism $\mathbf{m} \to \mathbf{n}$ in Δ and $\partial(y, g_1, ..., g_n) = (y^{'}, g_1^{'}, ..., g_m^{'})$ in $B(Y,G,*)$. Now $\beta = \phi(\alpha)$ iff $\tilde{y} = y, \tilde{g_1} = g_1, ..., \tilde{g_n} = g_n, \tilde{\partial} = \partial$ and $\tilde{x} = h^{-1}x^{'}$ where h is a product of the g_i which depends on ∂. It follows that for every β there is a unique α such that $\phi(\alpha) = \beta$, so ϕ is a continuous bijection. We will show that ϕ is proper, and then we will be done by Lemma 1.1.1. Using the existence of inverses, and the fact that Δ^n_m is a discrete space, we have that $\underline{B(Y,G,X)}_n \times \Delta^n \to B(Y,G,X) \times \underline{B(Y,G,*)}_n \times \Delta^n$ is the inclusion of a retract. It follows that if A is compact in $(P_n)_m$, then the closed subspace $\phi^{-1}(A)$ is contained within a compact subspace, and therefore is compact itself. □

When G is a topological monoid with a nondegenerate basepoint, it is straightforward to check that $B(Y,G,X) \to B(Y,G,*)$ is not necessarily a Hurewicz fibration. However, as explained by May in [5], it is possible to use an inductive argument of a similar nature to our proof of Theorem 1.3.2 to prove:

Theorem 1.3.4: *If G is a grouplike topological monoid with a nondegenerate basepoint, then $B(Y,G,X) \to B(Y,G,*)$ is a quasifibration.*

Proof. See [5]. □

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