Random walks and Bose gas

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Contents

1 Introduction

This project consists of three independent parts. The first one, Section 2, focuses on self-avoiding random walks, and the so-called bubble condition. Roughly, this is a simplified version of the lace expansion, and can be used to deduce the asymptotics of the susceptibility function for the strictly self-avoiding random walk. The main source for the first chapter is [Sla06].

Section 3 explains how a partition function, generated by a Hamiltonian of the Bose gas, can be represented as an integral over the paths of continuous-time random walks. This is the so-called Feynman-Kac representation. The main step in deriving this representation is expanding the exponential of the Laplacian operator (the latter is a term in the Hamiltonian) using the Lie-Trotter product formula. The walks that generate the paths interact with each other; in particular, they are weakly self-avoiding. There is hope that the Feynman-Kac representation might expand the set of techniques we can use to analyse the Bose gas. One of these is the bubble condition from Chapter 2, if generalised to the weakly self-avoiding random walks (this means $0 < \lambda < 1$). Another technique we might be able to use is the continuous-time lace expansion as presented in [BHH21].

Finally, Section 4 explores how the solutions to the heat, Laplace's and Poisson's equation can be expressed as functionals of the Brownian motion. For the heat equation, this is first done in $\mathbb{R}^n$, and then in any open domain $\Omega \subset \mathbb{R}^n$. The main source for this chapter is [Dur96].

2 Self-avoiding random walks

The main reference for this chapter is [Sla06]. A good suplementary source is [MS13].

Fix a finite set $\Omega \subset \mathbb{Z}^d$ that is invariant under the permutation of coordinates and transformations $x_i \mapsto -x_i$ for any $i = 1, \dots, d$. This will be the set of allowed steps. A random walk of length n or an n-step random walk is an $(n+1)$-tuple $\omega = (\omega(0), \omega(1), \dots, \omega(n))$ such that $\omega(i) - \omega(i-1) \in \Omega$ for all $i = 1, \dots, n$.

For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{Z}^d$ let $\mathcal{W}_n(y, x)$ and $\mathcal{S}_n(y, x)$ denote the set of all n-step random walks and n-step (strictly) self-avoiding random walks, respectively, starting at y and ending at x. For simplicity, denote $\mathcal{W}_n(x) = \mathcal{W}_n(0, x)$ and $\mathcal{S}_n(x) = \mathcal{S}_n(0, x)$. Let $\mathcal{W}(y, x)$ be the set of all random walks starting at y and ending at x, with no restriction on their length; that is $\mathcal{W}(y, x) = \cup_{n=0}^{\infty} \mathcal{W}_n(y, x)$. We define analogously the sets $\mathcal{W}(x)$, $\mathcal{S}(y, x)$, and $\mathcal{S}(x)$.

For any random walk ω and any $\lambda \in [0, 1]$ define

$$
\begin{aligned}
U_{st}(\omega) &= \begin{cases} -1 & \omega(s) = \omega(t), \\ 0 & \text{otherwise}, \end{cases} \\
\vartheta_\lambda(\omega) &= \prod_{0 \le s < t \le |\omega|} (1 + \lambda U_{st}), \\
c_n^{(\lambda)}(x) &= \sum_{\omega \in \mathcal{W}_n(x)} \vartheta_\lambda(\omega), \\
c_n^{(\lambda)} &= \sum_{x \in \mathbb{Z}^d} c_n^{(\lambda)}(x), \\
G_z^{(\lambda)}(x) &= \sum_{n=0}^{\infty} c_n^{(\lambda)}(x) z^n && \textit{Green's function}, \\
\chi_\lambda(z) &= \sum_{x \in \mathbb{Z}^d} G_z^{(\lambda)}(x) = \sum_{n=0}^{\infty} c_n^{(\lambda)} z^n && \textit{susceptibility}, \\
B_\lambda(z) &= \sum_{x \in \mathbb{Z}^d} \left(G_z^{(\lambda)}(x) \right)^2.
\end{aligned}
$$

Note that $c_n^{(1)}(x) = \sum_{\omega \in \mathcal{W}_n(x)} \vartheta_1(\omega) = \sum_{\omega \in \mathcal{S}_n(x)} 1$. Denote the radius of convergence of $G_z^{(\lambda)}(x)$ by $z_c^{(\lambda)}$. In any of these definitions that depend on λ, we will omit λ when $\lambda = 1$.

The value $\lambda = 0$ corresponds to the simple random walks, values $\lambda \in (0, 1)$ correspond to the weakly self-avoiding random walks, and value $\lambda = 1$ corresponds to the (strictly) self-avoiding random walks. For $\lambda \in \{0, 1\}$, the quantities $c_n^{(\lambda)}(x)$ and $c_n^{(\lambda)}$ are the number of n-step walks starting at the origin and ending at x, or with no prescribed ending, respectively.

Since any walk with $m+n$ steps can be split into two subwalks, one with m and one with n steps, it follows that

$$c_{m+n}^{(\lambda)} \leq c_m^{(\lambda)} c_n^{(\lambda)},$$

that is the sequence $\left(c_n^{(\lambda)}\right)_{n\in\mathbb{N}}$ is submultiplicative. This implies that the sequence $\left(\left(c_n^{(\lambda)}\right)^{1/n}\right)_{n\in\mathbb{N}}$ is convergent, and

$$\lim_{n\to\infty} \left(c_n^{(\lambda)}\right)^{1/n} = \inf_{n\in\mathbb{N}} \left(c_n^{(\lambda)}\right)^{1/n} =: \mu_\lambda. \tag{1}$$

This is a standard result. For proof see Lemma B.5 in Section B.1.3 of [FV18], and note that a sequence $(a_n)_{n\in\mathbb{N}}$ of positive numbers is submultiplicative if and only if the sequence $(\log a_n)_{n\in\mathbb{N}}$ is subadditive. From (1) we obtain $z_c^{(\lambda)} = \mu_\lambda^{(-1)}$.

2.1 The bubble condition

In this section we mostly focus on the case $\lambda = 1$. We will denote $B(z_c) = B_1(z_c)$, $\chi(z) = \chi_1(z)$, and $z_c = z_c^{(1)}$. We will show that the assumption $B(z_c) < \infty$ (the *bubble condition*) implies

$$\chi(z) \asymp \left(1 - \frac{z}{z_c}\right)^{-1} \tag{2}$$

for $z \in [0, z_c)$, where $f(x) \asymp g(x)$ means there exists a constant $c > 0$ such that

$$\frac{1}{c}\, g(x) \leq f(x) \leq c\, g(x).$$

We hope the result can be generalised to any $\lambda \in [0,1]$, so parts of the proof will be done for any $\lambda \in [0,1]$. We will prove (2) in two steps: we will derive a lower and an upper bound for $\chi(z)$. The former is fairly simple, and will be done for any $\lambda \in [0,1]$. The latter is much more involved, and will be done for $\lambda = 1$, although we will show at the end how a part of the derivation can be done for any $\lambda \in [0,1]$.

To establish a lower bound, we show that for all $\lambda \in [0,1]$ and $z \in \left[0, z_c^{(\lambda)}\right)$,

$$\left(1 - \frac{z}{z_c^{(\lambda)}}\right)^{-1} \leq \chi_\lambda(z). \tag{3}$$

To see that, recall that $z_c^{(\lambda)} = \mu_\lambda^{(-1)}$, and note that equation (1) implies $c_n^{(\lambda)} \geq \mu_\lambda^n$. Therefore,

$$\chi_\lambda(x) = \sum_{n=0}^{\infty} c_n^\lambda z^n \geq \sum_{n=0}^{\infty} \left(z_c^{(\lambda)}\right)^{-n} z^n = \left(1 - \frac{z}{z_c^{(\lambda)}}\right)^{-1}.$$

Let now $\lambda = 1$. We will prove the inequality

$$\chi(z) \leq 2B(z_c)\left(1 - \frac{z}{z_c}\right)^{-1}. \tag{4}$$

Together with (3) this will establish the asymptotics in (2). First, define

$$Q(z) = \frac{\mathrm{d}}{\mathrm{d}z}(z\chi(z)). \tag{5}$$

We will show that

$$\frac{\chi(z)^2}{B(z)} \leq Q(z). \tag{6}$$

Expand $Q(z)$:

$$\begin{aligned} Q(z) &= \chi(z) + z\chi'(z) \\ &= \sum_{n=0}^{\infty} c_n z^n + \sum_{n=0}^{\infty} n c_n z^n \\ &= \sum_{y\in\mathbb{Z}^d} \sum_{\omega\in\mathcal{S}(y)} (|\omega| + 1) z^{|\omega|}. \end{aligned} \tag{7}$$

Recall that the length of a walk ω is $|\omega| + 1$. For a self-avoiding walk, this is exactly the number of all the distinct vertices it visits. Thus, we can write:

$$Q(z) = \sum_{y\in\mathbb{Z}^d} \sum_{\omega\in\mathcal{S}(y)} \sum_{x\in\mathbb{Z}^d} \mathbb{1}[\omega(j) = x \text{ for some } j] z^{|\omega|}.$$

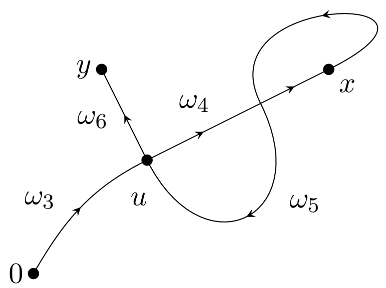

Figure 1: The walk from 0 to y split into 4 legs. Walk ω_4 and ω_5 intersect at least twice: at u and x.

Split each walk into two walks meeting at x:

$$Q(z) = \sum_{x,y\in\mathbb{Z}^d} \sum_{\substack{\omega_1\in\mathcal{S}(0,x)\\ \omega_2\in\mathcal{S}(x,y)}} z^{|\omega_1|} z^{|\omega_2|} \mathbb{1}[\omega_1 \cap \omega_2 = \{x\}] \tag{8}$$

$$= \sum_{x,y\in\mathbb{Z}^d} \sum_{\substack{\omega_1\in\mathcal{S}(0,x)\\ \omega_2\in\mathcal{S}(x,y)}} z^{|\omega_1|} z^{|\omega_2|} \left(1 - \mathbb{1}[\omega_1 \cap \omega_2 \neq \{x\}]\right)$$

$$= \chi(z)^2 - \sum_{x,y\in\mathbb{Z}^d} \sum_{\substack{\omega_1\in\mathcal{S}(0,x)\\ \omega_2\in\mathcal{S}(x,y)}} z^{|\omega_1|} z^{|\omega_2|} \mathbb{1}[\omega_1 \cap \omega_2 \neq \{x\}]. \tag{9}$$

Denote the second term by S. Since ω_1 and ω_2 both contain the point x, the indicator $\mathbb{1}[\omega_1 \cap \omega_2 \neq \{x\}]$ forces them to intersect at least twice. Let $u \in \mathbb{Z}^d$ be their last intersection as measured from ω_2. Note that $u \neq x$. Split ω_1 and ω_2 at u, and denote by ω_3, ω_4, ω_5, and ω_6 the first and second legs (subwalks) of ω_1 and ω_2, respectively. See figure 2.1. The sum S can then be rewritten as:

$$S = \sum_{\substack{x,y,u\in\mathbb{Z}^d\\ u\neq x}} \sum_{\substack{\omega_3\in\mathcal{S}(0,u)\\ \omega_4\in\mathcal{S}(u,x)\\ \omega_5\in\mathcal{S}(x,u)\\ \omega_6\in\mathcal{S}(u,y)}} z^{|\omega_3|+|\omega_4|+|\omega_5|+|\omega_6|} I(\omega_3,\omega_4,\omega_5,\omega_6),$$

where

$$I(\omega_3,\omega_4,\omega_5,\omega_6) = \mathbb{1}[\omega_3 \cap \omega_4 = \omega_3 \cap \omega_5 = \omega_3 \cap \omega_6 = \omega_4 \cap \omega_6 = \omega_5 \cap \omega_6 = \{u\}].$$

Ignore all of the mutual avoidances of the newly created walks, except for the mutual avoidance of ω_3 and ω_6. Algebraically, this corresponds to the inequality

$$I(\omega_3,\omega_4,\omega_5,\omega_6) \leq \mathbb{1}[\omega_3 \cap \omega_6 = \{u\}].$$

Substitute into the formula for S:

$$
\begin{aligned}
S &\leq \sum_{\substack{x,y,u\in\mathbb{Z}^d \\ u\neq x}} \sum_{\substack{\omega_3\in\mathcal{S}(0,u) \\ \omega_4\in\mathcal{S}(u,x) \\ \omega_5\in\mathcal{S}(x,u) \\ \omega_6\in\mathcal{S}(u,y)}} z^{|\omega_3|+|\omega_4|+|\omega_5|+|\omega_6|} \mathbb{1}[\omega_3\cap\omega_6=\{u\}] \\
&= \sum_{y,u\in\mathbb{Z}^d} \sum_{\substack{\omega_3\in\mathcal{S}(0,u) \\ \omega_6\in\mathcal{S}(u,y)}} z^{|\omega_3|+|\omega_6|} \mathbb{1}[\omega_3\cap\omega_6=\{u\}] \sum_{\substack{x\in\mathbb{Z}^d \\ x\neq u}} \sum_{\omega_4\in\mathcal{S}(u,x)} z^{|\omega_4|} \sum_{\omega_5\in\mathcal{S}(x,u)} z^{|\omega_5|}.
\end{aligned} \tag{10}
$$

Since the Ω (set of neighbours) is invariant with respect to the symmetry group of $\mathbb{Z}^d$,

$$
\sum_{\omega\in\mathcal{S}(x,u)} z^{|\omega|} = \sum_{\omega\in\mathcal{S}(u,x)} z^{|\omega|} = \sum_{\omega\in\mathcal{S}(u-x)} z^{|\omega|} = \sum_{n=0}^{\infty} c_n(u-x)z^n = G_z(u-x).
$$

Substituting this into (10), and setting $v=u-x$ yields

$$
S \leq \sum_{y,u\in\mathbb{Z}^d} \sum_{\substack{\omega_3\in\mathcal{S}(0,u) \\ \omega_6\in\mathcal{S}(u,y)}} z^{|\omega_3|+|\omega_6|} \mathbb{1}[\omega_3\cap\omega_6=\{u\}] \sum_{\substack{v\in\mathbb{Z}^d \\ v\neq 0}} G_z(v)^2.
$$

We now notice that the double and single sum can be summed independently of each other. The former is of the same form as the sum in (8), so it sums up to $Q(z)$. Also, note that $G_z(0)=1$ since there is exactly one self-avoiding walk from 0 to 0: an empty walk. We thus obtain

$$
\begin{aligned}
S &\leq Q(z) \sum_{\substack{v\in\mathbb{Z}^d \\ v\neq 0}} G_z(v)^2 \\
&= Q(z)\left(\sum_{v\in\mathbb{Z}^d} G_z(v)^2 - G_z(0)^2\right) \\
&= Q(z)(B(z)-1).
\end{aligned}
$$

Substituting this into (9) gives

$$
Q(z) \geq \chi(z)^2 - Q(z)(B(z)-1).
$$

Rearranging yields (6).

We will now show how this inequality implies (4), assuming that $B(z_c)<\infty$. Let $z_1\in[0,z_c)$

be arbitrary but fixed. Recall that (by definition)

$$Q(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(z\chi(z)\right) = \chi(z) + z\chi'(z).$$

Divide by $\chi(z)^2$ and rearrange to obtain

$$z\frac{\chi'(z)}{\chi(z)^2} = \frac{Q(z)}{\chi(z)^2} - \frac{1}{\chi(z)}.$$

By (6),

$$z\frac{\chi'(z)}{\chi(z)^2} \geq \frac{1}{B(z)} - \frac{1}{\chi(z)}. \tag{11}$$

Rewrite the left-hand side to get

$$z\left(-\frac{\mathrm{d}}{\mathrm{d}z}\chi^{-1}\right)(z) \geq \frac{1}{B(z)} - \frac{1}{\chi(z)},$$

where χ^{-1} is the reciprocal. Note that $\chi(z)$ and $B(z)$ are both increasing in $z \in [0, z_c)$. In particular, $\chi(z)^{-1}$ is decreasing, and so $-(\mathrm{d}/\mathrm{d}z)\chi^{(-1)}$ is non-negative. We can therefore conclude that

$$z_c\left(-\frac{\mathrm{d}}{\mathrm{d}z}\chi^{-1}\right)(z) \geq \frac{1}{B(z_c)} - \frac{1}{\chi(z_1)}$$

for $z \in [z_1, z_c]$. Now integrate with respect to z from z_1 to z_c (recall that $z_1 \in [0, z_c]$):

$$z_c\left(\frac{1}{\chi(z_1)} - \frac{1}{\chi(z_c)}\right) \geq \left(\frac{1}{B(z_c)} - \frac{1}{\chi(z_1)}\right)(z_c - z_1). \tag{12}$$

Note that $\chi(z)$ diverges to infinity as z increases to z_c, so $1/\chi(z_c) = 0$. Rearranging (12) gives

$$\begin{aligned}\frac{2z_c - z_1}{\chi(z_1)} &\geq \frac{z_c - z_1}{B(z_c)},\\ \chi(z_1) &\leq \frac{2z_c - z_1}{z_c - z_1}B(z_c)\\ &\leq \frac{2z_c}{z_c - z_1}B(z_c).\end{aligned} \tag{13}$$

Since the inequality holds for any $z_1 \in [0, z_c)$, we have derived the inequality

$$\chi(z) \leq \frac{2z_c}{z_c - z} B(z_c),$$

which is equivalent to (4).

It is worth mentioning that for any $\lambda \in [0, 1]$, $Q_\lambda(z)$ can be bounded from above by

$$Q_\lambda(z) \leq \chi_\lambda(z)^2, \tag{14}$$

where Q_λ is defined analogously to Q in (5):

$$Q_\lambda(z) = \frac{\mathrm{d}}{\mathrm{d}z}(z\chi_\lambda(z)).$$

Bound (14) implies that

$$z\frac{\chi'_\lambda(z)}{\chi_\lambda(z)^2} \leq 1 - \frac{1}{\chi_\lambda(z)};$$

compare to (11). By following the same procedure as from (11) to (13), one can derive

$$\begin{aligned} \chi_\lambda(z_1) &\geq \frac{2z_c - z_1}{z_c - z_1} \\ &\geq \frac{z_c}{z_c - z_1}. \end{aligned}$$

Since this again holds for all $z_1 \in [0, z_c)$, we have found an alternative way of establishing (3). To show (14) holds, first expand Q_λ as in (7). We get

$$Q_\lambda(z) = \sum_{y \in \mathbb{Z}^d} \sum_{\omega \in \mathcal{W}(y)} (|\omega| + 1)\vartheta_\lambda(\omega) z^{|\omega|}.$$

Express the length of a walk as the number of sites (not necessarily distinct) it visits:

$$Q_\lambda(z) = \sum_{y \in \mathbb{Z}^d} \sum_{\omega \in \mathcal{W}(y)} \sum_{x \in \mathbb{Z}^d} \sum_{j=0}^{|\omega|} \mathbb{1}[\omega(j) = x] \vartheta_\lambda(\omega) z^{|\omega|}$$

Similarly as before, we want to split each walk into two subwalks. But to do that, we first need to fix the length of the walk. We therefore write

$$\begin{aligned} Q_\lambda(z) &= \sum_{y\in\mathbb{Z}^d}\sum_{n=0}^{\infty}\sum_{\omega\in\mathcal{W}_n(y)}\sum_{x\in\mathbb{Z}^d}\sum_{j=0}^{n}\mathbb{1}[\omega(j)=x]\vartheta_\lambda(\omega)z^n \\ &= \sum_{x,y\in\mathbb{Z}^d}\sum_{n,j=0}^{\infty}\sum_{\omega\in\mathcal{W}_n(y)}\mathbb{1}[j\le n]\mathbb{1}[\omega(j)=x]\vartheta_\lambda(\omega)z^n. \end{aligned}$$

We now split each walk ω into two walks ω_1 and ω_2 meeting at x. Note that $\vartheta_\lambda(\omega) \le \vartheta_\lambda(\omega_1)\vartheta_\lambda(\omega_2)$. This yields:

$$\begin{aligned} Q_\lambda(z) &\le \sum_{x,y\in\mathbb{Z}^d}\sum_{n,j=0}^{\infty}\sum_{\substack{\omega_1\in\mathcal{W}_j(x)\\ \omega_2\in\mathcal{W}_{n-j}(x,y)}}\mathbb{1}[j\le n]\vartheta_\lambda(\omega_1)\vartheta_\lambda(\omega_2)z^j z^{n-j} \\ &= \sum_{x\in\mathbb{Z}^d}\sum_{j=0}^{\infty}\sum_{\omega_1\in\mathcal{W}_j(x)}\vartheta_\lambda(\omega_1)z^j\sum_{y\in\mathbb{Z}^d}\sum_{n=j}^{\infty}\sum_{\omega_2\in\mathcal{W}_{n-j}(x,y)}\vartheta_\lambda(\omega_2)z^{n-j}. \end{aligned}$$

Set $k=n-j$. Recall that due to the translational invariance of the random walks, summing over $\mathcal{W}_k(x,y)$ is equivalent to summing over $\mathcal{W}_k(y-x)$ in this case. Set $u=y-x$. As y ranges over $\mathbb{Z}^d$, and x is held fixed, u ranges over $\mathbb{Z}^d$. We therefore obtain the following:

$$\begin{aligned} Q_\lambda(z) &\le \sum_{x\in\mathbb{Z}^d}\sum_{j=0}^{\infty}\sum_{\omega_1\in\mathcal{W}_j(x)}\vartheta_\lambda(\omega_1)z^j\sum_{u\in\mathbb{Z}^d}\sum_{k=0}^{\infty}\sum_{\omega_2\in\mathcal{W}_k(u)}\vartheta_\lambda(\omega_2)z^k \\ &= \chi_\lambda(z)^2. \end{aligned}$$

This establishes (14).

3 The Feynman-Kac formula for Bose gas

In this section we show how the partition function for a Bose gas system (defined in (18) with the Hamiltonian defined in (17)) can be expressed as an integral over the paths of a continuous-time Markov walk; see (21). This kind of representation is known as the Feynman-Kac formula.

We first need to formally formulate our problem. Let Λ be a finite subset of $\mathbb{Z}^d$ (we denote this by $\Lambda \Subset \mathbb{Z}^d$). For any $n \in \mathbb{N}$ define $\mathcal{H}_{\Lambda,n} = l^2(\Lambda^n)$ and

$$\mathcal{H}_{\Lambda,n}^{(+)} = \left\{ \varphi \in l^2(\Lambda^n) : \varphi(x_1, \ldots, x_n) = \varphi\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right) \ \forall \sigma \in S_n \right\}. \tag{15}$$

Both are Hilbert spaces; the latter is the space of symmetric functions of n arguments. Note that $\dim \mathcal{H}_{\Lambda,n} = |\Lambda|^n$ and $\dim \mathcal{H}_{\Lambda,n}^{(+)} = \binom{|\Lambda|+n-1}{n}$. The latter is the number of combinations of n elements in Λ with repetitions. Finally, define

$$\mathcal{F}_{\Lambda}^{(+)} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_{\Lambda,n}^{(+)}.$$

For any $n \in \mathbb{N}$ and $i = 1, \ldots, n$ define the ith component discrete Laplacian operator on $\mathcal{H}_{\Lambda,\varphi}$ and $\mathcal{H}_{\Lambda,\varphi}^{(+)}$ by

$$\Delta_i \varphi(x_1, \ldots, x_n) = \sum_{y \sim x_i} \left(\varphi(x_1, \ldots, y, \ldots, x_n) - \varphi(x_1, \ldots, x_i, \ldots, x_n)\right), \tag{16}$$

where $y \sim x$ means y and x are neighbours, and only the ith argument of φ changes in the definition above. The dependence on n will be left implicit. On the same spaces define an operator $\delta(i,j)$ by

$$\delta(i,j)\varphi(x_1, \ldots, x_n) = \delta_{x_i,x_j}\varphi(x_1, \ldots, x_n),$$

where δ_{x_i,x_j} is just the usual Kronecker delta. Finally, we define the Hamiltonian. Fix any τ, u, $\mu \in \mathbb{R}$ and define (on $\mathcal{H}_{\Lambda,\varphi}$ and $\mathcal{H}_{\Lambda,\varphi}^{(+)}$)

$$H_{\Lambda,n} = -\tau \sum_{i=1}^{n} \Delta_i + u \sum_{1 \le i < j \le n} \delta(i,j) - \mu n \,\mathrm{id}, \tag{17}$$

where id is the identity operator. We can now naturally define the Hamiltonian on $\mathcal{F}_\Lambda^{(+)}$ by

$$H_\Lambda = \bigoplus_{n\in\mathbb{N}} H_{\Lambda,n},$$

that is, for $\varphi \in \mathcal{F}_\Lambda^{(+)}$, we can write $\varphi = \bigoplus_{n\in\mathbb{N}} \varphi_n$ with $\varphi_n \in \mathcal{H}_{\Lambda,n}^{(+)}$, and

$$H_\Lambda \varphi = \bigoplus_{n\in\mathbb{N}} H_{\Lambda,n}\varphi_n.$$

The Hamiltonian H_Λ represents the energy of the Bose gas. Fix an inverse temperature $\beta > 0$. The partition function of the system is defined by

$$Z_{\Lambda,\beta} = \mathrm{Tr}_{\mathcal{F}_\Lambda^{(+)}} \mathrm{e}^{-\beta H_\Lambda} = \sum_{n\in\mathbb{N}} \mathrm{Tr}_{\mathcal{H}_{\Lambda,n}^{(+)}} \mathrm{e}^{-\beta H_{\Lambda,n}}, \tag{18}$$

where for a Hilbert space $\mathcal{H}$, $\mathrm{Tr}_{\mathcal{H}}$ denotes the trace with respect to any orthonormal basis.

We now show how $Z_{\Lambda,\beta}$ can be expressed as an integral over random paths. For every $x \in \Lambda$ denote by d_x the number of its neighbours. Define a continuous-time Markov walk X on Λ such that at each site $x \in \Lambda$, its jump intensity is d_x (meaning that the holding time of the walk in state x is distributed exponentially with parameter d_x), and its transition rate to a site $y \in \Lambda$ is 1 for neighbouring sites and 0 otherwise. The generator matrix for X is the negative of the standard Laplacian matrix for Λ (this can hint to why there even is a connection between the Hamiltonian and random walks). The infinitesimal behaviour of X satisfies the following: for every $s \geq 0$ and $t > 0$ small enough, and for all $x,\ y \in \Lambda$,

$$\mathbb{P}(X(s+t) = y \mid X(s) = x) = \begin{cases} 1 - d_x t + O(t^2)\,, & \text{if } y = x, \\ t + O(t^2)\,, & \text{if } y \sim x, \\ O(t^2)\,, & \text{otherwise.} \end{cases} \tag{19}$$

The term $O(t^2)$ comes from the probability of X jumping at least twice in time t; see Section 3.1. Note that $y \sim x$ means that x and y are neighbours.

Define $\mathcal{W}^{\beta\tau} = \{\omega : [0, \beta\tau] \to \Lambda\,;\ \omega \text{ càdlàg}\}$, i.e. every element of $\mathcal{W}^{\beta\tau}$ is a right-continuous path on Λ with left limits, and of length $\beta\tau$. Define $W^{\beta\tau}$ to be the probability measure on $\mathcal{W}^{\beta\tau}$

induced by the random walk X. The following formula gives an interpretation of $W^{\beta\tau}$:

$$W^{\beta\tau}(\{\omega \in \mathcal{W}^{\beta\tau}\,;\, \omega(t) = x\}) = \mathbb{P}(X(t) = x). \tag{20}$$

For any x, $y \in \Lambda$ define $W^{\beta\tau}_{x,y}$ to be the measure on $\mathcal{W}^{\beta\tau}$ that corresponds to the probability measure $W^{\beta\tau}$ conditioned on the event $A_x = \{\omega \in \mathcal{W}^{\beta\tau} : \omega(0) = x\}$, and restricted on the event $B_y = \{\omega \in \mathcal{W}^{\beta\tau} : \omega(\beta\tau) = y\}$. That is, for any $C \subset \mathcal{W}^{\beta\tau}$: $W^{\beta\tau}_{x,y}(C) = W^{\beta\tau}(C \cap B_y \mid A_x)$. Note that $\mathcal{W}_{x,y}$ is not a probability measure. Similarly, let $\mathbb{P}^{\beta\tau}_{x,y}$ be the probability measure $\mathbb{P}$ conditioned on $X(0) = x$ and restricted to $X(\beta\tau) = y$. Finally, define

$$\begin{aligned}\tilde{Z}_{\Lambda,\beta} = \sum_{n\in\mathbb{N}} \frac{\mathrm{e}^{\beta\mu n}}{n!} \sum_{x_1,\ldots,x_n} \sum_{\sigma\in S_n} \int_{\mathcal{W}^{\beta\tau}} \mathrm{d}W^{\beta\tau}_{x_1,x_{\sigma(1)}}(\omega_1) \cdots \\ \cdots \int_{\mathcal{W}^{\beta\tau}} \mathrm{d}W^{\beta\tau}_{x_n,x_{\sigma(n)}}(\omega_n) \exp\left(-u \sum_{1\le i<j\le n} \int_0^{\beta\tau} \mathrm{d}t \delta_{\omega_i(t),\omega_j(t)}\right),\end{aligned} \tag{21}$$

where $\omega_1, \ldots, \omega_n$ are paths of independent copies of X; denote these copies by $X_1, \ldots, X_n$. We claim that $\tilde{Z}_{\Lambda,\beta} = Z_{\Lambda,\beta}$. The rest of this section is devoted to the proof of this claim.

We will manipulate both (18) and (21) to show they are equal. We will use the Lie-Trotter product formula on $\exp(\sum_{i=1}^n \Delta_i)$ in (18) to interpret the action of the operator as generating discrete-time random walks. Similarly, we will discretise the time integral in (21) to express the integrals over the continuous-time random walks as sums over the discrete-time random walks.

We start with the latter. The time integral in (21) can be interpreted as a Riemann integral, meaning we can write

$$\int_0^{\beta\tau} \mathrm{d}t \delta_{\omega_i(t),\omega_j(t)} = \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m} \delta_{\omega_i(k\beta\tau/m),\omega_j(k\beta\tau/m)}.$$

Substitute this into (21) to obtain

$$\tilde{Z}_{\Lambda,\beta} = \sum_{n\in\mathbb{N}} \frac{\mathrm{e}^{\beta\mu n}}{n!} \sum_{x_1,\ldots,x_n} \sum_{\sigma\in S_n} \lim_{m\to\infty} I_{n,\boldsymbol{x},\sigma,m}, \tag{22}$$

where $\boldsymbol{x} = (x_1, \dots, x_n) \in \Lambda^n$, and

$$\begin{aligned} I_{n,\boldsymbol{x},\sigma,m} = \int_{\mathcal{W}^{\beta\tau}} \mathrm{d}W^{\beta\tau}_{x_1,x_{\sigma(1)}}(\omega_1) \cdots \\ \cdots \int_{\mathcal{W}^{\beta\tau}} \mathrm{d}W^{\beta\tau}_{x_n,x_{\sigma(n)}}(\omega_n) \exp\left(-\frac{u}{m} \sum_{1\le i<j\le n} \sum_{k=0}^{m} \delta_{\omega_i(k\beta\tau/m),\omega_j(k\beta\tau/m)} \right). \end{aligned} \tag{23}$$

Fix n, $\boldsymbol{x}$, σ, and m. The integrand in (23) only assumes finitely many values, meaning we can express the integral as a sum. Define $x_{i,k} \in \Lambda$ to be the location of X_i at time $k\beta\tau/m$ for $k = 0, \dots, m$ and $i = 1, \dots, n$. Write

$$\underline{\boldsymbol{x}} = \left(x_{i,k}\right)_{k=1,\dots,m}^{i=1,\dots,n} \in \Lambda^{mn}.$$

Then (23) simplifies to

$$\begin{aligned} I_{n,\boldsymbol{x},\sigma,m} = \sum_{\underline{\boldsymbol{x}}\in\Lambda^{mn}} \prod_{i=1}^{n} \mathbb{P}^{\beta\tau}_{x_i,x_{\sigma(i)}}\left(X_i(k\beta\tau/m) = x_{i,k} \text{ for } k = 1, \dots, m\right) \\ \cdot \exp\left(-\frac{u}{m} \sum_{1\le i<j\le n} \sum_{k=0}^{m} \delta_{x_{i,k},x_{j,k}} \right). \end{aligned} \tag{24}$$

Note that

$$\mathbb{P}^{\beta\tau}_{x_i,x_{\sigma(i)}}\left(X_i(0) = x_{i,0}\right) = \delta_{x_{i,0},x_i},$$

and

$$\mathbb{P}^{\beta\tau}_{x_i,x_{\sigma(i)}}\left(X_i(\beta\tau) = x_{i,m}\right) = \delta_{x_{i,m},x_{\sigma(i)}} \mathbb{P}\left(X_i(\beta\tau) = x_{i,m} \mid X_i(0) = x_i\right).$$

Before moving forward, note that since β and τ are fixed throughout, $O\left((\beta\tau/m)^2\right) = O(m^{-2})$. Since the walks $X_1, \dots, X_n$ are Markov, we have that for every $i = 1, \dots, n$, and every $k =$

$1, \ldots, m$ with m large enough,

$$\begin{aligned}
&\mathbb{P}^{\beta\tau}_{x_i, x_{\sigma(i)}}\big(X_i(k\beta\tau/m) = x_{i,k}\big) \text{ for } k = 1, \ldots, m\big) \\
&\qquad = \delta_{x_{i,m}, x_{\sigma(i)}} \prod_{k=1}^{m} \mathbb{P}\big(X_i(k\beta\tau/m) = x_{i,m} \mid X_i((k-1)\beta\tau/m) = x_{i,k-1}\big) \\
&\qquad \overset{(19)}{=} \delta_{x_{i,m}, x_{\sigma(i)}} \prod_{k=1}^{m} \begin{cases} 1 - d_{x_{i,k}} \frac{\beta\tau}{m} + O\big(m^{-2}\big)\,, & \text{if } x_{i,k} = x_{i,k-1}, \\ \frac{\beta\tau}{m} + O\big(m^{-2}\big)\,, & \text{if } x_{i,k} \sim x_{i,k-1}, \\ O\big(m^{-2}\big)\,, & \text{otherwise.} \end{cases}
\end{aligned} \tag{25}$$

Recall that the probability of any walk jumping more than once in a time interval of length $\beta\tau/m$ is $O\big(m^{-2}\big)$. Thus, equation (25) allows us to replace the continuous-time random walks $X_1, \ldots, X_n$ in (24) with n independent copies of a discrete-time random walk with m steps, and with transition probabilities given by the braces in (25). Denote this walk by X^m, and its copies by $X_1^m, \ldots, X_n^m$. The elements of Λ^{mn} now represent the possible paths of $(X_1^m, \ldots, X_n^m)$, that is all the possible permutations of paths of the n copies of X^m. For clarity, we will index the steps of X^m (and its copies) by $k\beta\tau/m$, where $k = 0, \ldots, m$. We say that X^m jumps at step $k\beta\tau/m$ for $0 \leq k \leq m-1$ if $X^m((k+1)\tau\beta/m) \neq X^m(k\beta\tau/m)$.

Substitute (25) into (24). We will expand the sum over Λ^{mn} and group the elements of Λ^{mn} by the number of jumps that the n copies of X^m perform.

Referring back to (25), for any $0 \leq k \leq m-1$, the probability of observing no jumps at step $k\beta\tau/m$, conditioned on $X_i(k\beta\tau/m) = x_{i,k}$ for $i = 1, \ldots, n$, is

$$\prod_{i=1}^{n} \left(1 - d_{x_{i,k}} \frac{\beta\tau}{m} + O\Big(m^{-2}\Big)\right) = 1 - \frac{\beta\tau}{m} \sum_{i=1}^{n} d_{x_{i,k}} + O\Big(m^{-2}\Big)\,. \tag{26}$$

For any $i = 1, \ldots, n$, and every $y \sim x_i$ the probability of observing exactly one jump at step $k\beta\tau/m$: the jump of X_i from x_i to y, conditioned on $X_i(k\beta\tau/m) = x_{i,k}$, is

$$\left(\frac{\beta\tau}{m} + O\Big(m^{-2}\Big)\right) \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(1 - d_{x_{j,k}} \frac{\beta\tau}{m} + O\Big(m^{-2}\Big)\right) = \frac{\beta\tau}{m} + O\Big(m^{-2}\Big)\,. \tag{27}$$

Sum over all neighbours $y \sim x_1$ and all $i = 1, \ldots, n$ to get that the probability of observing exactly one jump to a neighbouring site, and no other jumps at step $k\beta\tau/m$, conditioned on

$X_i(k\beta\tau/m) = x_{i,k}$ for $i = 1, \dots, n$, is

$$\sum_{i=1}^{n} \sum_{y \sim x_i} \frac{\beta\tau}{m} + O\left(m^{-2}\right) = \frac{\beta\tau}{m} \sum_{i=1}^{n} d_{x_{i,k}} + O\left(m^{-2}\right). \tag{28}$$

The probability of any other event (observing any number of jumps to non-neighbouring sites, or observing at least two jumps simultaneously) is of the magnitude $O(m^{-2})$. This coincides with the fact that the above two probabilities sum up to $1 + O(m^{-2})$. Note that the number of ways such events can occur (at any one step) depends only on n, and not on m.

We will now show that the probability of ever observing at least two simultaneous jumps, or at least one jump to a non-neighbouring site, vanishes as m approaches infinity. At any step the probability of observing any of the two is of the magnitude $O(m^{-2})$. Say we observe any of the two at l different steps for $1 \le l \le m$. The probability of this event is bounded from above by

$$\begin{aligned}\binom{m}{l} O\left(m^{-2}\right)^{l} &= \frac{1}{l!}\, m\, O\left(m^{-2}\right) \cdot (m-1)\, O\left(m^{-2}\right) \cdots (m-l+1)\, O\left(m^{-2}\right) \\ &\le m^{l}\, O\left(m^{-2l}\right) \\ &= O\left(m^{-l}\right). \end{aligned} \tag{29}$$

Sum up over all $1 \le l \le m$ to get that the probability of simultaneous jumps or jumps to non-neighbouring sites occurring any non-zero number of times is less than

$$\sum_{l=1}^{m} O\left(m^{-l}\right) \le O\left(m^{-1}\right) + mO\left(m^{-2}\right) = O\left(m^{-1}\right), \tag{30}$$

which goes to 0 as m approaches infinity.

We can therefore ignore all those paths in Λ^{mn} which consist of any combination of simultaneous jumps and jumps to non-neighbouring sites. Denote the set of all the remaining paths as $\mathcal{M}$. Define $I'_{n,\boldsymbol{x},\sigma,m}$ by the right-hand side of see (24), but sum over $\mathcal{M}$ instead of Λ^{mn}. We've shown that

$$\lim_{m\to\infty} I_{n,\boldsymbol{x},\sigma,m} = \lim_{m\to\infty} I'_{n,\boldsymbol{x},\sigma,m}.$$

In combination with (22), this implies that

$$\tilde{Z}_{\Lambda,\beta} = \sum_{n\in\mathbb{N}} \frac{\mathrm{e}^{\beta\mu n}}{n!} \sum_{x_1,\ldots,x_n} \sum_{\sigma\in S_n} \lim_{m\to\infty} I'_{n,\boldsymbol{x},\sigma,m}. \tag{31}$$

This is as far as we will manipulate the probabilistic (Feynman-Kac) form of the partition function. We now turn our attention to (18):

$$Z_{\Lambda,\beta} = \mathrm{Tr}_{\mathcal{F}_\Lambda^{(+)}} \mathrm{e}^{-\beta H_\Lambda} = \sum_{n\in\mathbb{N}} \mathrm{Tr}_{\mathcal{H}_{\Lambda,n}^{(+)}} \mathrm{e}^{-\beta H_{\Lambda,n}},$$

where the trace is taken with respect to the basis of $\mathcal{H}_{\Lambda,n}^{(+)}$ defined in (15). We would prefer to work with the basis of $\mathcal{H}_{\Lambda,n}$. We can replace $\mathrm{Tr}_{\mathcal{H}_{\Lambda,n}^{(+)}}$ with $\mathrm{Tr}_{\mathcal{H}_{\Lambda,n}}$ if we project its argument to $\mathcal{H}_{\Lambda,n}^{(+)}$. To this end, define a projection operator $S_{+,n} : \mathcal{H}_{\Lambda,n} \to \mathcal{H}_{\Lambda,n}^{(+)}$ by

$$S_{+,n}\varphi(x_1,\ldots,x_n) = \frac{1}{n!} \sum_{\sigma\in S_n} \varphi\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right).$$

Then

$$Z_{\Lambda,\beta} = \sum_{n\in\mathbb{N}} \mathrm{Tr}_{\mathcal{H}_{\Lambda,n}} S_{+,n} \mathrm{e}^{-\beta H_{\Lambda,n}}. \tag{32}$$

Recall that $\mathcal{H}_{\Lambda,n}$ is a Hilbert space of functions from Λ^n to $\mathbb{C}$. Every function is uniquely determined by its values on the points of Λ^n, so a natural choice of basis for $\mathcal{H}_{\Lambda,n}$ are the functions

$$\varphi_{x_1,\ldots,x_n} : \Lambda^n \to \mathbb{C},$$
$$\varphi_{x_1,\ldots,x_n}(y_1,\ldots,y_n) = \delta_{x_1,y_1} \cdots \delta_{x_n,y_n},$$

for $x_1,\ldots,x_n \in \Lambda$. We will denote these functions using the Dirac (bra-ket) notation:

$$\varphi_{x_1,\ldots,x_n} \equiv |x_1,\ldots,x_n\rangle.$$

Expand the trace in (32):

$$
\begin{aligned}
Z_{\Lambda,\beta} &= \sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{x_1,\ldots,x_n\in\Lambda} \sum_{\sigma\in S_n} \left\langle x_1,\ldots,x_n \,\middle|\, \mathrm{e}^{-\beta H_{\Lambda,n}} \,\middle|\, x_{\sigma(1)},\ldots,x_{\sigma(n)} \right\rangle \\
&= \sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{x_1,\ldots,x_n\in\Lambda} \sum_{\sigma\in S_n} \left\langle x_{\sigma(1)},\ldots,x_{\sigma(n)} \,\middle|\, \mathrm{e}^{-\beta H_{\Lambda,n}} \,\middle|\, x_1,\ldots,x_n \right\rangle ;
\end{aligned}
\tag{33}
$$

the last equality holds since $\exp(-\beta H_{\Lambda,n})$ is positive-semidefinite. Recall the definition of $H_{\Lambda,n}$ in (17):

$$
H_{\Lambda,n} = -\tau \sum_{i=1}^{n} \Delta_i + u \sum_{1\le i<j\le n} \delta(i,j) - \mu n \,\mathrm{id}\,.
$$

Since id commutes with all the operators,

$$
\exp\left(-\beta H_{\Lambda,n}\right) = \exp(\mu n \,\mathrm{id}) \cdot \exp\left(\beta\tau \sum_{i=1}^{n} \Delta_i - \beta u \sum_{1\le i<j\le n} \delta(i,j)\right). \tag{34}
$$

We now use the Lie-Trotter product formula: for any finite-dimensional operators (matrices) A and B,

$$
\begin{aligned}
\exp(A+B) &= \lim_{m\to\infty} \left(\exp\left(\frac{1}{m}A\right)\exp\left(\frac{1}{m}B\right)\right)^m \\
&= \lim_{m\to\infty} \left(\left(1+\frac{1}{m}A\right)\exp\left(\frac{1}{m}B\right)\right)^m .
\end{aligned}
\tag{35}
$$

Using (35), expand the second factor in (34):

$$
\begin{aligned}
\exp\left(-\beta H_{\Lambda,n}\right) = \lim_{m\to\infty} &\left(\left(1+\frac{\beta\tau}{m}\sum_{i=1}^{n}\Delta_i\right)\exp\left(-\frac{\beta u}{m}\sum_{1\le i<j\le n}\delta(i,j)\right)\right)^m \\
&\cdot \exp(\mu n \,\mathrm{id}).
\end{aligned}
\tag{36}
$$

It is sensible to leave the operators $(\delta(i,j))_{i,j}$ and id in the exponent, as they are diagonal in the basis $(|x_1,\ldots,x_n\rangle)_{x_1,\ldots,x_n\in\Lambda}$. Substituting (36) into (33) yields

$$
Z_{\Lambda,\beta} = \sum_{n\in\mathbb{N}} \frac{\mathrm{e}^{\beta\mu n}}{n!} \sum_{x_1,\ldots,x_n\in\Lambda} \sum_{\sigma\in S_n} \lim_{m\to\infty} T_{n,\boldsymbol{x},\sigma,m}, \tag{37}
$$

where $\boldsymbol{x} = (x_1, \dots, x_n) \in \Lambda^n$, and

$$T_{n,\boldsymbol{x},\sigma,m} = \left\langle x_{\sigma(1)}, \dots, x_{\sigma(n)} \,\middle|\, L^m \,\middle|\, x_1, \dots, x_n \right\rangle, \tag{38}$$
$$L = \left(1 + \frac{\beta\tau}{m} \sum_{i=1}^{n} \Delta_i\right) \exp\left(-\frac{\beta u}{m} \sum_{1 \le i < j \le n} \delta(i,j)\right)$$

We first examine how L acts on a vector. Recall the definition of Δ_i in (16). It is straightforward to see that

$$\Delta_i \left|x_1, \dots, x_n\right\rangle = -d_{x_i} \left|x_1, \dots, x_n\right\rangle + \sum_{y \sim x_i} \left|x_1, \dots y, \dots, x_n\right\rangle,$$

where y is in the ith component. With this in mind,

$$L \left|x_1, \dots, x_n\right\rangle = \exp\left(-\frac{\beta u}{m} \sum_{1 \le i < j \le n} \delta_{x_i, x_j}\right) \cdot \left(\left(1 - \frac{\beta\tau}{m} \sum_{i=1}^{n} d_{x_i}\right) \left|x_1, \dots, x_n\right\rangle \right.$$
$$\left. + \frac{\beta\tau}{m} \sum_{i=1}^{n} \sum_{y \sim x_i} \left|x_1, \dots, y, \dots, x_n\right\rangle \right), \tag{39}$$

where y is always in the ith component.

We will now show how this connects to the random-walks representation. Recall the n independent copies of a discrete-time random walk X^m with m steps. We can interpret $\left|x_1, \dots, x_n\right\rangle$ as the initial state of these walks: for every $1 \le i \le n$, let $X_i^m(0) = x_i$. Then $L \left|x_1, \dots, x_n\right\rangle$ is a linear combination of all the possible states that these walks can occupy after one step, excluding those that would require more than one walk to jump (i.e. simultaneous jumps) or any walk to jump to a non-neighbouring site. Compare the coefficients in front of the kets in (39) to (26) and (27): they are products of an exponential factor and the probabilities of $X_1^m, \dots, X_n^m$ occupying the states represented by the respective kets, up to an error term of size $O(m^{-2})$, conditioned on $X_i^m(0) = x_i$ for $i = 1, \dots n$. More generally, let $\left|\boldsymbol{x}(k)\right\rangle = \left|x_{1,k}, \dots, x_{n,k}\right\rangle$ represent the state of the random walks after k steps (at time $k\beta\tau/m$). Then $L\left|\boldsymbol{x}(k)\right\rangle$ has the same interpretation as before, but we are now observing the step from $k\beta\tau/m$ to $(k+1)\beta\tau/m$, and we are conditioning on $X_i^m(k\beta\tau/m) = x_{i,k}$ for $i = 1, \dots n$.

We can now conclude that $L^m \left|x_1, \dots, x_n\right\rangle$ is a linear combination of all the possible states that $X_1^m, \dots, X_n^m$ can occupy after m steps (at time $\beta\tau$), if they are allowed at most one jump per step in total, where each jump can only be to a neighbouring site. The coefficients in front of the kets are of the following form:

$$\begin{aligned}&\exp\left(-\frac{\beta u}{m}\sum_{1\le i<j\le n}\sum_{k=0}^{m}\delta_{x_{i,k},x_{j,k}}\right)\prod_{k=0}^{m-1}\left(p_k+O\left(m^{-2}\right)\right)\\ =&\exp\left(-\frac{\beta u}{m}\sum_{1\le i<j\le n}\sum_{k=0}^{m}\delta_{x_{i,k},x_{j,k}}\right)\prod_{k=0}^{m-1}p_k+O\left(m^{-1}\right),\end{aligned}$$

where

$$\begin{aligned}p_k=\prod_{k=0}^{m-1}\Big(&\mathbb{P}[X_i((k+1)\beta\tau/m)=x_{i,k+1}\text{ for }i=1,\ldots,n\\ &\mid X_i(k\beta\tau/m)=x_{i,k}\text{ for }i=1,\ldots,n]\Big),\end{aligned}$$

for a choice of $x_{i,k}\in\Lambda,\ k=0,\ldots,m,\ i=1,\ldots,n$, that satisfies the aforementioned restrictions on the jumps of the walks. Note that the final $O(m^{-1})$ term was obtained in the same way as in (29) and (30); the exponential factor in the equation above is at most 1, so it cannot enlarge the big Oh terms.

Multiply $L^m\left|x_1,\ldots,x_n\right\rangle$ by $\left\langle x_{\sigma(1)},\ldots,x_{\sigma(n)}\right|$ from the left to obtain $T_{n,\boldsymbol{x},\sigma,m}$ in (38). Since

$$\left\langle x_{\sigma(1)},\ldots,x_{\sigma(n)}\,\middle|\,x_{1,m},\ldots,x_{n,m}\right\rangle=\prod_{i=1}^{n}\delta_{x_{\sigma(i)},x_{i,m}},$$

$T_{n,\boldsymbol{x},\sigma,m}$ only contains those terms for which $x_{i,m}=x_{\sigma(i)}$ for all $i=1,\ldots,n$. Therefore,

$$T_{n,\boldsymbol{x},\sigma,m}=I'_{n,\boldsymbol{x},\sigma,m}+O\left(m^{-1}\right).\tag{40}$$

Substituting (40) into (37), and comparing it to (31) yields $\tilde{Z}_{\Lambda,\beta}=Z_{\Lambda,\beta}$. This concludes the proof.

3.1 Note

We claimed in the paragraph under (19) that for a continuous-time Markov walk, for $t>0$ small enough, the probability of observing at least two jumps in a time interval of length t is $O(t^2)$. Note that in general, this probability can only be estimated as $o(t)$; this bound is stated in most literature, for example in Proposition 15.30 in [Bre92]. For generality, denote the jump intensity of the walk at site $x \in \Lambda$ by $\lambda(x)$. The following proof of the $O(t^2)$ bound works when Λ is finite. If Λ is infinite, a sufficient condition for the proof to work is that the jump intensities are uniformly bounded for the following proof to work. If we only have that $\mathbb{E}_x[\lambda] < \infty$, we can prove the $o(t^\alpha)$ bound for any $1 \leq \alpha < 2$. We comment after the proof how this can be done.

Start the Markov walk at time 0, and denote by T the time of the first jump; this is a random variable. For any $s \geq 0$, denote by θ_s the shift operator that shifts random processes by time s:

$$(\omega \circ \theta_s)(t) = \omega(t+s).$$

Then, the time of the second jump can be represented as the random variable $T \circ \theta_T$. We want to show that $\mathbb{P}_x(T \circ \theta_T \leq t) = O(t^2)$ for any $x \in \Lambda$, where $\mathbb{P}_x$ is the probability $\mathbb{P}$ conditioned on $X(0) = x$. Indeed:

$$\begin{aligned}
\mathbb{P}_x(T \circ \theta_T \leq t) &\leq \mathbb{P}_x(T \leq t, T \circ \theta_T \leq t) \\
&= \mathbb{P}_x(T \circ \theta_T \leq t \mid T \leq t)\mathbb{P}_x(T \leq t) \\
&= \mathbb{E}_x[\mathbb{1}[T \circ \theta_T \leq t] \mid T \leq t]\mathbb{P}_x(T \leq t).
\end{aligned}$$

By the law of total expectation,

$$\mathbb{P}_x(T \circ \theta_T \leq t) \leq \mathbb{E}_x\left[\mathbb{E}_x[\mathbb{1}[T \circ \theta_T \leq t] \mid T \leq t, X_T] \mid T \leq t\right]\mathbb{P}_x(T \leq t).$$

Since T and X_T are independent (implicitily by definition), and by the strong Markov property,

$$\begin{aligned}
\mathbb{P}_x(T \circ \theta_T \leq t) &\leq \mathbb{E}_x\left[\mathbb{E}_{X_T}[\mathbb{1}[T \leq t]]\right]\mathbb{P}_x(T \leq t) \\
&= \mathbb{E}_x\left[\mathbb{P}_{X_T}(T \leq t)\right]\mathbb{P}_x(T \leq t) \\
&= \sum_{y \sim x} \mathbb{P}_y(T \leq t)\mathbb{P}_x(X_T = y)\mathbb{P}_x(T \leq t) \\
&= \sum_{y \sim x} \left(1 - \mathrm{e}^{-\lambda(y)t}\right)\mathbb{P}_x(X_T = y)\left(1 - \mathrm{e}^{-\lambda(x)t}\right).
\end{aligned}$$

Taylor expand (here we use that λ is uniformly bounded in case Λ is infinite, otherwise $|\lambda(y)t|$ might be unbounded and consequently the Taylor series divergent):

$$\begin{aligned}
\mathbb{P}_x(T \circ \theta_T \leq t) &= \sum_{y \sim x} \left(\lambda(y)t + O\left(t^2\right)\right) \mathbb{P}_x(X_T = y) \left(\lambda(x)t + O\left(t^2\right)\right) \\
&= \sum_{y \sim x} \mathbb{P}_x(X_T = y) O\left(t^2\right) \\
&= O\left(t^2\right).
\end{aligned}$$

This concludes the proof. In case of infinite Λ and $\mathbb{E}_x[\lambda] < \infty$, split the expectation into two parts: one over a set where λ is small enough for the Taylor expansion to be valid (say $\lambda(y) < t^{-\alpha}$ for $\alpha < 1$), and the rest. Use the Markov inequality to bound the second one.

4 Brownian motion in the heat and Poisson's equation

In this section we show how the solutions to the heat and Poisson's equation can be expressed as functionals of the Brownian motion. In particular, we first state how this can be done for the heat equation in $\mathbb{R}^n$ for any $n \in \mathbb{N}$, and for Laplace's and Poisson's equation in an open bounded region $\Omega \subset \mathbb{R}^n$ (we also note how the conditions on Ω can be relaxed). We then use these results to solve the heat equation in a region with a boundary. We will not bother with imposing the necessary or succifient conditions on the initial and boundary conditions. These conditions can be extracted from the cited references.

We first introduce some notation. For any $n \in \mathbb{N}$ denote by $\boldsymbol{B}_t$ the Brownian motion in $\mathbb{R}^n$ evaluated at time $t \geq 0$. The dependence on n will be left implicit. For any $\boldsymbol{x} \in \mathbb{R}^n$ denote by $\mathbb{P}_{\boldsymbol{x}}$ the probability for which $\mathbb{P}_{\boldsymbol{x}}(\boldsymbol{B}_0 = \boldsymbol{x}) = 1$. Denote by $\mathbb{E}_{\boldsymbol{x}}$ the expectation with respect to $\mathbb{P}_{\boldsymbol{x}}$.

For any set $A \subset \mathbb{R}^n$ define $\tau_A = \inf\{t > 0 \mid \boldsymbol{B}_t \notin A\}$. If A is open, τ_A is the first time $\boldsymbol{B}$ leaves A. Note that τ_A is a random variable. In particular, if A is "nice enough", τ_A is a stopping time.

4.1 The heat equation in $\mathbb{R}^n$ & Poisson's equation

The following two propositions are extracted from Chapter 4.A in [Dur96]. The first one is proved in Chapter 4.1, and the second one in Chapter 4.2 of the same book.

Proposition 4.1. *The solution to the homogeneous heat equation*

$$\begin{aligned} \frac{\partial u}{\partial t}(t, \boldsymbol{x}) - \frac{1}{2}\Delta u(t, \boldsymbol{x}) &= 0, && \textit{for } (t, \boldsymbol{x}) \in (0, \infty) \times \mathbb{R}^n, \\ u(0, \boldsymbol{x}) &= f(\boldsymbol{x}), && \textit{for } \boldsymbol{x} \in \mathbb{R}^n, \end{aligned}$$

can be expressed as

$$u(t, \boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}}\left(f(\boldsymbol{B}_t)\right).$$

Proposition 4.2. *The solution to the inhomogeneous heat equation*

$$\begin{aligned} \frac{\partial u}{\partial t}(t,\boldsymbol{x}) - \frac{1}{2}\Delta u(t,\boldsymbol{x}) &= h(t,\boldsymbol{x}), && \textit{for } (t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,\boldsymbol{x}) &= 0, && \textit{for } \boldsymbol{x}\in\mathbb{R}^n, \end{aligned}$$

can be expressed as

$$u(t,\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}}\left(\int_0^t h(t-s,\boldsymbol{B}_s)\,\mathrm{d}s\right).$$

The following two propositions are extracted from Chapter 4.B in [Dur96]. The first one is proved in Chapter 4.4, and the second one in Chapter 4.5 of the same book. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set.

Proposition 4.3. *The solution to Laplace's equation*

$$\begin{aligned} \Delta u(\boldsymbol{x}) &= 0, && \textit{for } \boldsymbol{x}\in\Omega\subset\mathbb{R}^n, \\ u(\boldsymbol{x}) &= g(\boldsymbol{x}), && \textit{for } \boldsymbol{x}\in\partial\Omega, \end{aligned}$$

can be expressed as

$$u(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}}\left(g(\boldsymbol{B}_{\tau_\Omega})\right). \tag{41}$$

Proposition 4.4. *The solution to Poisson's equation*

$$\begin{aligned} -\Delta u(\boldsymbol{x}) &= h(\boldsymbol{x}), && \textit{for } \boldsymbol{x}\in\Omega\subset\mathbb{R}^n, \\ u(\boldsymbol{x}) &= 0, && \textit{for } \boldsymbol{x}\in\partial\Omega, \end{aligned}$$

can be expressed as

$$u(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}}\left(\int_0^{\tau_\Omega} h(\boldsymbol{B}_t)\mathrm{d}t\right). \tag{42}$$

The homogeneous heat equation is linear, meaning its inhomogeneous version can be expressed as integrals of the solutions to the homogeneous equations. This is Duhamel's principle (see 4.3.2). The homogeneous Laplace's equation is linear as well, but it is unclear how exactly an analogue of Duhamel's principle would apply in this case. It seems this has to do with the geometry of the region one would have to integrate with respect to: a time interval $[0,t]$ in the case of the heat equation, versus an arbitrary $\Omega \subset \mathbb{R}^n$ in the case of Laplaces's equation. Nevertheless, in both cases the integral representation of the solution to the inhomogeneous equation can be derived using probabilistic methods.

Using the propositions above we can solve the inhomogeneous heat equation with a non-zero initial condition, and Poisson's equation with non-zero boundary condition. In both cases, we split the original equation into two subequations: the homogeneous equation with a non-zero initial or boundary condition, respectively, and the inhomogeneous equation with a homogeneous (i.e. zero) initial or boundary condition, respectively. The solution to the original equation is then the sum of the solutions to the subequations.

In Proposition 4.3 and Proposition 4.4, we required $\Omega \subset \mathbb{R}^n$ to be open and bounded, but as mentioned earlier, these conditions can be relaxed. We will however stick to the original conditions for simplicity. The boundness condition on Ω can be replaced by the condition that for every $\boldsymbol{x} \in \Omega$, $\mathbb{P}_{\boldsymbol{x}}(\tau_\Omega < \infty) = 1$. If this condition is not met, we need to add an additional indicator function $\mathbb{1}_{\{\tau_\Omega < \infty\}}$ into the expectations in (41) and (42). Next, we need to assume that for every point $\boldsymbol{x} \in \partial\Omega$, $\mathbb{P}_{\boldsymbol{x}}(\tau_\Omega = 0) = 1$. (such $\boldsymbol{x}$ is said to be *regular*). Every open subset of $\mathbb{R}$ satisfies this condition; see 4.3.1 for proof. A punctured disk in $\mathbb{R}^d$ for $d \geq 2$ is an example of a set that does not satisfy it. See Example 4.1 in Section 4.4 of [Dur96].

4.2 The heat equation in a bounded region

Let $\Omega \subset \mathbb{R}^n$ be open and bounded.

Proposition 4.5. *The solution to the homogeneous heat equation*

$$\begin{aligned} \frac{\partial u}{\partial t}(t,\boldsymbol{x}) - \frac{1}{2}\Delta u(t,\boldsymbol{x}) &= 0, && \textit{for } (t,\boldsymbol{x}) \in (0,\infty) \times \Omega, \\ u(0,\boldsymbol{x}) &= f(\boldsymbol{x}), && \textit{for } \boldsymbol{x} \in \Omega, \\ u(t,\boldsymbol{x}) &= 0, && \textit{for } (t,\boldsymbol{x}) \in (0,\infty) \times \partial\Omega, \end{aligned}$$

can be expressed as

$$u(t,\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}}\left(f(\boldsymbol{B}_{t\wedge\tau_\Omega})\mathbb{1}_{\{t<\tau_\Omega\}}\right).$$

We make no claim as to how f must behave on the boundary.

Proof. We follow Section 4.1 in [Dur96]. In Theorem 1.2 in [Dur96], redefine the random process

M_s as

$$M_s = u(t-s, \boldsymbol{B}_{s\wedge\tau_\Omega}).$$

The proof of the theorem plays out in the same way with one correction: instead of integrating from 0 to s, we integrate from 0 to $s \wedge \tau_\Omega$, but leave the integrands and integrators unchanged ($\boldsymbol{B}_r$ in not replaced by $\boldsymbol{B}_{r\wedge\tau_\Omega}$). All of the integrals still make sense: When integrating with respect to $\mathrm{d}r$, the value of $u(t-r, \boldsymbol{B}_{r\wedge\tau_\Omega})$ is constant in time on $[\tau_\Omega, s]$, and so its time derivative is zero. When integrating with respect to the Brownian motion, note that both $\mathrm{d}B^i_{r\wedge\tau_\Omega}$ and $\mathrm{d}\langle B^i_{\cdot\wedge\tau_\Omega}, B^j_{\cdot\wedge\tau_\Omega}\rangle$ are identically 0 when $s > \tau_\Omega$ (since they both consist of differences of Brownian motions), and we can simply drop τ_Ω when $s \leq \tau_\Omega$. No other arguments in the proof change. Note that at the very end of the proof, the "second term on the right-hand side" is a local martingale because it is of bounded variation. We conclude that M is a continuous local martingale.

We now check that Theorem 1.3 in [Dur96] still holds:

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{x}}(M_t) &= \mathbb{E}_{\boldsymbol{x}}\left(u\left(0, \boldsymbol{B}_{t\wedge\tau_\Omega}\right)\right) = \mathbb{E}_{\boldsymbol{x}}\left(f(\boldsymbol{B}_{t\wedge\tau_\Omega})\mathbb{1}_{\{t<\tau_\Omega\}}\right), \\
\mathbb{E}_{\boldsymbol{x}}(M_0) &= \mathbb{E}_{\boldsymbol{x}}\left(u\left(t, \boldsymbol{B}_0\right)\right) = u(t, \boldsymbol{x}).
\end{aligned}$$

All the other arguments remain unchanged. Regularity of solution inside Ω follows in the same manner as in the original setting (note that the stopped Brownian motion is bounded). □

Now, using Duhamel's principle (see Section 4.3.2) and Proposition 4.5, we get the following:

Proposition 4.6. *The solution to the inhomogeneous heat equation*

$$\begin{aligned}
\frac{\partial u}{\partial t}(t, \boldsymbol{x}) - \frac{1}{2}\Delta u(t, \boldsymbol{x}) = g(t, \boldsymbol{x}), &\qquad \textit{for } (t, \boldsymbol{x}) \in (0, \infty) \times \Omega, \\
u(0, \boldsymbol{x}) = 0, &\qquad \textit{for } \boldsymbol{x} \in \Omega, \\
u(t, \boldsymbol{x}) = 0, &\qquad \textit{for } (t, \boldsymbol{x}) \in (0, \infty) \times \partial\Omega,
\end{aligned}$$

can be expressed as

$$u(t, \boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}}\left(\int_0^{t\wedge\tau_\Omega} g(t-s, \boldsymbol{B}_s)\,\mathrm{d}s\right).$$

Proof. Observe that

$$\mathbb{E}_{\boldsymbol{x}}\left(\int_0^{t\wedge\tau_\Omega} g(t-s,\boldsymbol{B}_s)\,\mathrm{d}s\right)=\mathbb{E}_{\boldsymbol{x}}\left(\int_0^{t} g(t-s,\boldsymbol{B}_{s\wedge\tau_\Omega})\,\mathbb{1}_{\{s<\tau_\Omega\}}\mathrm{d}s\right).$$

□

As in Section 4.1, we can solve the inhomogeneous equation with a non-zero initial condition by splitting it into two subequations as dictated by Proposition 4.5 and Proposition 4.6, solving them separately, and then adding the solutions together.

We are now able to solve the following equation:

$$\begin{aligned}\frac{\partial u}{\partial t}(t,\boldsymbol{x})-\frac{1}{2}\Delta u(t,\boldsymbol{x})&=h(t,\boldsymbol{x}), &&\text{for } (t,\boldsymbol{x})\in(0,\infty)\times\Omega,\\ u(0,\boldsymbol{x})&=f(\boldsymbol{x}), &&\text{for } \boldsymbol{x}\in\Omega,\\ u(t,\boldsymbol{x})&=g(t,\boldsymbol{x}), &&\text{for } (t,\boldsymbol{x})\in(0,\infty)\times\partial\Omega,\end{aligned}\tag{43}$$

which is the main focus of this section. The formula for the solution u is stated in (51). If g is only a function of space, that is $g(t,\boldsymbol{x})=g(\boldsymbol{x})$, the formula simplifies to (52).

To solve equation (43), first fix $t\geq 0$, and split the equation into three parts. Let $u_1,\ u_2\colon[0,\infty)\times\Omega\to\mathbb{R}$ and $v_t:\Omega\to\mathbb{R}$ satisfy the following equations, where we define

$v(t, \boldsymbol{x}) = v_t(\boldsymbol{x})$:

$$
\begin{aligned}
\Delta v_t(\boldsymbol{x}) &= 0, && \text{for } \boldsymbol{x} \in \Omega, \\
v_t(\boldsymbol{x}) &= g(t, \boldsymbol{x}), && \text{for } \boldsymbol{x} \in \partial\Omega,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u_1}{\partial t}(t, \boldsymbol{x}) - \frac{1}{2}\Delta u_1(t, \boldsymbol{x}) &= 0, && \text{for } (t, \boldsymbol{x}) \in (0, \infty) \times \Omega, \\
u_1(0, \boldsymbol{x}) &= f(\boldsymbol{x}) - v(0, \boldsymbol{x}), && \text{for } \boldsymbol{x} \in \Omega, \\
u_1(t, \boldsymbol{x}) &= 0, && \text{for } (t, \boldsymbol{x}) \in (0, \infty) \times \partial\Omega,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u_2}{\partial t}(t, \boldsymbol{x}) - \frac{1}{2}\Delta u_2(t, \boldsymbol{x}) &= h(t, \boldsymbol{x}) - \frac{\partial v}{\partial t}(t, \boldsymbol{x}), && \text{for } (t, \boldsymbol{x}) \in (0, \infty) \times \Omega, \\
u_2(0, \boldsymbol{x}) &= 0, && \text{for } \boldsymbol{x} \in \Omega, \\
u_2(t, \boldsymbol{x}) &= 0, && \text{for } (t, \boldsymbol{x}) \in (0, \infty) \times \partial\Omega.
\end{aligned}
$$

Then the solution to equation (43) is

$$
u(t, \boldsymbol{x}) = v(t, \boldsymbol{x}) + u_1(t, \boldsymbol{x}) + u_2(t, \boldsymbol{x}). \tag{44}
$$

Indeed,

$$
\begin{aligned}
\frac{\partial u}{\partial t}(t, \boldsymbol{x}) - \frac{1}{2}\Delta u(t, \boldsymbol{x}) &= \left(\frac{\partial v}{\partial t} - \frac{1}{2}\Delta v + \frac{\partial u_1}{\partial t} - \frac{1}{2}\Delta u_1 + \frac{\partial u_2}{\partial t} - \frac{1}{2}\Delta u_2\right)(t, \boldsymbol{x}) \\
&= \frac{\partial v}{\partial t}(t, \boldsymbol{x}) - \frac{1}{2}\Delta v_t(\boldsymbol{x}) + h(t, \boldsymbol{x}) - \frac{\partial v}{\partial t}(t, \boldsymbol{x}) \\
&= h(t, \boldsymbol{x}),
\end{aligned}
$$

with the initial value

$$
u(0, \boldsymbol{x}) = v(0, \boldsymbol{x}) + f(\boldsymbol{x}) - v(0, \boldsymbol{x}) = f(\boldsymbol{x}),
$$

and the boundary value on $(t, \boldsymbol{x}) \in (0, \infty) \times \partial\Omega$ of

$$
u(t, \boldsymbol{x}) = v_t(\boldsymbol{x}) = g(t, \boldsymbol{x}).
$$

By Proposition 4.3, Proposition 4.5, and Proposition 4.6, the functions v, u_1, and u_2, respectively, can be expressed as functionals of Brownian motion:

$$v(t, \boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}} \left(g\left(t, \boldsymbol{B}_{\tau_\Omega}\right)\right), \tag{45}$$

$$u_1(t, \boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}} \left(\left(f(\boldsymbol{B}_{t\wedge\tau_\Omega}) - v(0, \boldsymbol{B}_{t\wedge\tau_\Omega})\right) \mathbb{1}_{\{t<\tau_\Omega\}} \right), \tag{46}$$

$$u_2(t, \boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}} \left(\int_0^{t\wedge\tau_\Omega} \left(h(t-s, \boldsymbol{B}_s) - \frac{\partial v}{\partial t}(t-s, \boldsymbol{B}_s) \right) \mathrm{d}s \right), \tag{47}$$

where $\boldsymbol{B}$ is Brownian motion, and $\tau_\Omega = \inf\{t > 0 \mid \boldsymbol{B}_t \notin \Omega\}$. By substituting (45) into (46) and (47), we get:

$$u_1(t, \boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}} \left(\left(f(\boldsymbol{B}_{t\wedge\tau_\Omega}) - \mathbb{E}_{\boldsymbol{B}_{t\wedge\tau_\Omega}} \left(g\left(0, \boldsymbol{B}_{\tau_\Omega}\right)\right) \right) \mathbb{1}_{\{t<\tau_\Omega\}} \right), \tag{48}$$

$$u_2(t, \boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}} \left(\int_0^{t\wedge\tau_\Omega} \left(h(t-s, \boldsymbol{B}_s) - \frac{\partial}{\partial t} \mathbb{E}_{\boldsymbol{B}_s} \left(g\left(t-s, \boldsymbol{B}_{\tau_\Omega}\right)\right) \right) \mathrm{d}s \right). \tag{49}$$

The right-hand side of (48) can be simplified. Firstly, the indicator function $\mathbb{1}_{\{t<\tau_\Omega\}}$ enables us to replace $E_{\boldsymbol{B}_{t\wedge\tau_\Omega}}$ with $\mathbb{E}_{\boldsymbol{B}_t}$. Let $\mathcal{F}$ be the Brownian filtration of $\boldsymbol{B}$. Rewrite

$$\mathbb{E}_{\boldsymbol{x}} \left(\mathbb{E}_{\boldsymbol{B}_t} \left(g\left(0, \boldsymbol{B}_{\tau_\Omega}\right)\right) \mathbb{1}_{\{t<\tau_\Omega\}} \right) = \mathbb{E}_{\boldsymbol{x}} \left(\mathbb{E}\left(g\left(0, \boldsymbol{B}_{\tau_\Omega}\right) \middle| \mathcal{F}_t \right) \mathbb{1}_{\{t<\tau_\Omega\}} \right).$$

The indicator function $\mathbb{1}_{\{t<\tau_\Omega\}}$ is measurable with respect to $\mathcal{F}_t$, so

$$\mathbb{E}_{\boldsymbol{x}} \left(\mathbb{E}\left(g\left(0, \boldsymbol{B}_{\tau_\Omega}\right) \middle| \mathcal{F}_t \right) \mathbb{1}_{\{t<\tau_\Omega\}} \right) = \mathbb{E}_{\boldsymbol{x}} \left(\mathbb{E}\left(g\left(0, \boldsymbol{B}_{\tau_\Omega}\right) \mathbb{1}_{\{t<\tau_\Omega\}} \middle| \mathcal{F}_t \right) \right).$$

By the tower rule (i.e. the law of total expectation),

$$\mathbb{E}_{\boldsymbol{x}} \left(\mathbb{E}\left(g\left(0, \boldsymbol{B}_{\tau_\Omega}\right) \mathbb{1}_{\{t<\tau_\Omega\}} \middle| \mathcal{F}_t \right) \right) = \mathbb{E}_{\boldsymbol{x}} \left(g\left(0, \boldsymbol{B}_{\tau_\Omega}\right) \mathbb{1}_{\{t<\tau_\Omega\}} \right).$$

Thus, (48) simplifies to

$$u_1(t, \boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}} \left(\left(f(\boldsymbol{B}_{t\wedge\tau_\Omega}) - g\left(0, \boldsymbol{B}_{\tau_\Omega}\right)\right) \mathbb{1}_{\{t<\tau_\Omega\}} \right). \tag{50}$$

Now substitute (45), (49), and (50) into (44) to get:

$$\begin{aligned} u(t, \boldsymbol{x}) &= \mathbb{E}_{\boldsymbol{x}} \left(g\left(t, \boldsymbol{B}_{\tau_\Omega}\right) + \left(f\left(\boldsymbol{B}_{t\wedge\tau_\Omega}\right) - g\left(0, \boldsymbol{B}_{\tau_\Omega}\right)\right) \mathbb{1}_{\{t<\tau_\Omega\}} \right) \\ &\quad + \mathbb{E}_{\boldsymbol{x}} \left(\int_0^{t\wedge\tau_\Omega} \left(h(t-s, \boldsymbol{B}_s) - \frac{\partial}{\partial t} \mathbb{E}_{\boldsymbol{B}_s} \left(g\left(t-s, \boldsymbol{B}_{\tau_\Omega}\right)\right) \right) \mathrm{d}s \right). \end{aligned} \tag{51}$$

This is as far as we will simplify this expression in the general case.

Consider now a special case when the boundary condition g is not time-dependent, that is $g(t, \boldsymbol{x}) = g(\boldsymbol{x})$. Then (51) simplifies to

$$u(t, \boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}} \left(g(\boldsymbol{B}_{\tau_\Omega}) + \left(f\left(\boldsymbol{B}_{t\wedge\tau_\Omega}\right) - g\left(\boldsymbol{B}_{\tau_\Omega}\right)\right) \mathbb{1}_{\{t<\tau_\Omega\}} + \int_0^{t\wedge\tau_\Omega} h(t-s, \boldsymbol{B}_s)\, \mathrm{d}s \right).$$

Note that $g\left(\boldsymbol{B}_{\tau_\Omega}\right) - g\left(\boldsymbol{B}_{\tau_\Omega}\right) \mathbb{1}_{\{t<\tau_\Omega\}} = g\left(\boldsymbol{B}_{\tau_\Omega}\right) \mathbb{1}_{\{t\geq\tau_\Omega\}}$, so the last formula further simplifies to

$$u(t, \boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}} \left(f(\boldsymbol{B}_{t\wedge\tau_\Omega}) \mathbb{1}_{\{t<\tau_\Omega\}} + g\left(\boldsymbol{B}_{\tau_\Omega}\right) \mathbb{1}_{\{t\geq\tau_\Omega\}} + \int_0^{t\wedge\tau_\Omega} h(t-s, \boldsymbol{B}_s)\, \mathrm{d}s \right). \tag{52}$$

We can rewrite this as

$$u(t, \boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}} \left(F(\boldsymbol{B}_{t\wedge\tau_\Omega}) + \int_0^{t\wedge\tau_\Omega} h(t-s, \boldsymbol{B}_s)\, \mathrm{d}s \right),$$

where

$$F(\boldsymbol{x}) = \begin{cases} f(\boldsymbol{x}), & \text{if } \boldsymbol{x} \in \Omega, \\ g(\boldsymbol{x}), & \text{if } \boldsymbol{x} \in \partial\Omega. \end{cases}$$

4.3 Notes

4.3.1 Boundary points of an open subset of $\mathbb{R}$ are regular

Quick sketch why every boundary point of an open subset of $\mathbb{R}$ is regular. First, let's state Blumenthal's 0-1 law. Let $\mathcal{F}_t = \sigma\{B_s \mid 0 \leq s \leq t\}$ be the natural filtration of the Brownian motion and $\mathcal{F}_t^+ = \cap_{s>t}\mathcal{F}_s$ its right-continuous extension. Then for any set $A \in \mathcal{F}_0^+$ and for all $\boldsymbol{x} \in \mathbb{R}^n$,

$$\mathbb{P}_{\boldsymbol{x}}(A) \in \{0, 1\}.$$

Sketch of the proof. First, suppose $Z \in \mathcal{C}$ is bounded where $\mathcal{C}$ is the sigma algebra generated by all continuous Brownian paths (so think $\cup_{t\geq 0}\mathcal{F}_t$). Then for any $s \geq 0$ and any $\boldsymbol{x} \in \mathbb{R}^n$,

$$\mathbb{E}_{\boldsymbol{x}}\left(Z|\mathcal{F}_s^+\right) = \mathbb{E}_{\boldsymbol{x}}\left(Z|\mathcal{F}_s\right). \tag{53}$$

It suffices to show this for $Z = \prod_{m=1}^{k} f_m(\boldsymbol{B}_{t_m})$ for any k, where f are measurable and bounded, and $t_i \geq 0$ for all i. Briefly, any other random variable can be approximated by an appropriate sequence of variables of the above type. We can write such Z as $Z = X(Y \circ \theta_s)$, where $X \in \mathcal{F}_s$, $Y = \mathcal{C}$, and θ_s is the shift operator. Random variable X contains factors with $t_i \leq s$, and the rest can be restarted at s. Then, by standard properties of conditional expectation and the Markov property,

$$\mathbb{E}_{\boldsymbol{x}}\left(Z|\mathcal{F}_s^+\right) = X\mathbb{E}_{\boldsymbol{x}}\left(Z \circ \theta_s|\mathcal{F}_s^+\right) = X\mathbb{E}_{\boldsymbol{B}_s}Y \in \mathcal{F}_s.$$

This implies that

$$X\mathbb{E}_{\boldsymbol{B}_s}Y = \mathbb{E}_{\boldsymbol{x}}\left(X\mathbb{E}_{\boldsymbol{B}_s}Y|\mathcal{F}_s\right) = \mathbb{E}_{\boldsymbol{x}}\left(\mathbb{E}_{\boldsymbol{x}}\left[Z|\mathcal{F}_s^+\right]\Big|\,\mathcal{F}_s\right) = \mathbb{E}_{\boldsymbol{x}}\left(Z|\mathcal{F}_s\right),$$

which proves the claim.

Now let $A \in \mathcal{F}_0^+$. Then

$$\mathbb{1}_A = \mathbb{E}_{\boldsymbol{x}}(\mathbb{1}_A|\mathcal{F}_0^+).$$

By equation (53),

$$\mathbb{E}_{\boldsymbol{x}}(\mathbb{1}_A|\mathcal{F}_0^+) = \mathbb{E}_{\boldsymbol{x}}(\mathbb{1}_A|\mathcal{F}_0).$$

But $\mathcal{F}_0$ is trivial up to the null sets under $\mathbb{P}_{\boldsymbol{x}}$ (since $\mathbb{P}_{\boldsymbol{x}}(\boldsymbol{B}_0 = \boldsymbol{x}) = 1$), so

$$\mathbb{E}_{\boldsymbol{x}}(\mathbb{1}_A|\mathcal{F}_0) = \mathbb{E}_{\boldsymbol{x}}(\mathbb{1}_A).$$

Combining all of the above, we get

$$\mathbb{1}_A = \mathbb{E}_{\boldsymbol{x}}(\mathbb{1}_A) = \mathbb{P}_{\boldsymbol{x}}(\mathbb{1}_A).$$

Since the left-hand side can only be either 0 or 1, so can the right one. This finishes the proof of Blumenthal's 0-1 law.

Let now B_t be centred one-dimensional Brownian motion. Define $\tau_\Omega = \inf\{t \geq 0 \mid B_t > 0\}$. Note that in earlier definitions, the infinimum was taken over $t > 0$. Clearly $\mathbb{P}_0(\tau_\Omega \leq t) \geq \mathbb{P}_0(B_t > 0) = 1/2$ for every t. Let $t \searrow 0$. By the Bounded Convergence Theorem,

$$\mathbb{P}_0(\tau_\Omega = 0) = \lim_{t \searrow 0} \mathbb{P}_0(\tau_\Omega \leq t) \geq \frac{1}{2}.$$

Since $\{\tau_\Omega = 0\} \in \mathcal{F}_0^+$, Blumenthal's 0-1 Theorem and the above inequality imply that $\mathbb{P}_0(\tau_\Omega = 0) = 1$. This means that Brownian motion starting from 0 immediately hits $(0, \infty)$. By symmetry, it must also immediately hit $(-\infty, 0)$. Define $T_0 = \inf\{t > 0 \mid B_t = 0\}$. Continuity of Brownian motion implies that $\mathbb{P}_0(T_0 = 0) = 1$.

This can be trivially generalised from 0 to any point $a \in \mathbb{R}$, but not to any dimension. While a higher-dimensional Brownian motion can be represented as a vector of one-dimensional independent Brownian motions, applying the last conclusion to each component of the vector gives us no control over when each component will hit 0.

Let now $\Omega \subset \mathbb{R}$ be an open set and $x \in \partial\Omega$. Define $\tau_\Omega = \inf\{t > 0 \mid B_t \notin \Omega\}$ and $T_x = \inf\{t > 0 \mid B_t = x\}$. Then, by previous paragraph, $\mathbb{P}_x(T_x = 0) = 1$, and since $\tau_\Omega \leq T_x$, we conclude that $\mathbb{P}_x(\tau_\Omega = 0) = 1$. This concludes the sketch of the proof that every boundary point of an open subset in $\mathbb{R}$ is regular.

4.3.2 Duhamel's principle

We state Duhamel's principle in its general form.

Proposition 4.7. *Consider an inhomogeneous evolution equation*

$$\begin{aligned} \frac{\partial u}{\partial t}(t, \boldsymbol{x}) - Lu(t, \boldsymbol{x}) &= g(t, \boldsymbol{x}), && \textit{for } (t, \boldsymbol{x}) \in (0, \infty) \times \mathbb{R}^n, \\ u(0, \boldsymbol{x}) &= 0, && \textit{for } \boldsymbol{x} \in \mathbb{R}^n, \end{aligned}$$

where L is a linear differential operator. Its solution u can be expressed as

$$u(t, \boldsymbol{x}) = \int_0^t u_s(t - s, \boldsymbol{x}) \mathrm{d}s = \int_0^t u_{t-s}(s, \boldsymbol{x}) \mathrm{d}s, \tag{54}$$

where u_s is a solution to the homogeneous evolution equation

$$\begin{aligned} \frac{\partial u_s}{\partial t}(t,\boldsymbol{x}) - Lu_s(t,\boldsymbol{x}) &= 0, && \textit{for}(t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^n, \\ u_s(0,\boldsymbol{x}) &= g(s,\boldsymbol{x}), && \textit{for } \boldsymbol{x}\in\mathbb{R}^n. \end{aligned} \tag{55}$$

Note that the two integrals in (54) differ by a change of variable $s \mapsto t-s$.

Proof. Differentiate (54):

$$\begin{aligned} \frac{\partial u}{\partial t}(t,\boldsymbol{x}) &= \frac{\partial}{\partial t}\int_0^t u_s(t-s,\boldsymbol{x})\mathrm{d}s \\ &= u_t(0,\boldsymbol{x}) + \int_0^t \frac{\partial u_s}{\partial t}(t-s,\boldsymbol{x})\mathrm{d}s. \end{aligned}$$

By (55),

$$\frac{\partial u}{\partial t}(t,\boldsymbol{x}) = g(t,\boldsymbol{x}) + \int_0^t Lu_s(t-s,\boldsymbol{x})\mathrm{d}s.$$

Since L is linear, we can formally exchange the order of integration and differentiation:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,\boldsymbol{x}) &= g(t,\boldsymbol{x}) + L\int_0^t u_s(t-s,\boldsymbol{x})\mathrm{d}s \\ &= g(t,\boldsymbol{x}) + Lu. \end{aligned}$$

The initial condition on u is trivially satisfied. This concludes the proof. □

We can immediately generalise this to an evolution equation on any subregion $\Omega \subset \mathbb{R}^n$ with homogeneous boundary conditions:

Proposition 4.8. *Consider an inhomogeneous evolution equation*

$$\begin{aligned} \frac{\partial u}{\partial t}(t,\boldsymbol{x}) - Lu(t,\boldsymbol{x}) &= g(t,\boldsymbol{x}), && \textit{for } (t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,\boldsymbol{x}) &= 0, && \textit{for } \boldsymbol{x}\in\mathbb{R}^n, \\ u(t,\boldsymbol{x}) &= 0, && \textit{for } (t,\boldsymbol{x}) \in (0,\infty)\times\partial\Omega, \end{aligned}$$

where L is a linear differential operator. Its solution u can be expressed as

$$u(t,\boldsymbol{x}) = \int_0^t u_s(t-s,\boldsymbol{x})\,\mathrm{d}s = \int_0^t u_{t-s}(s,\boldsymbol{x})\,\mathrm{d}s,$$

where u_s is a solution to the homogeneous evolution equation

$$\begin{aligned} \frac{\partial u_s}{\partial t}(t, \boldsymbol{x}) - Lu_s(t, \boldsymbol{x}) = 0, \qquad & \textit{for } (t, \boldsymbol{x}) \in (0, \infty) \times \mathbb{R}^n, \\ u_s(0, \boldsymbol{x}) = g(s, \boldsymbol{x}), \qquad & \textit{for } \boldsymbol{x} \in \mathbb{R}^n, \\ u_s(t, \boldsymbol{x}) = 0, \qquad & \textit{for } (t, \boldsymbol{x}) \in (0, \infty) \times \partial\Omega. \end{aligned}$$

Proof. Since $u_s(t, \boldsymbol{x}) = 0$ for all $t \geq 0$ and all $x \in \partial\Omega$, the boundary condition on u is trivially satisfied. The rest of the proof is the same as the previous proof. □

References

[BHH21] David Brydges, Tyler Helmuth and Mark Holmes. 'The Continuous-Time Lace Expansion'. In: *Communications on Pure and Applied Mathematics* 74.11 (2021), pp. 2251–2309.

[Bre92] Leo Breiman. *Probability.* Classics In Applied Mathematics. SIAM, 1992.

[Dur96] Richard Durrett. *Stochastic calculus: a practical introduction.* CRC Press, 1996.

[FV18] Sacha Friedli and Yvan Velenik. *Statistical mechanics of lattice systems: a concrete mathematical introduction.* Cambridge University Press, 2018.

[MS13] Neal Madras and Gordon Slade. *The Self-Avoiding Walk.* Birkhäuser, 2013.

[Sla06] Gordon Slade. *The Lace Expansion and its Applications.* Lecture Notes in Mathematics 1879: Ecole d'Eté de Probabilités de Saint-Flour XXIV – 2004. New York: Springer, 2006.