Classification of Transformations of Equivalent Kernels of DPPs, arXiv:2302.02471

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Setup

X is a DPP in Λ (a set) if $\exists\ K : \Lambda^2 \to \mathbb{F}(= \mathbb{R}$ or $\mathbb{C})$ such that

$$\boxed{\text{"Prob}(X \text{ has particles at } x_1, \ldots, x_n) \;=\; \det(K(x_i, x_j))_{i,j=1}^{n}\text{"}},$$

for every $n \in \mathbb{N}$ and $x_k \in \Lambda$.

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Symmetry, non-symmetry, counterexamples

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- $K(x, y) = K(y, x)$ and $Q(x, y) = Q(y, x) \;\; \forall x, y \in \Lambda$

$$\Longrightarrow \text{ conjecture holds.}$$

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$$\begin{vmatrix} K(1,3) & K(1,4) \\ K(2,3) & K(2,4) \end{vmatrix} \neq 0 \neq \begin{vmatrix} K(3,1) & K(3,2) \\ K(4,1) & K(4,2) \end{vmatrix}$$

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- $K \neq 0$ and $\begin{vmatrix} K(x,y) & K(x,w) \\ K(z,y) & K(z,w) \end{vmatrix} \neq 0 \;\; \forall x,y,z,w \in \Lambda$ distinct

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  $\underset{\text{Sapranidis, '23}}{\Longrightarrow}$ conjecture holds.

Idea of Proof

Theorem (Sapranidis, '23)

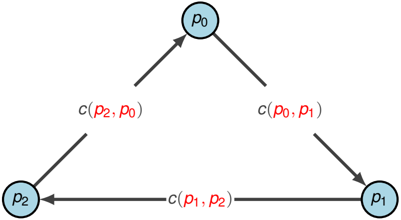
$K \neq 0$ *and* $\begin{vmatrix} K(x,y) & K(x,w) \\ K(z,y) & K(z,w) \end{vmatrix} \neq 0 \;\; \forall x, y, z, w \in \Lambda$ *distinct and* $Q \equiv K$

$\implies$ Q *is in one of the following two forms, for some function* $g \neq 0$*:*

$$Q(x,y) = \frac{g(x)}{g(y)} K(x,y) \qquad \text{or} \qquad Q(x,y) = \frac{g(x)}{g(y)} K(y,x)$$

Proof (outline):

Show $S(x,y) = \frac{Q(x,y)}{K(x,y)}$ or $\tilde{S}(x,y) = \frac{Q(x,y)}{K(y,x)}$ satisfies the 3-*cocycle property*: for every simple 3-cycle $p = (p_i)_{i=0}^{3} \in \Lambda^4$,



$\equiv 1.$

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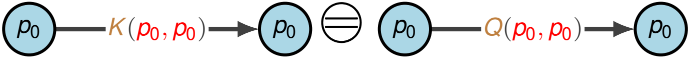
Define $S(x,y) = \frac{Q(x,y)}{K(x,y)}$, $\tilde{S}(x,y) = \frac{Q(x,y)}{K(y,x)}$.

Let $p = (p_i)_{i=0}^{1} \in \Lambda^2$ be a 1-cycle. Then,

$$\det(K(x_i, x_j))_{i,j=1}^{n} = \det(Q(x_i, x_j))_{i,j=1}^{n}$$

with $n = 1$ yields



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Theorem (Sapranidis, '23)

$K \neq 0$ and $\begin{vmatrix} K(x,y) & K(x,w) \\ K(z,y) & K(z,w) \end{vmatrix} \neq 0$ $\forall x,y,z,w \in \Lambda$ distinct and $Q \equiv K$
$\implies$ Q is in one of the following two forms, for some function $g \neq 0$:

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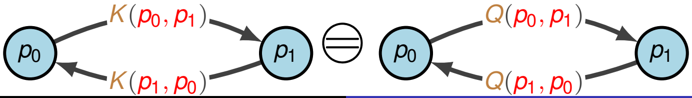
Let $p = (p_i)_{i=0}^{2} \in \Lambda^3$ be a (simple) 2-cycle. Then,

$$\det(K(x_i,x_j))_{i,j=1}^{n} = \det(Q(x_i,x_j))_{i,j=1}^{n}$$

with $n = 2$ yields

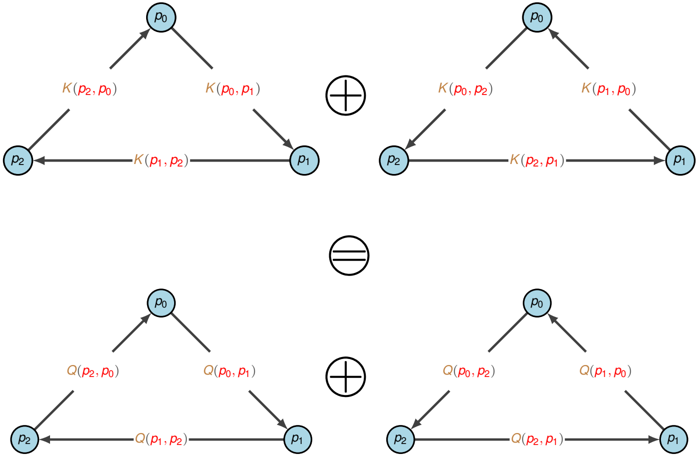


Idea of Proof

Proof (outline):

Let $p = (p_i)_{i=0}^{3} \in \Lambda^4$ be a simple 3-cycle.

The Leibniz formula applied to $Q \equiv K$ with $n = 3$ yields



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Define $S(x, y) = \frac{Q(x,y)}{K(x,y)}$, $\tilde{S}(x, y) = \frac{Q(x,y)}{K(y,x)}$.

The previous slides imply the following lemma:

Theorem

For every (simple) cycle $p := (p_i)_{i=0}^{3}$ *of length* 3 *in* Λ*, it is either the case that*

$$\textbf{\textit{Case 1:}}\ S[p] = 1;$$

or

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Finally, show that every (simple) cycle $p := (p_i)_{i=0}^3$ of length 3 in Λ is in the same case (non-trivial: 4-cycles, graph-theoretic magic tricks, see arXiv:2302.02471). □