Limit Order Book Modeling

Stability of the Markovian Model

Harry Sapranidis Mantelos

University of Warwick

March 8, 2021

Contents...

Introduction to LOB

- Essentially, LOB is a file in a computer that contains all the orders sent to the market with their characteristics such as the **sign** of the order (buy or sell), the **price**, the **quantity**, and a **timestamp** giving the time the order was recorded by the market.

BUY ORDERS				SELL ORDERS			
TIME	MPID	SHARES	BID	ASK	SHARES	MPID	TIME
10:58:01.021	NSDQ	300	576.42	576.52	80	NSDQ	10:58:01.367
9:30:13.227	CDRG	75	576.40	576.58	100	NSDQ	10:58:01.367
10:45:35.954	UBSS	500	576.40	576.67	100	NSDQ	10:58:00.289
10:58:00.901	NSDQ	10	576.37	576.73	100	TMBR	10:58:00.223
10:58:00.899	NSDQ	10	576.36	576.74	100	NSDQ	10:58:01.369
10:58:00.225	NSDQ	8	576.26	576.80	100	NSDQ	10:58:01.213
9:30:13.227	CDRG	2	576.25	576.85	100	NSDQ	10:58:00.225

Structure of LOB

1. A *buy limit order* is an order to buy a security at a specific price
2. A *sell limit order* is an order to sell a security at a specific price

BUY ORDERS				SELL ORDERS			
TIME	MPID	SHARES	BID	ASK	SHARES	MPID	TIME
10:58:01.021	NSDQ	300	576.42	576.52	80	NSDQ	10:58:01.367
9:30:13.227	CDRG	75	576.40	576.58	100	NSDQ	10:58:01.367
10:45:35.954	UBSS	500	576.40	576.67	100	NSDQ	10:58:00.289
10:58:00.901	NSDQ	10	576.37	576.73	100	TMBR	10:58:00.223
10:58:00.899	NSDQ	10	576.36	576.74	100	NSDQ	10:58:01.369
10:58:00.225	NSDQ	8	576.26	576.80	100	NSDQ	10:58:01.213
9:30:13.227	CDRG	2	576.25	576.85	100	NSDQ	10:58:00.225

Types of Orders

- A trader in a LOB of a particular stock may submit:
  1. *A limit order* (bid/ask) to specify the price at which they're willing to buy/sell a certain number of shares of the stock;
  2. *A market order* (bid/ask) to immediately buy/sell a certain quantity of shares of the stock (at the best available opposite quote i.e. top level);
  3. *A cancellation order* to cancel a limit order they had submitted earlier.

Set up

1. Each side (bid and ask) of the order book will have a finite number $K > 0$ of levels where limit orders can be placed, ranging from 1 to K ticks away from the best available opposite quote.
2. Price of a limit order will be given in number of ticks.
3. Model the LOB as a $2K$-dimensional Markov process $\boldsymbol{X}(t) := (\boldsymbol{a}(t); \boldsymbol{b}(t)) = (a_1(t), ..., a_K(t); b_1(t), ..., b_K(t))$
   - $\boldsymbol{a}(t)$ represents the ask side of the LOB, and $a_i(t)$ is the number of shares available, at a price i ticks away from the best bid price $P^B(t)$, from all the ask limit orders submitted at this price, at time t.
   - $\boldsymbol{b}(t)$ represents the bid side of the LOB, and $b_i(t)$ is the number of shares available, at a price i ticks away from the best ask price $P^A(t)$, from all the bid limit orders submitted at this price, at time t.

Set up

4. By convention, $a_i(t) \geq 0$ and $b_i(t) \leq 0$
5. Assume each limit order is of constant size $q \in \mathbb{Z}$ i.e. $a_i(t) \in \{mq : m \in \mathbb{Z}_+\}$ and $b_i(t) \in \{mq : m \in \mathbb{Z}_-\}$
6. By construction, we adopt a finite moving frame for the LOB, where different events (incoming orders) cause the $2K$ price levels to change

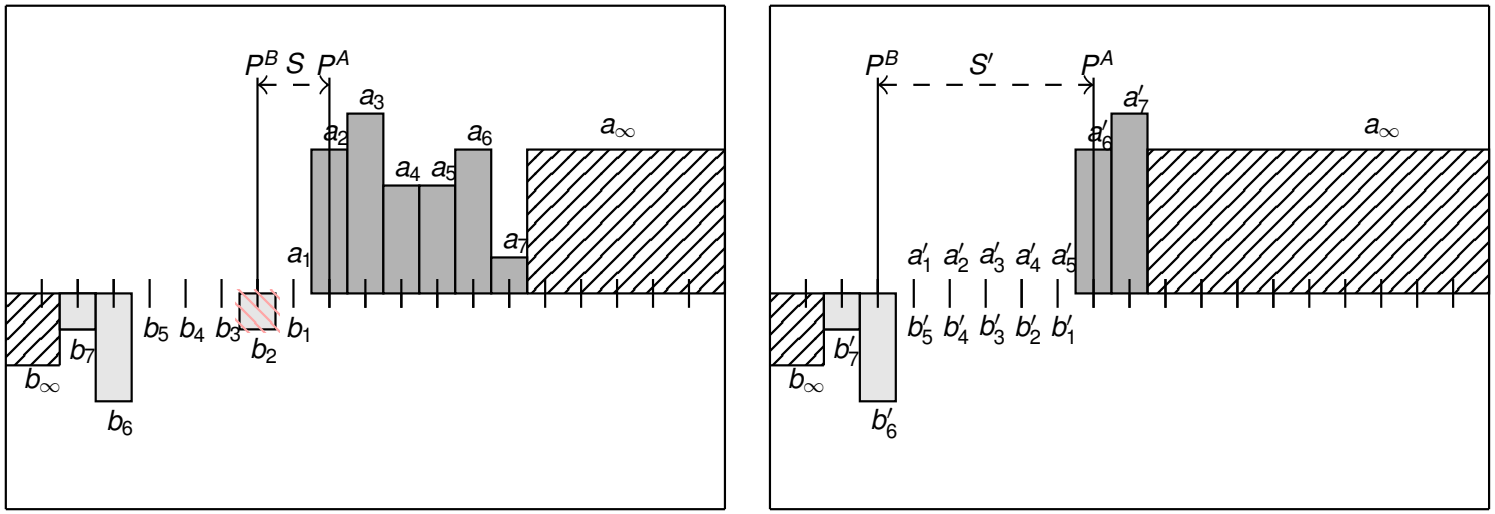

Figure: K=7

Set up

7. Impose the following boundary conditions on the frame:
   - Every time the moving frame leaves a certain price level, the no. of shares at that level is set to a preset constant a_∞ or b_∞ depending on the side of the order book the moving frame leaves from
   - $a_\infty \in \{q, 2q, 3q, ...\}$ and $b_\infty \in \{-q, -2q, -3q, ...\}$

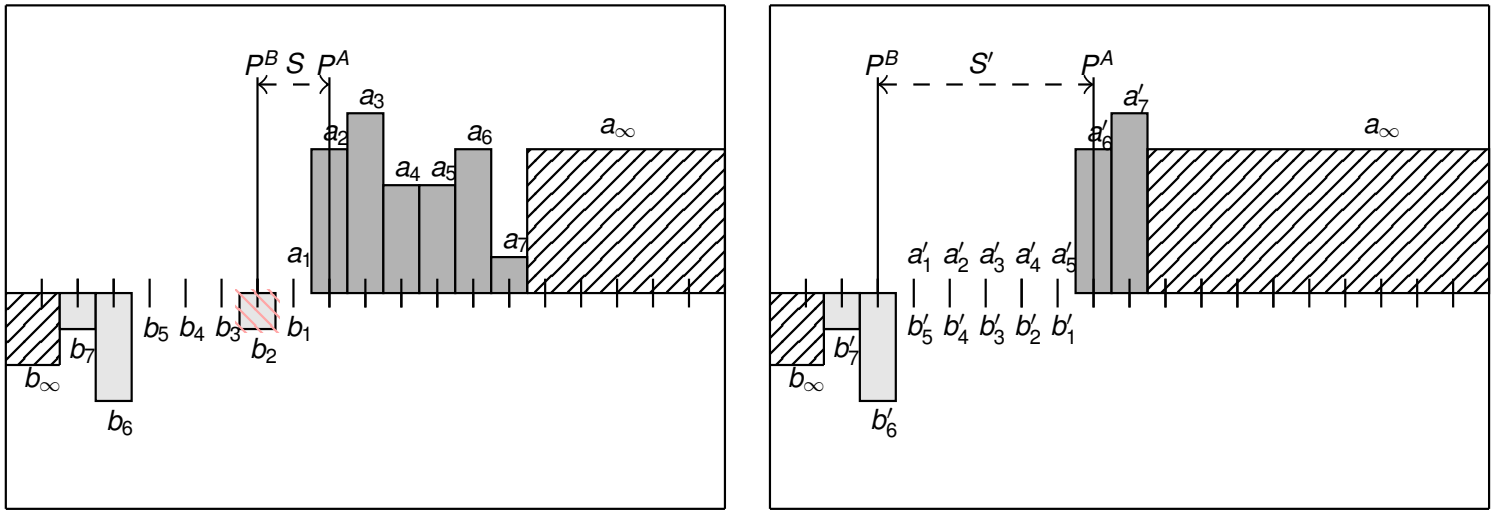

Figure: K=7

Set up

- Ensures the Markovianity of our model: we don't keep track of the price levels that have been visited and then left by the moving frame at some prior time.
- Eliminates the possibility of the moving frame leaving all the current price levels and hence causing the order book to become empty.
- This, in turn, ensures that P^A and P^B are always well-defined.

Notations and Expressions

1. The *cumulative depth* of the LOB:
   - $$A_i(t) := \sum_{k=1}^{i} a_k(t), \quad i \in \{1, ..., K\} \quad \text{for the ask side}$$
   - $$B_i(t) := \sum_{k=1}^{i} |b_k(t)|, \quad i \in \{1, ..., K\} \quad \text{for the bid side}$$
2. The respective inverses of $A_i(t)$ and $B_i(t)$:
   - $A(t)^{-1}(\gamma) := \inf\{p \in \{1, ..., K\} : A_p(t) = \gamma\}$ for some γ amount of shares
   - $B(t)^{-1}(\gamma) := \inf\{p \in \{1, ..., K\} : B_p(t) = \gamma\}$ for some γ amount of shares

Notations and Expressions

3. $S(t) := P^A(t) - P^B(t)$ is the *spread* of the LOB at time t
   - $S(t) > 0$ because $P^A(t) > P^B(t)$

4. We denote the *tick size* by $\triangle P$. This is the difference between a price level and the next in our LOB. Hence, each of our price levels $1, ..., K$ in both the bid and ask side are multiples of $\triangle P$.

5. $i_{S(t)} := A(t)^{-1}(0) = B(t)^{-1}(0) = \frac{S(t)}{\triangle P}$ is the index referring to the first non-empty level on both the bid and ask side of the order book i.e. it's the spread given in number of ticks.

Notations and Expressions

6. $(M^{\pm}(t) : t \geq 0)$ is the counting process for the number of buy/sell market orders at time t;

7. $(L_i^{\pm}(t) : t \geq 0)$ is the counting process for the number of sell/buy limit orders (of size q shares) at level $i \in \{1, ..., K\}$ at time t;

8. $(C_i^{\pm}(t) : t \geq 0)$ is the counting process for the number of cancellations of sell/buy limit orders at level $i \in \{1, ..., K\}$ at time t.

9. We model the above processes as independent Poisson processes with intensities $\lambda^{M^{\pm}}$, $\lambda_i^{L^{\pm}}$, $a_i(t) \cdot \lambda_i^{C^+}$, $|b_i(t)| \cdot \lambda_i^{C^+}$ respectively.

correct expressions?

Dynamics

We express the dynamics of the ask side, $\boldsymbol{a}(t)$, of our order book at time t by the following SDE:

$$
\begin{aligned}
da_i(t) = & -\mathbb{1}_{\{a_i(t)\neq 0\}}(q - A_{i-1}(t))_+ dM^+(t) + q dL_i^+(t) - \mathbb{1}_{\{a_i(t)\neq 0\}} q dC_i^+ \\
& + \left(J^{M^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dM^-(t) + \sum_{j=1}^{K} \left(J^{L_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dL_j^-(t) \\
& + \sum_{j=1}^{K} \left(J^{C_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dC_j^-(t)
\end{aligned}
$$

where $x_+ := \max\{0, x\}$ for $x \in \mathbb{R}$, $\boldsymbol{x}_i$ denotes the i-th entry of a vector $\boldsymbol{x}$, and J is a shift operator corresponding to the renumbering of the ask side of the book after an event that affects the bid side of the book, and vice versa.

Explanation and Illustration

We give an illustration of the dynamics expressed by the SDE

$$
\begin{aligned}
da_i(t) = & -\mathbb{1}_{\{a_i(t)\neq 0\}}(q - A_{i-1}(t))_+ dM^+(t) + q dL_i^+(t) - \mathbb{1}_{\{a_i(t)\neq 0\}} q dC_i^+ \\
& + \left(J^{M^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dM^-(t) + \sum_{j=1}^{K} \left(J^{L_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dL_j^-(t) \\
& + \sum_{j=1}^{K} \left(J^{C_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dC_j^-(t)
\end{aligned}
$$

Explanation and Illustration

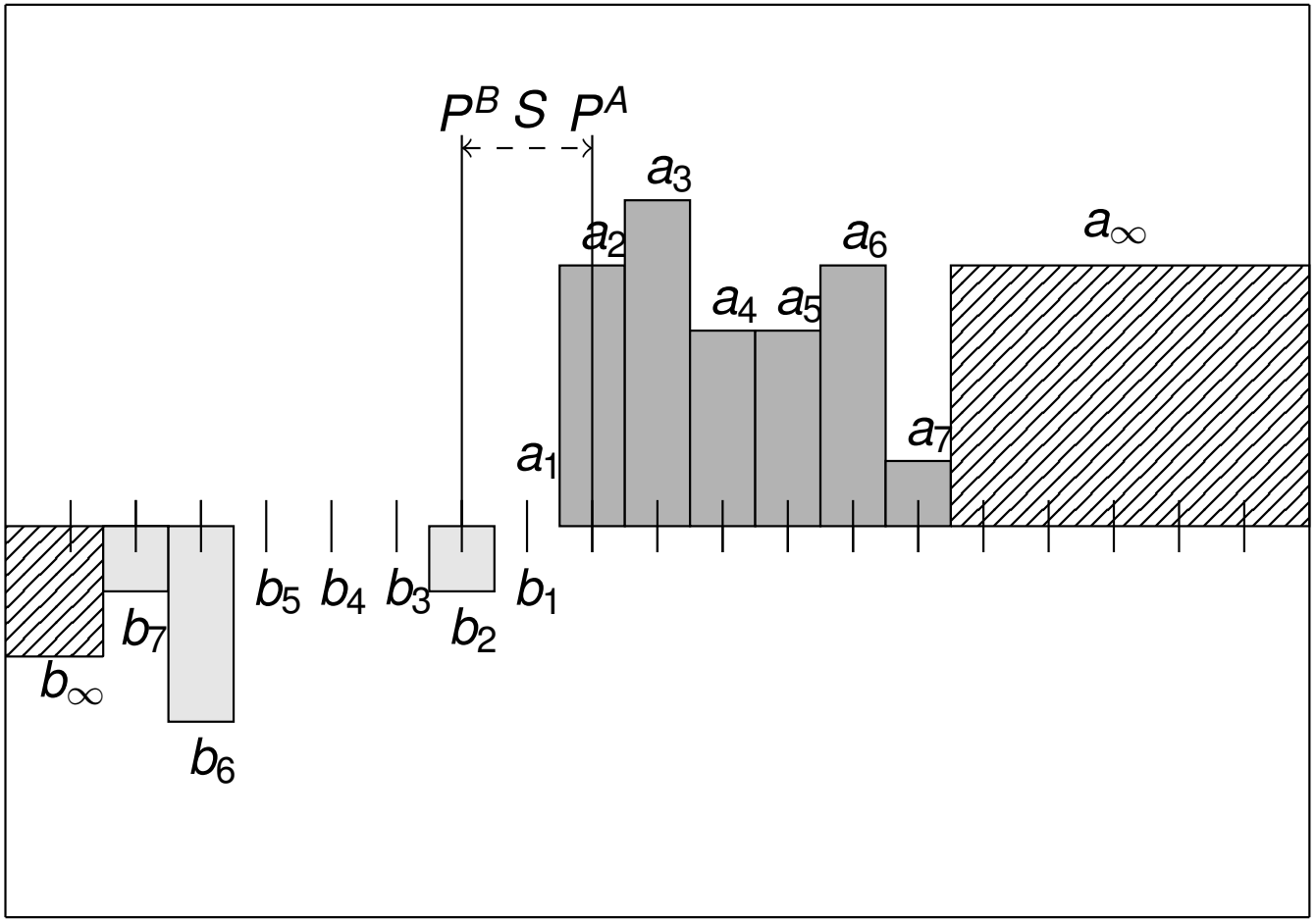

Figure: Order book with $K = 7$ and $q = 1$

Explanation and Illustration

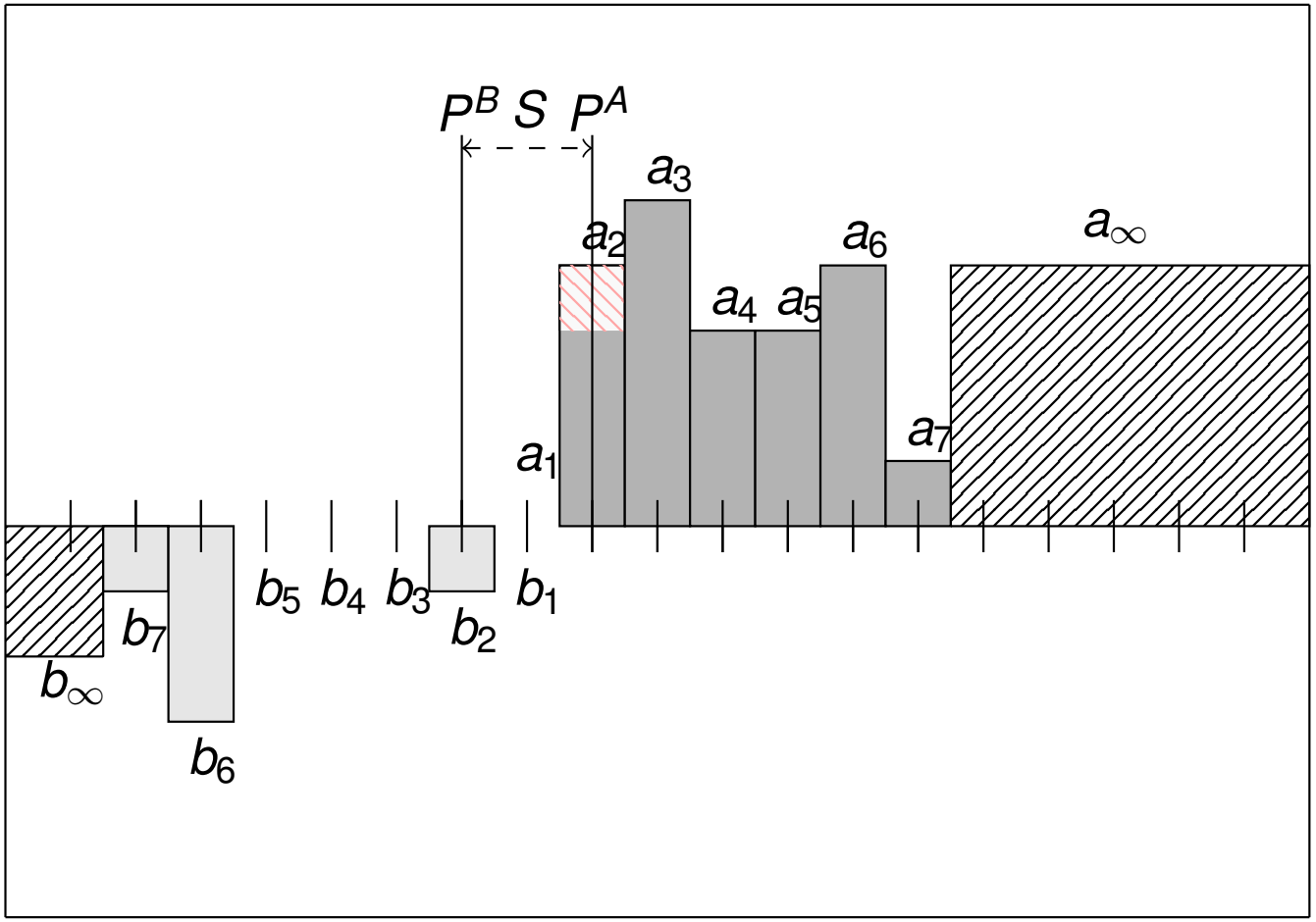

Figure: Incoming buy market order dM^+

Explanation and Illustration

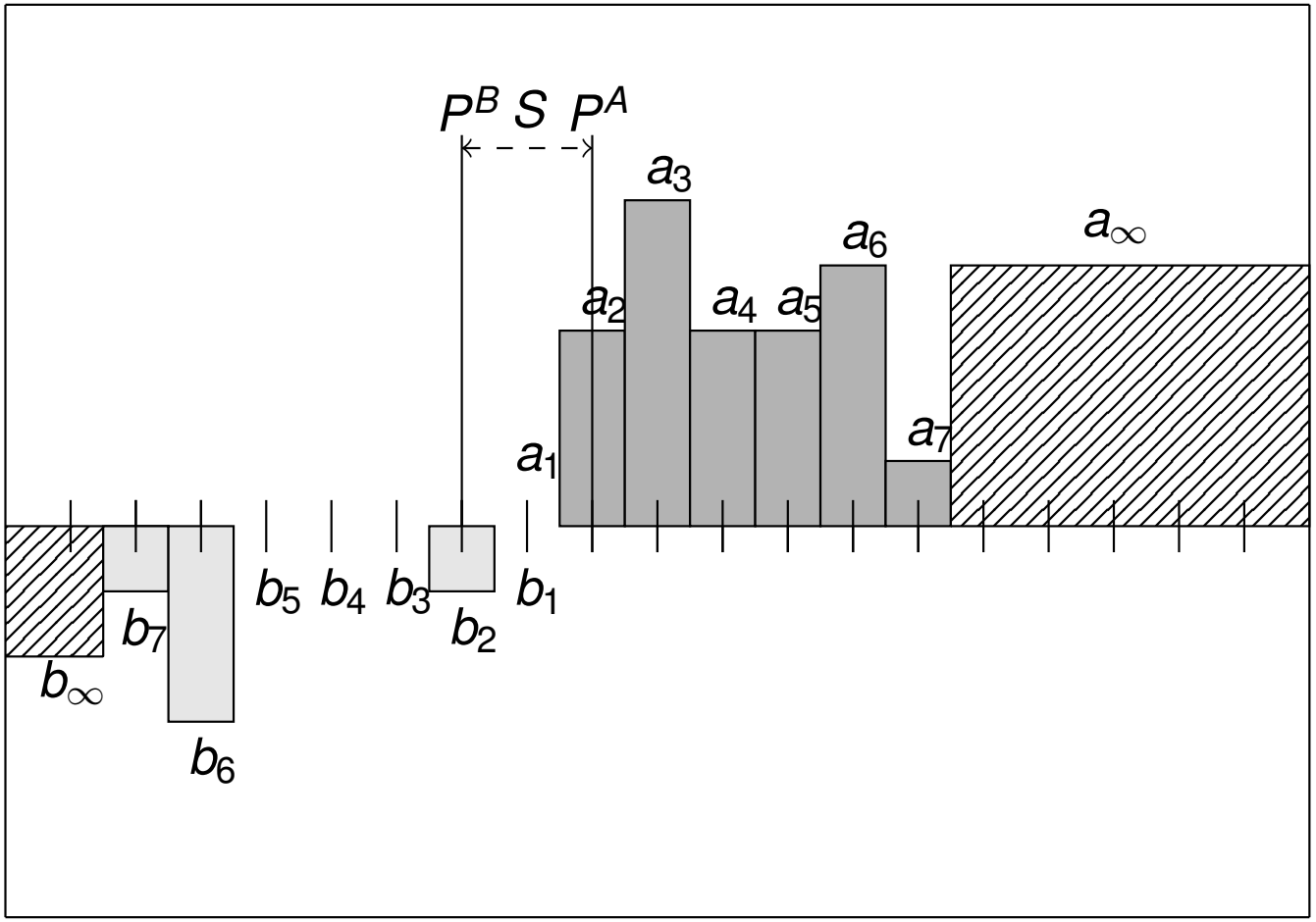

Figure: Resulting order book after dM^+

Explanation and Illustration

We give an illustration of the dynamics expressed by the SDE

$$da_i(t) = -\mathbb{1}_{\{a_i(t)\neq 0\}}(q - A_{i-1}(t))_+ dM^+(t) + qdL_i^+(t) - \mathbb{1}_{\{a_i(t)\neq 0\}} qdC_i^+$$
$$+ \left(J^{M^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dM^-(t) + \sum_{j=1}^{K} \left(J^{L_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dL_j^-(t)$$
$$+ \sum_{j=1}^{K} \left(J^{C_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dC_j^-(t)$$

Explanation and Illustration

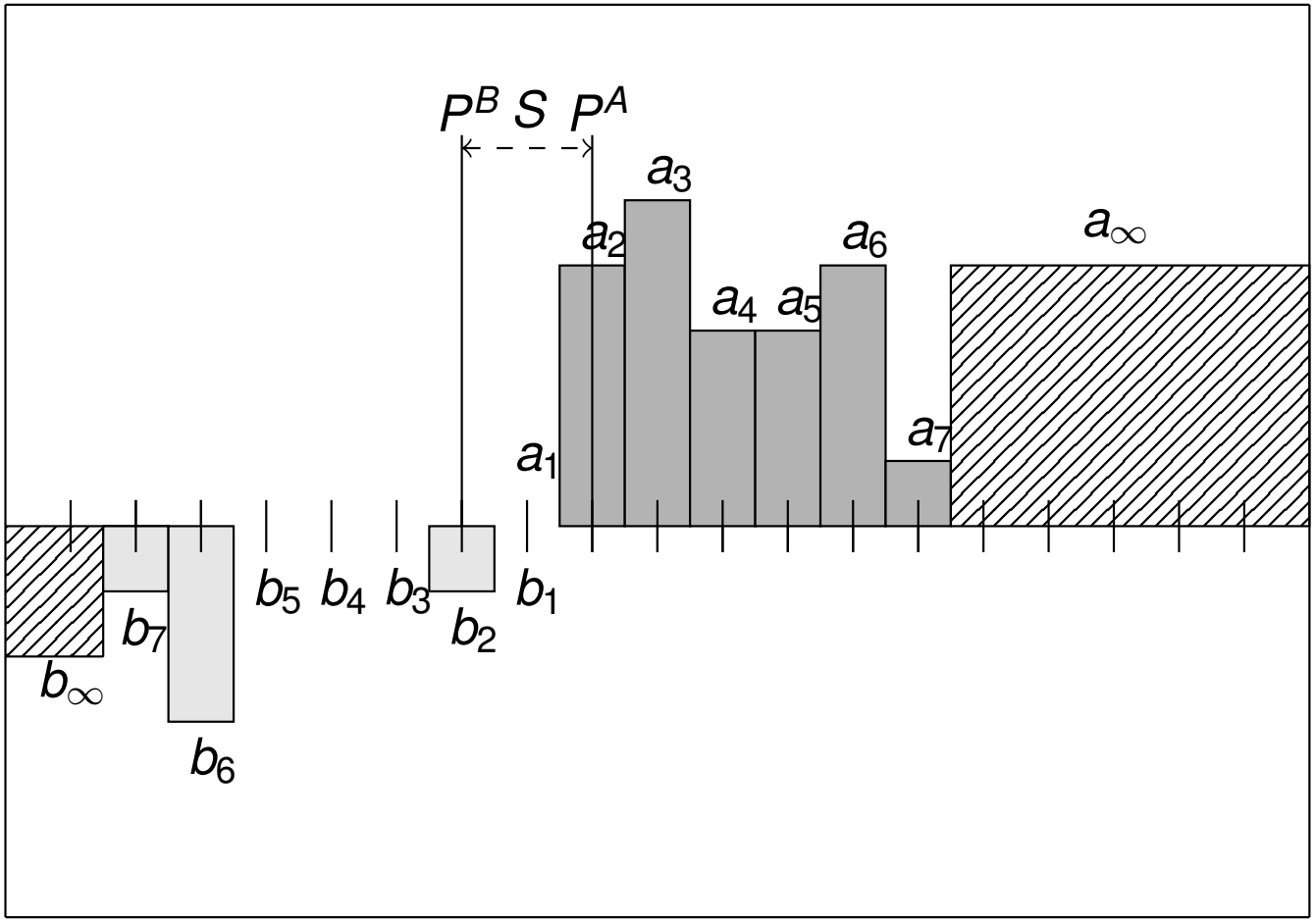

Figure: Order book with $K = 7$ and $q = 1$

Explanation and Illustration

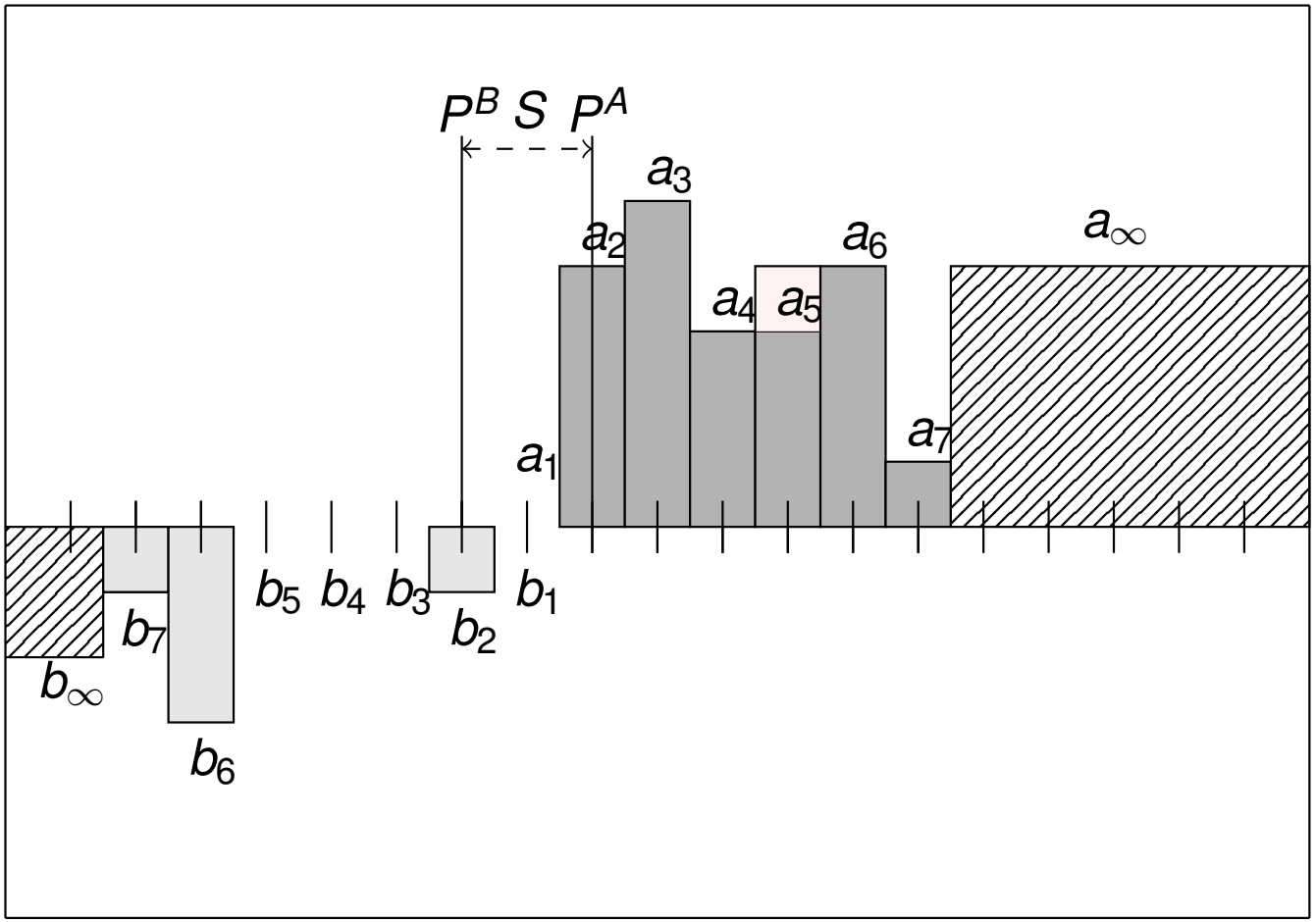

Figure: Incoming ask limit order at level $i = 5$, dL_5^+

Explanation and Illustration

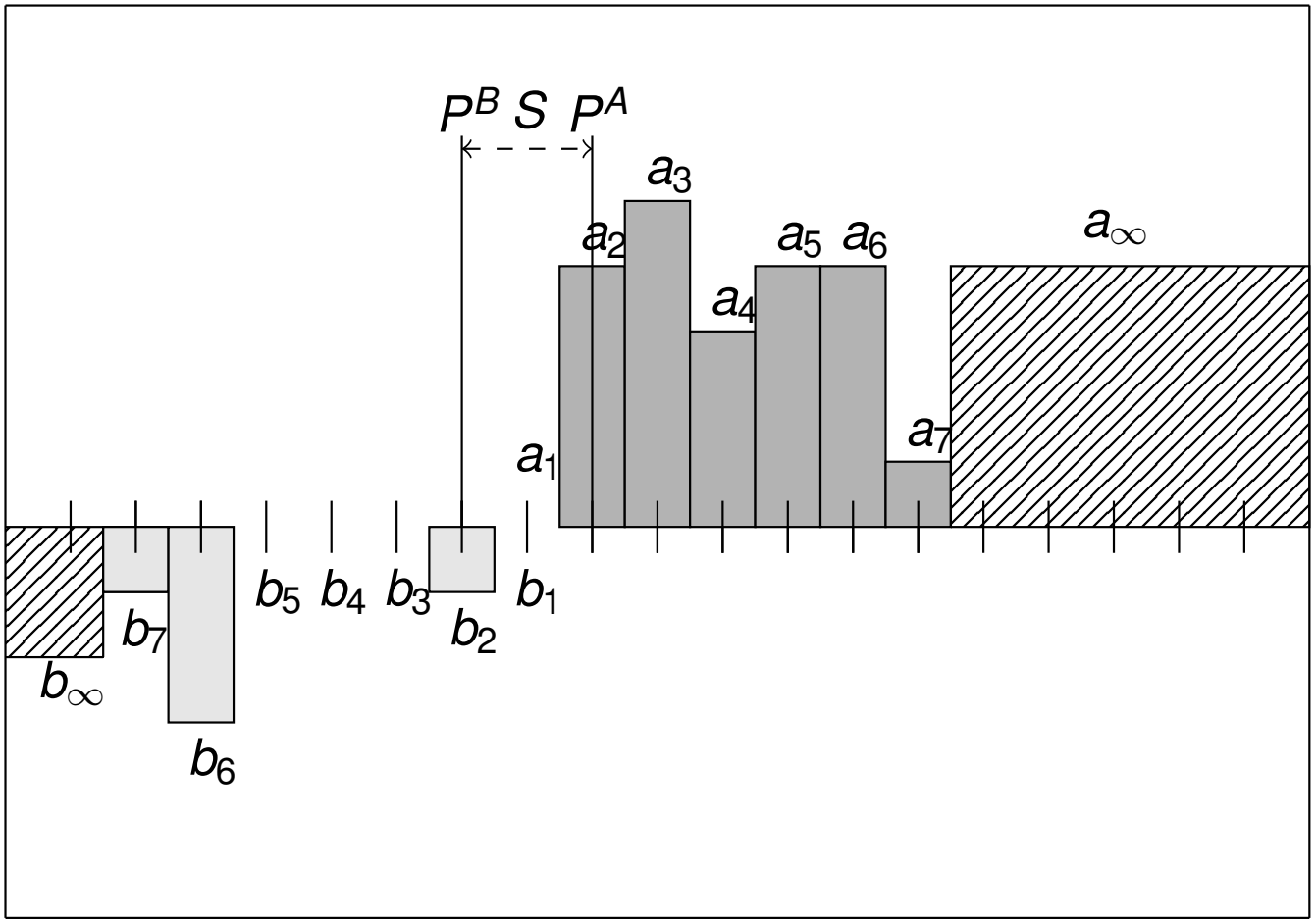

Figure: Resulting order book after dL_5^+

Explanation and Illustration

We give an illustration of the dynamics expressed by the SDE

$$
\begin{aligned}
da_i(t) = & -\mathbb{1}_{\{a_i(t)\neq 0\}}(q - A_{i-1}(t))_+ dM^+(t) + q dL_i^+(t) - \mathbb{1}_{\{a_i(t)\neq 0\}} q dC_i^+ \\
& + \left(J^{M^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dM^-(t) + \sum_{j=1}^{K} \left(J^{L_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dL_j^-(t) \\
& + \sum_{j=1}^{K} \left(J^{C_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dC_j^-(t)
\end{aligned}
$$

Explanation and Illustration

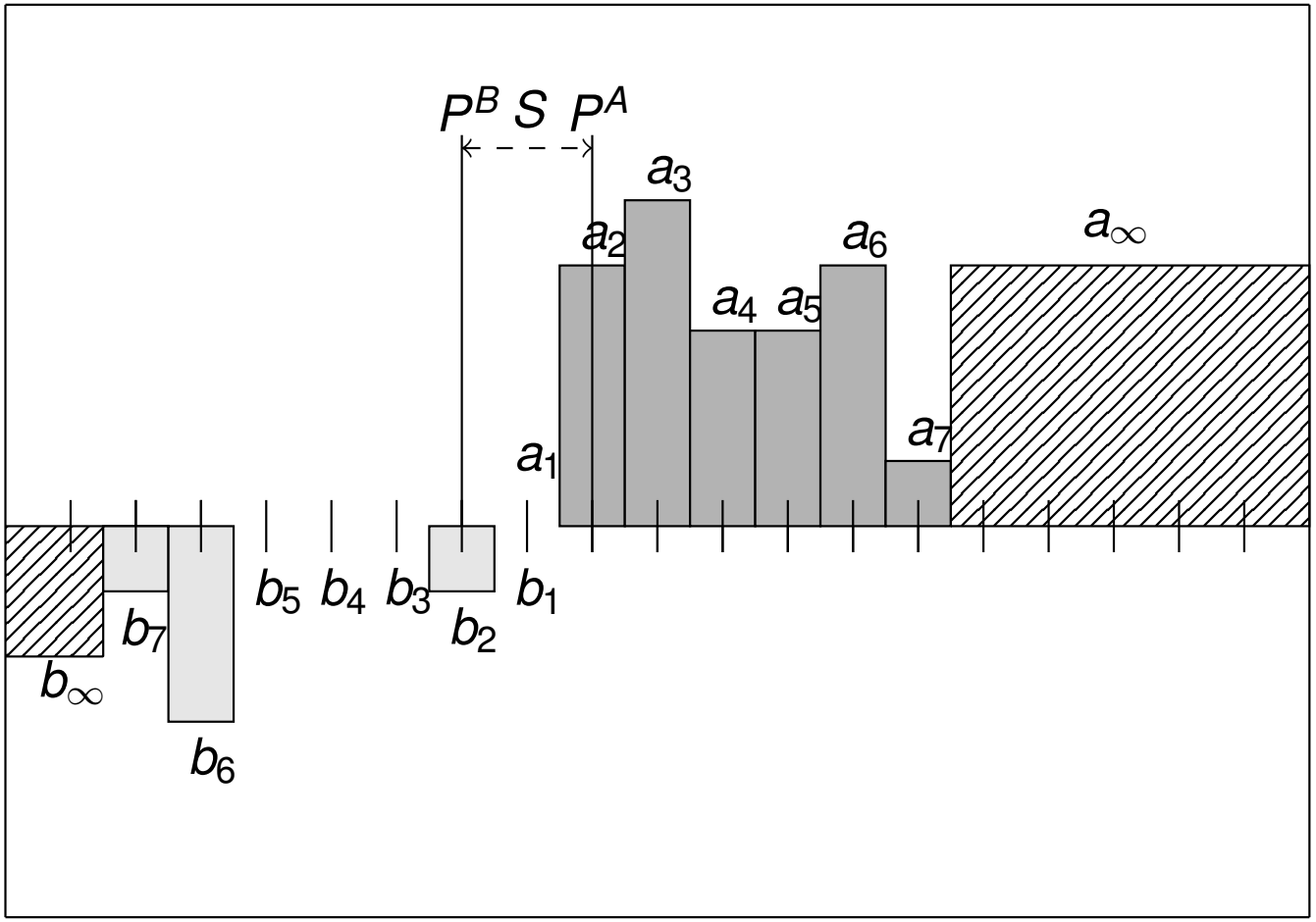

Figure: Order book with $K = 7$ and $q = 1$

Explanation and Illustration

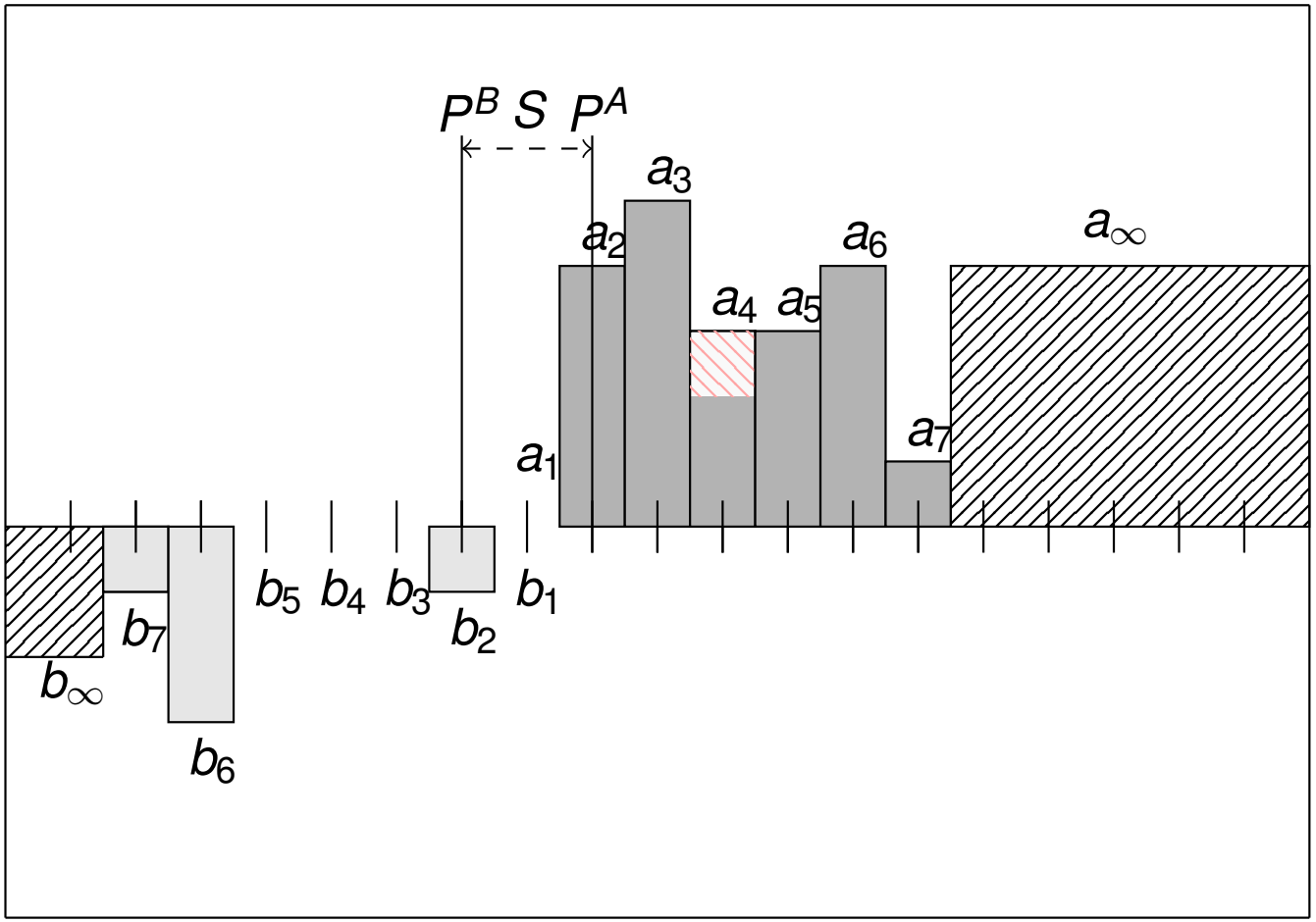

Figure: Incoming cancellation of an ask limit order at level $i = 4$, dC_4^+

Explanation and Illustration

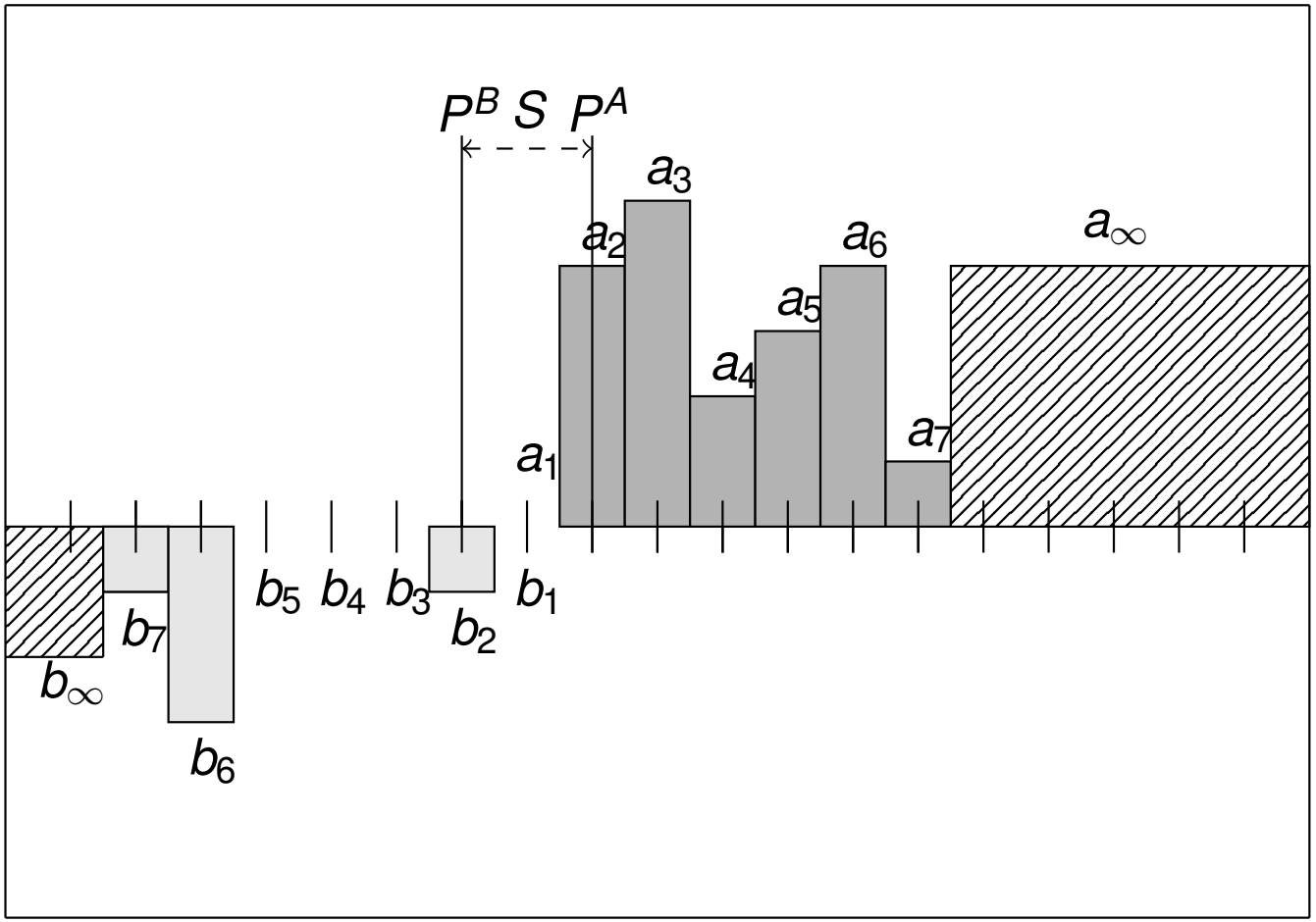

Figure: Resulting order book after dC_4^+

Explanation and Illustration

We give an illustration of the dynamics expressed by the SDE

$$da_i(t) = -\mathbb{1}_{\{a_i(t)\neq 0\}}(q - A_{i-1}(t))_+ dM^+(t) + q dL_i^+(t) - \mathbb{1}_{\{a_i(t)\neq 0\}} q dC_i^+($$

$$+ \left(J^{M^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dM^-(t) + \sum_{j=1}^{K} \left(J^{L_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dL_j^-(t)$$

$$+ \sum_{j=1}^{K} \left(J^{C_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dC_j^-(t)$$

Explanation and Illustration

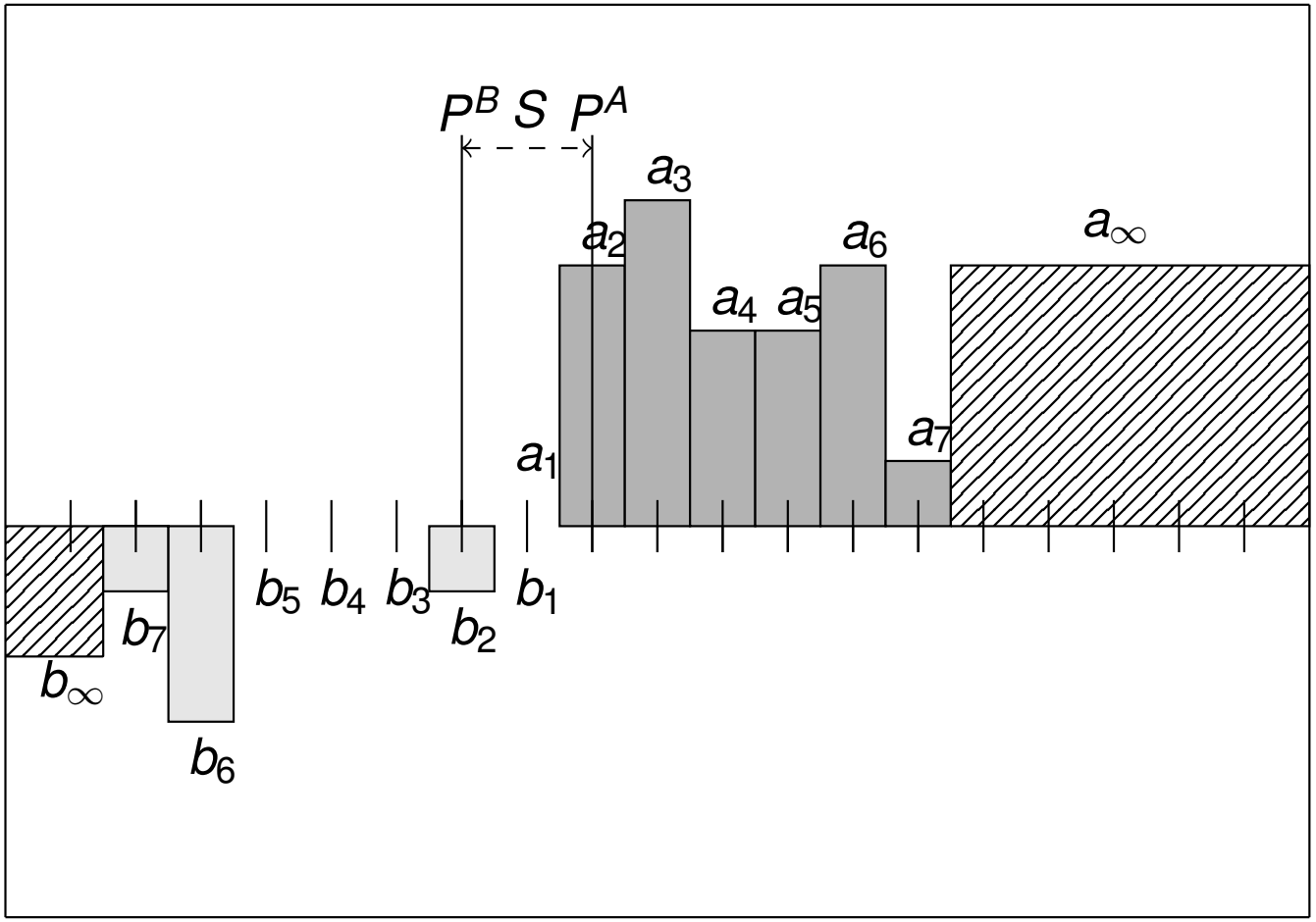

Figure: Order book with $K = 7$ and $q = 1$

Explanation and Illustration

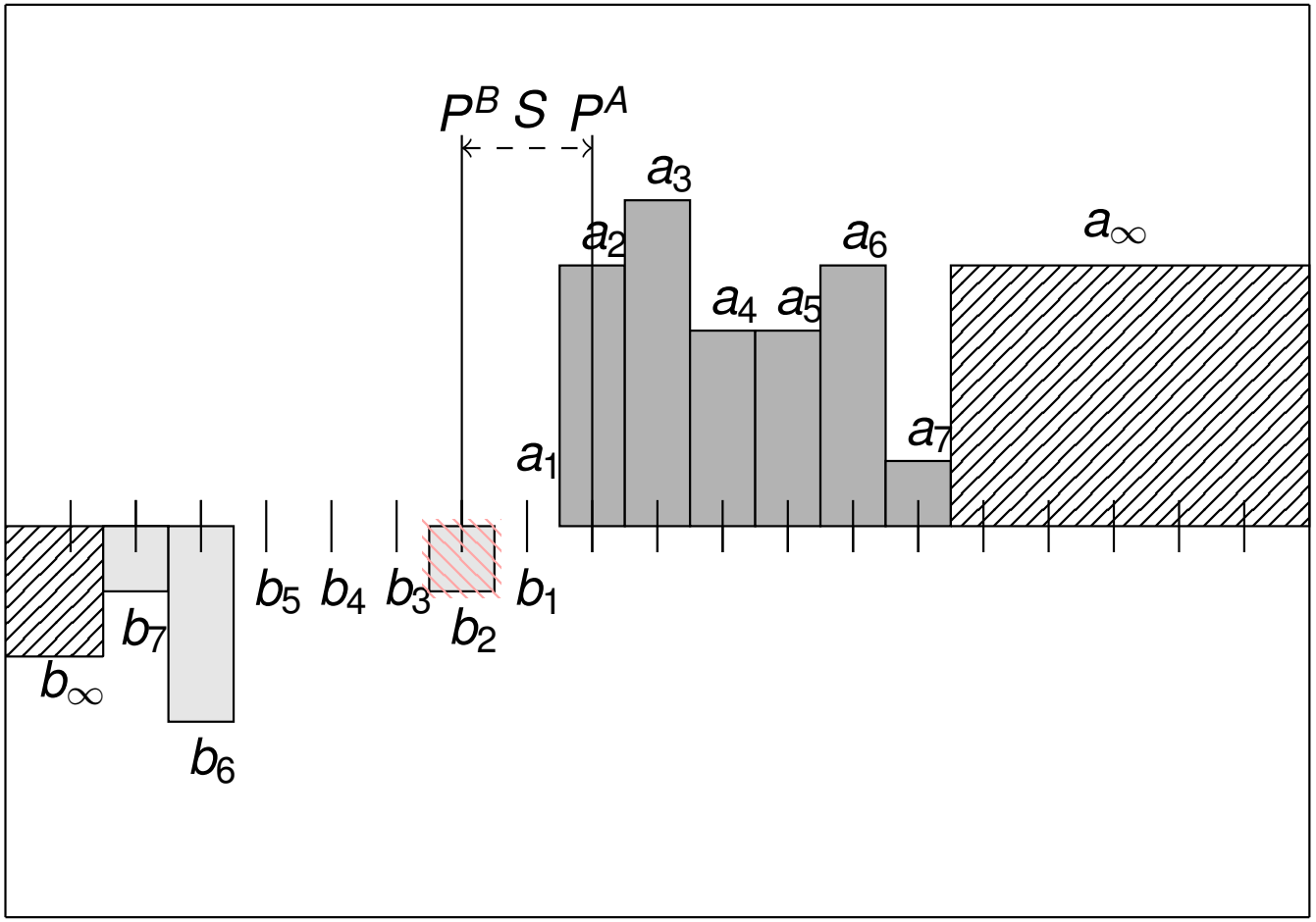

Figure: Incoming sell market order dM^-

Explanation and Illustration

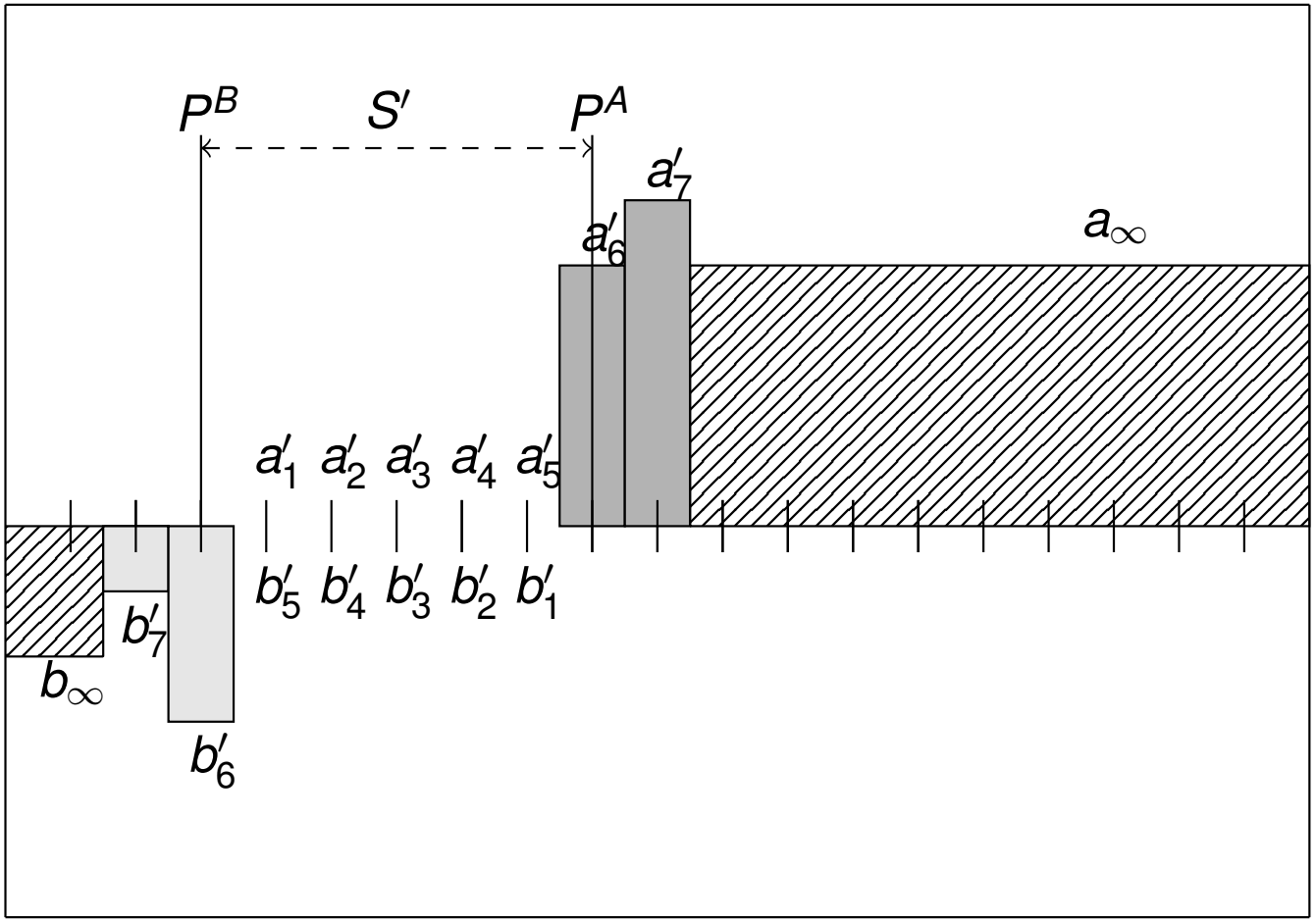

Figure: Resulting order book after dM^-

Explanation and Illustration

We give an illustration of the dynamics expressed by the SDE

$$da_i(t) = -\mathbb{1}_{\{a_i(t)\neq 0\}}(q - A_{i-1}(t))_+ dM^+(t) + q dL_i^+(t) - \mathbb{1}_{\{a_i(t)\neq 0\}} q dC_i^+($$

$$+ \left(J^{M^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dM^-(t) + \sum_{j=1}^{K} \left(J^{L_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dL_j^-(t)$$

$$+ \sum_{j=1}^{K} \left(J^{C_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dC_j^-(t)$$

Explanation and Illustration

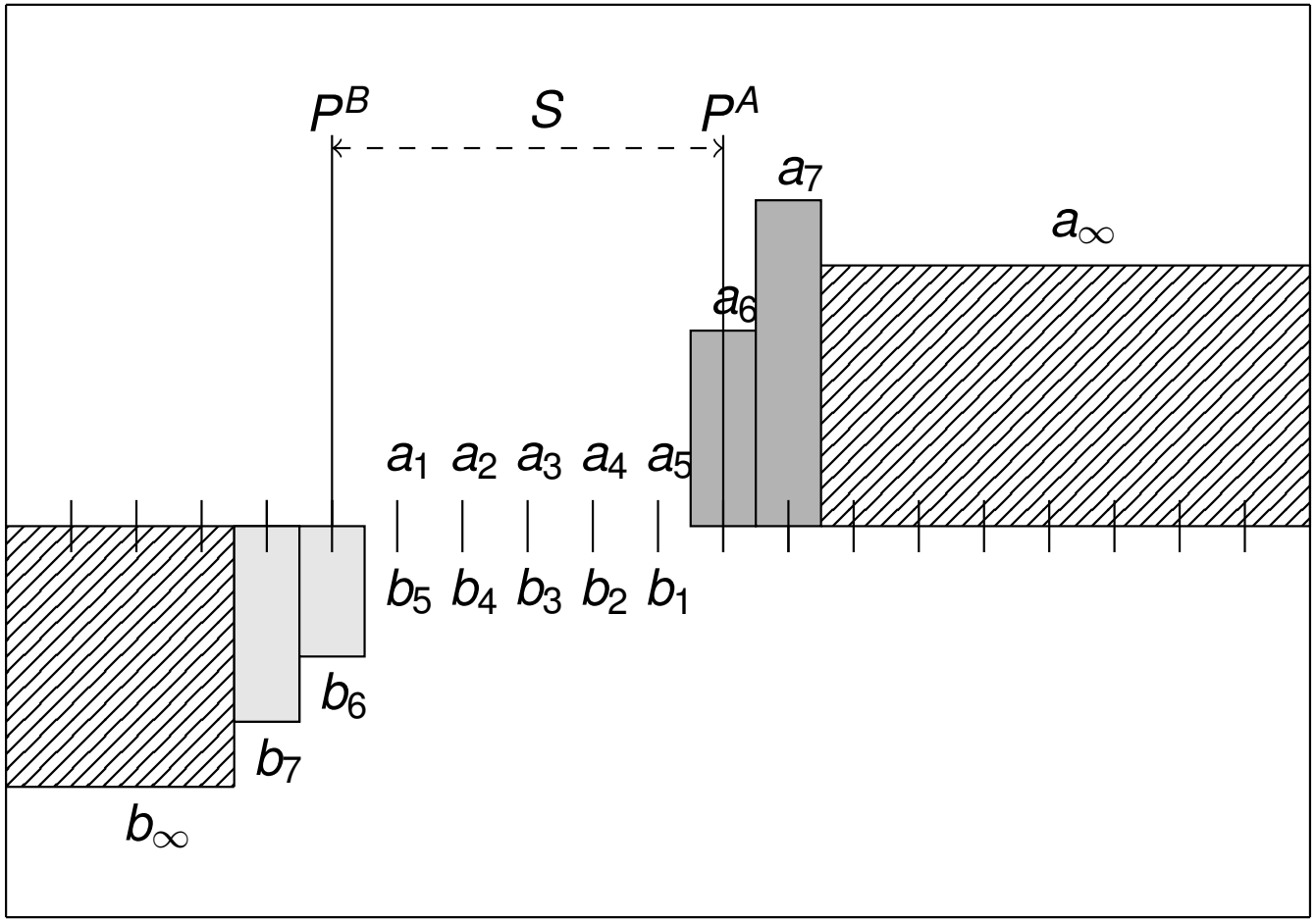

Figure: Order book with $K = 7$ and $q = 1$

Explanation and Illustration

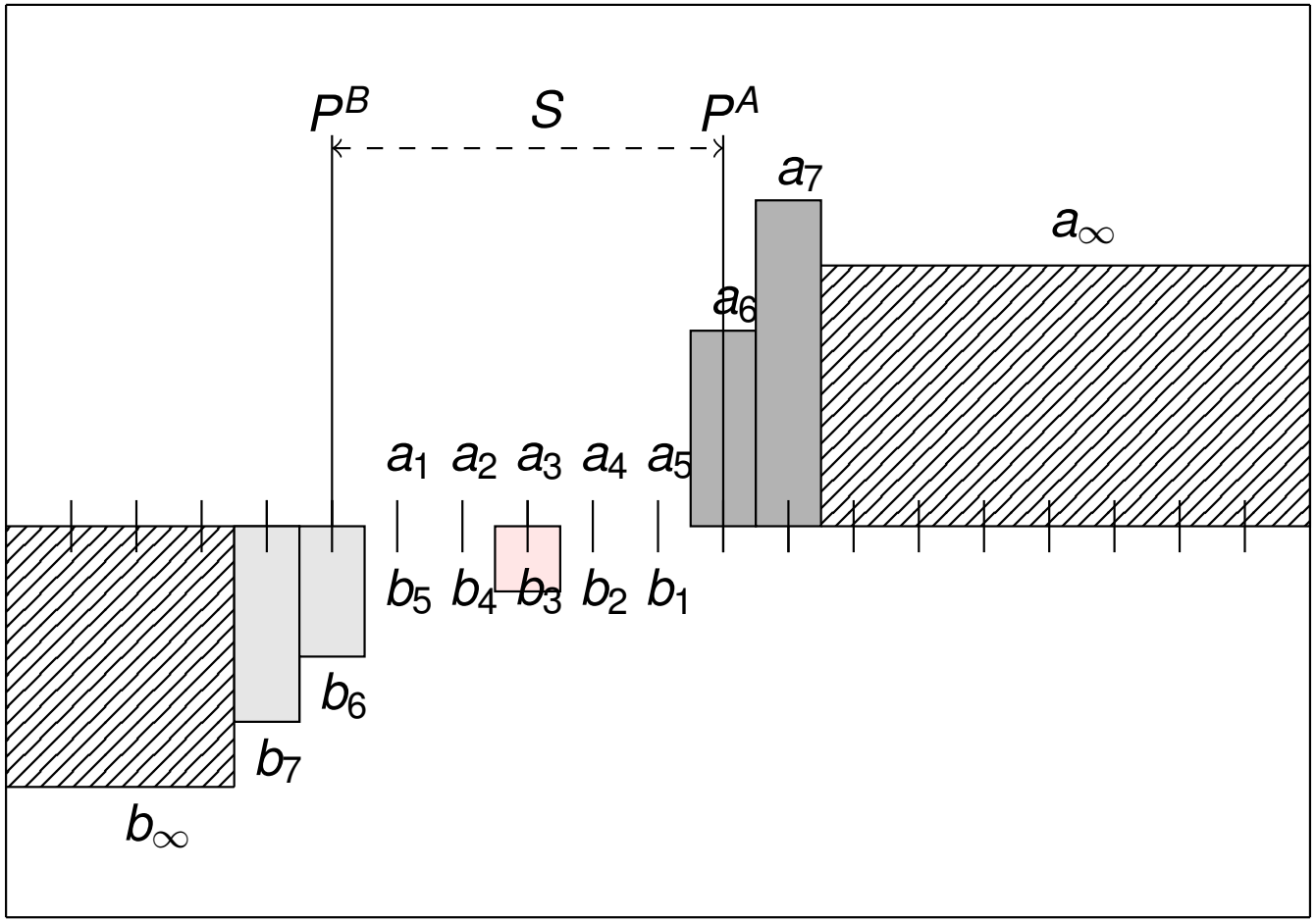

Figure: Incoming buy limit order at level $i = 3$, dL_3^-

Explanation and Illustration

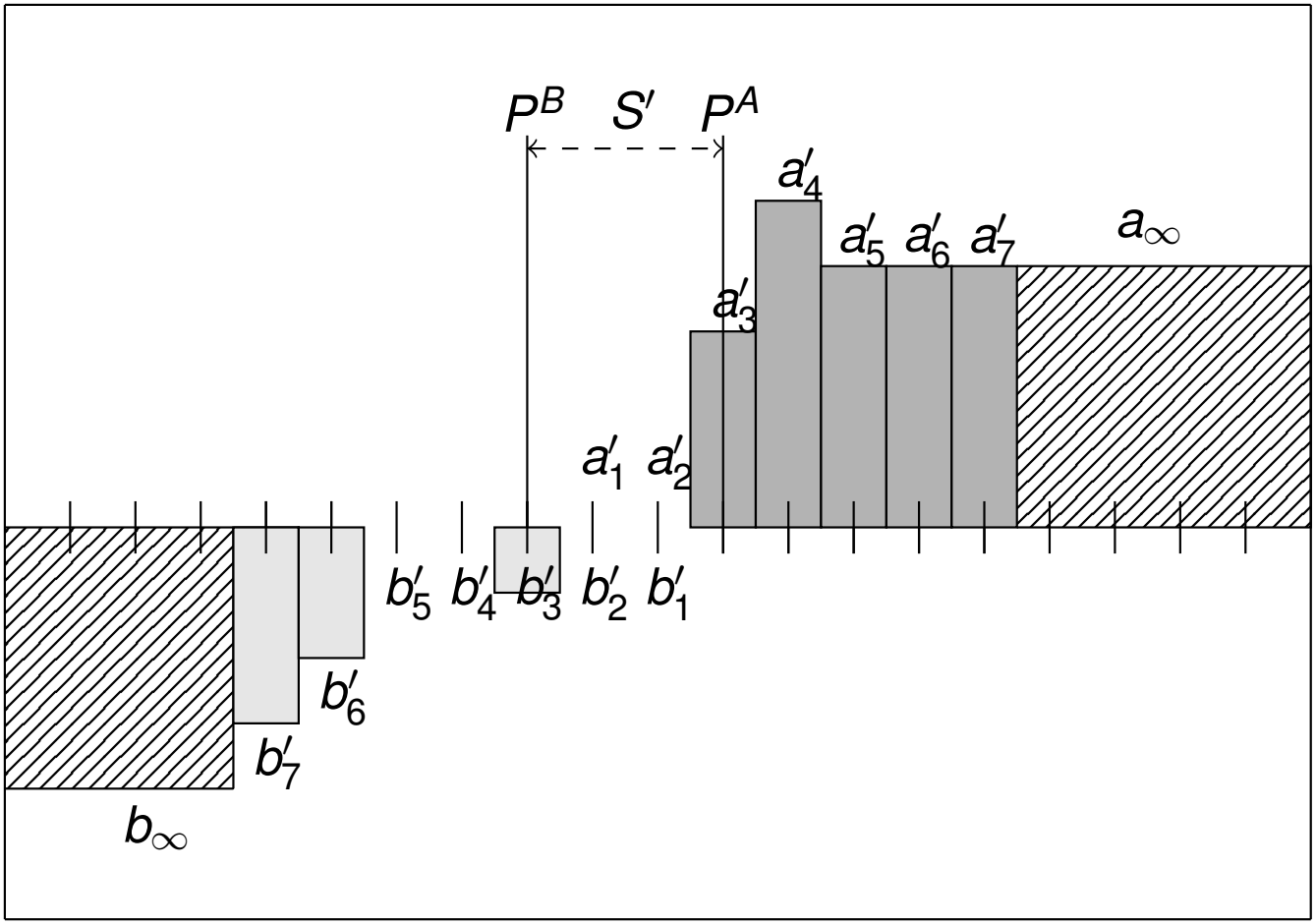

Figure: Resulting order book after dL_3^-

Explanation and Illustration

We give an illustration of the dynamics expressed by the SDE

$$da_i(t) = -\mathbb{1}_{\{a_i(t)\neq 0\}}(q - A_{i-1}(t))_+ dM^+(t) + q dL_i^+(t) - \mathbb{1}_{\{a_i(t)\neq 0\}} q dC_i^+($$

$$+ \left(J^{M^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dM^-(t) + \sum_{j=1}^{K} \left(J^{L_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dL_j^-(t)$$

$$+ \sum_{j=1}^{K} \left(J^{C_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t)\right)_i dC_j^-(t)$$

Explanation and Illustration

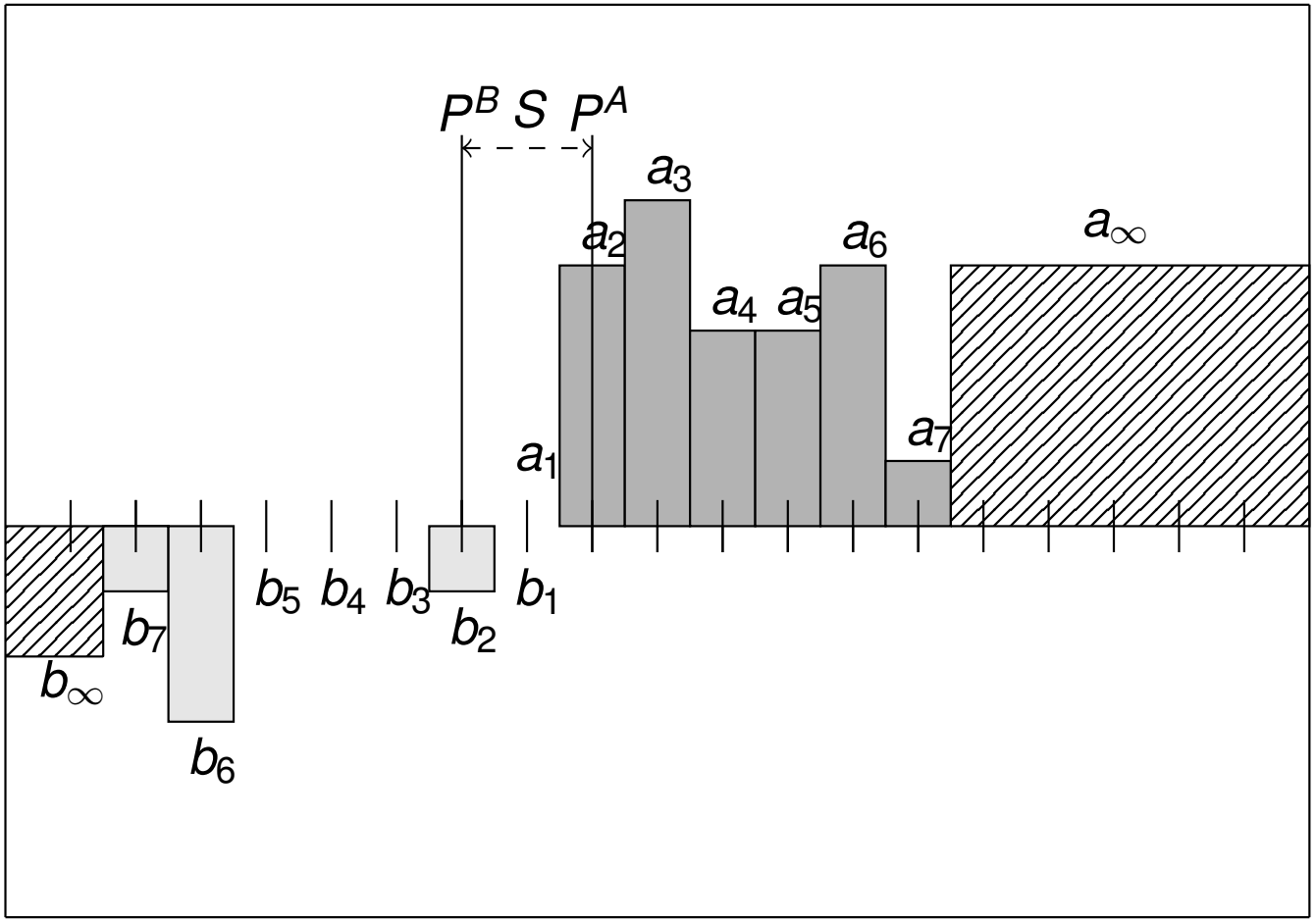

Figure: Order book with $K = 7$ and $q = 1$

Explanation and Illustration

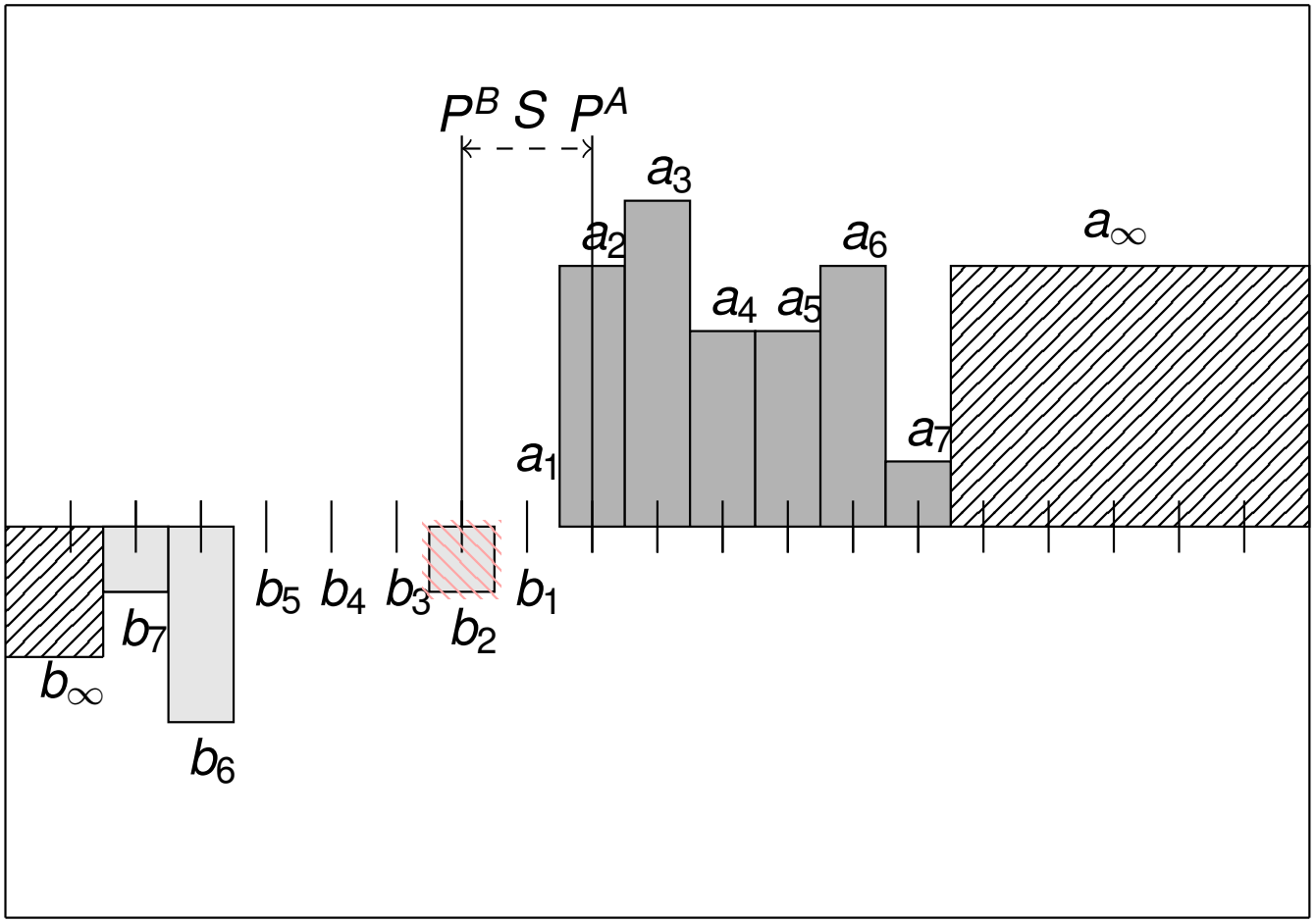

Figure: Incoming cancellation of a buy limit order at level $i = 2$, dC_2^-

Explanation and Illustration

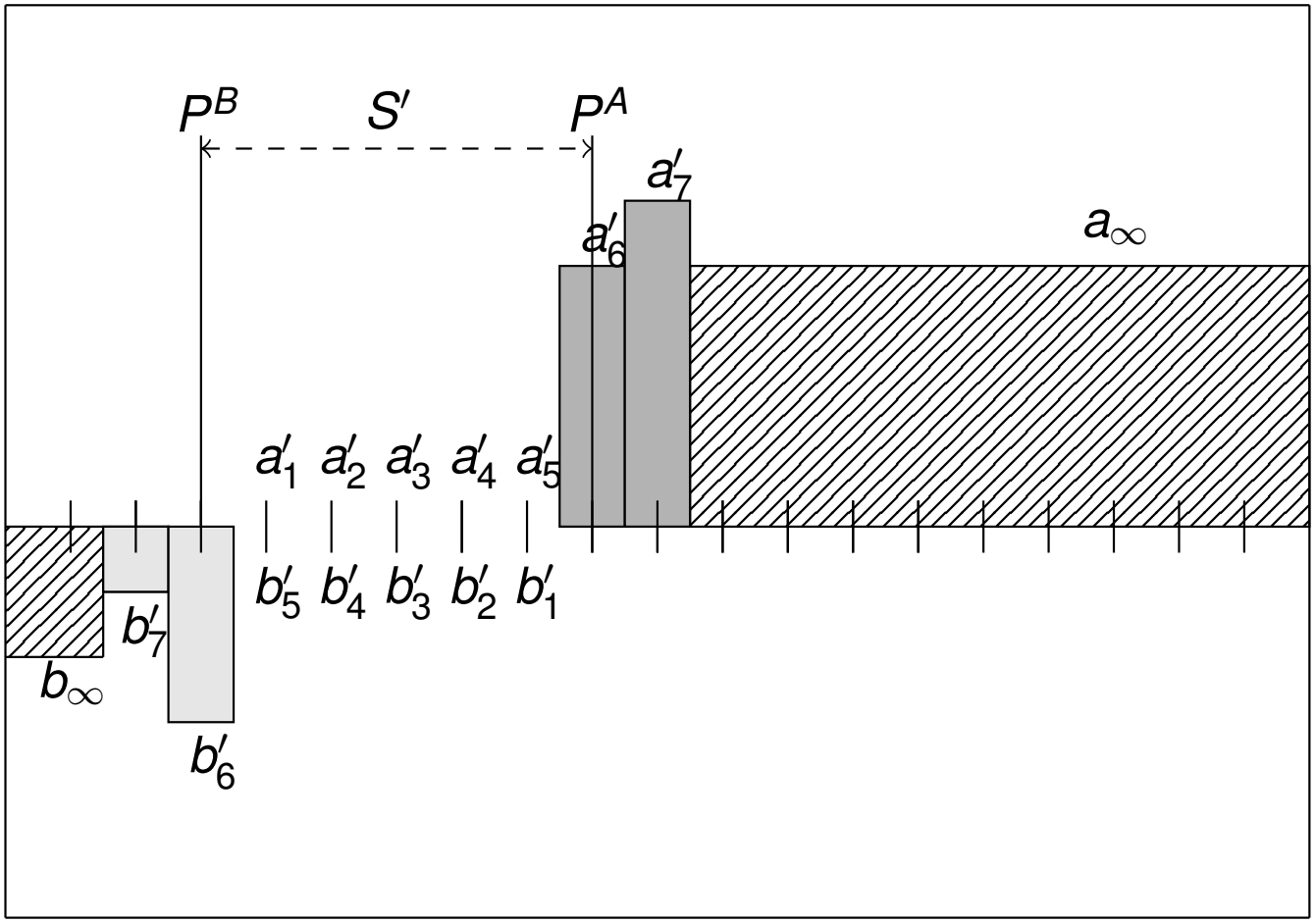

Figure: Resulting order book after dC_2^-

Dynamics

We now express the dynamics of the bid side, $\boldsymbol{b}(t)$, of our order book at time t by the following SDE:

$$
\begin{aligned}
db_i(t) = {} & \mathbb{1}_{\{b_i(t)\neq 0\}}(q - B_{i-1}(t))_+ dM^-(t) - q dL_i^-(t) + \mathbb{1}_{\{b_i(t)\neq 0\}} q dC_i^-(t) \\
& + \left(J^{M^+}(\boldsymbol{b}(t)) - \boldsymbol{b}(t)\right)_i dM^+(t) + \sum_{j=1}^{K} \left(J^{L_j^+}(\boldsymbol{b}(t)) - \boldsymbol{b}(t)\right)_i dL_j^+(t) \\
& + \sum_{j=1}^{K} \left(J^{C_j^+}(\boldsymbol{b}(t)) - \boldsymbol{b}(t)\right)_i dC_j^+(t)
\end{aligned}
$$

The signs of the above expression are as they are because we have to account for the fact that by convention the $b_i(t)$'s are non-positive.

Preliminaries

Definition

A family of mappings $(T^t : t \geq 0)$ from S to S, parameterized by non-negative numbers t is said to form a *semigroup* if:

1. T^0 is the identity mapping in S.
2. $T^t T^s = T^{t+s} \qquad$ for all $t, s \geq 0$

Definition

A semigroup $(T^t : t \geq 0)$ of bounded linear operators on a Banach space V is called *strongly continuous* if

$$\| T^t(f) - f \| \longrightarrow 0 \text{ as t} \rightarrow 0 \qquad \text{for all } f \in V$$

where $\| \cdot \|$ is the norm of the Banach space V.

Preliminaries

Definition

Let $(T_t : t \geq 0)$ be a strongly continuous semigroup of linear operators on a Banach space V. The *infinitesimal generator* (or simply, *generator*) of $(T_t : t \geq 0)$ is defined as the operator:

$$A(f) = \lim_{t \to 0+} \frac{T_t(f) - f}{t}, \qquad f \in V \tag{1}$$

which is defined on the linear subspace $D_A \subset V$ where this limit exists.

Some Definitions

Definition

A *transition kernel* from a measurable space $(X, \mathcal{F})$ to a measurable space $(Y, \mathcal{G})$ is a function of two variables $\mu(x, A)$, $x \in X, A \in \mathcal{G}$ such that

1. For fixed $A \in \mathcal{G}$, $\mu(x, A)$ as a function of $x \in X$ is $\mathcal{F}$-measurable.
2. For fixed $x \in X$, $\mu(x, A)$ as a function of $A \in \mathcal{G}$ is a measure for the measurable space $(Y, \mathcal{G})$.

The transition kernel is *bounded* if $\sup_{x \in X} \|\mu(x, \cdot)\|$ where $\|\cdot\|$ is a norm on the space of measures of $(Y, \mathcal{G})$. Moreover, if all measures $\mu(x, \cdot)$ on $(Y, \mathcal{G})$ are probability measures, then we say that $\mu(x, A)$ is a *transition probability kernel*.

Some Definitions

Definition

An adapted process $X = (X_t : t \geq 0)$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$ with values in $\mathbb{R}$ is said to satisfy the *Markov Property* if

$$\mathbb{E}(f(X_t)|\mathcal{F}_s) = \mathbb{E}(f(X_t)|X_s) \text{ almost surely}$$

for all $f \in B(\mathbb{R})$ and $0 \leq s \leq t$, where $B(\mathbb{R})$ denotes the (Banach) space of bounded Borel functions on $\mathbb{R}$.

Some Definitions

Definition

a *Markov Process* in $\mathbb{R}$ is a family of processes $((X^{s,x}_{t\geq s}) : s \in \mathbb{R}_+, x \in \mathbb{R})$ depending on $s \in \mathbb{R}_+$ and $x \in \mathbb{R}$ as parameters (the process $(X^{s,x}_{t\geq s})$ is a process with initial value x at time s) such that there exists a family of transition probability kernels $p_{s,t}(x, A)$ from $\mathbb{R}$ to $\mathbb{R}$, $0 \leq s \leq t$ where $A \in \mathscr{B}(\mathbb{R})$ (where $\mathscr{B}(\mathbb{R})$ denotes the Borel sigma algebra of $\mathbb{R}$) which we'll call *transition probabilities*, such that

$$\mathbb{E}(f(X^{s,x}_t)|\mathcal{F}_u)) = \mathbb{E}(f(X^{s,x}_t)|X^{s,x}_u)) \tag{2}$$

$$= \int_{\mathbb{R}} f(y) p_{u,t}(X^{s,x}_u, dy) \text{ almost surely} \tag{3}$$

for all $f \in B(\mathbb{R})$ and $0 \leq s \leq u \leq t$.

Some Definitions

Definition

The operator

$$\Phi^{s,t}(f(x)) = \int_{\mathbb{R}} f(y) p_{s,t}(x, dy)$$

from (3) is called the *transition operator* of the Markov Process $((X^{s,x}_{t \geq s}) : s \in \mathbb{R}_+, x \in \mathbb{R})$.

Definition

A Markov Process is called *time-homogeneous* if $\Phi^{s,t}$ and $p_{s,t}(x, A)$ depend only on the difference $t - s$.

Important Theorem

Theorem

The family of the transition operators $(\Phi^{s,t} : s \leq t)$ *of a Markov process in* $\mathbb{R}$ *forms a Markov propagator in* $B(\mathbb{R})$.

Moreover, if this Markov process is time-homogeneous, the family $(\Phi_t(f(x))) = \mathbb{E}_x(f(X_t)), \;\; t \geq 0$ *forms a Markov semi-group.*

Generator of $(\boldsymbol{X}(t) : t \geq 0)$

$$
\begin{aligned}
\mathcal{L}f(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(f([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - f(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(f(a_i + q; J^{L_i^+}(\boldsymbol{b})) - f(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(f(a_i - q; J^{C_i^+}(\boldsymbol{b})) - f(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(f(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - f(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(f(J^{L_i^-}(\boldsymbol{a}); b_i - q) - f(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(f(J^{C_i^-}(\boldsymbol{a}); b_i + q) - f(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

Proof

Sketch proof.

Recall that $(X(t) : t \geq 0)$ where $X(t) := (\boldsymbol{a}(t); \boldsymbol{b}(t)) = (a_1(t), ..., a_K(t); b_1(t), ..., b_K(t))$, is the $2K$-dimensional Markovian model of the LOB.

Let $\boldsymbol{x} = (\boldsymbol{a}; \boldsymbol{b}) = (a_1, ..., a_K; b_1, ..., b_K)$ where $a_i \in q\mathbb{Z}_+$ and $b_i \in q\mathbb{Z}_-$.

By the construction of our Markov process, our process is time-homogeneous. Hence, Theorem tells us that the family $(\Phi_t(f(\boldsymbol{x})) : t \geq 0)$ where $\Phi_t(f(\boldsymbol{x})) = \mathbb{E}(f(X(t))|X(0) = \boldsymbol{x})$, forms a Markov semigroup.

Theorem

The family of the transition operators $(\Phi^{s,t} : s \leq t)$ *of a Markov process in* $\mathbb{R}$ *forms a Markov propagator in* $B(\mathbb{R})$.

Moreover, if this Markov process is time-homogeneous, the family $(\Phi_t(f(x))) = \mathbb{E}_x(f(X_t)), \quad t \geq 0$ *forms a Markov semi-group.*

Proof

Sketch proof.

The regularity assumption of f we have assumed allows us to compute the generator of our process.

$$\begin{aligned}\mathcal{L}f(\boldsymbol{x}) = \mathcal{L}f(\boldsymbol{a};\boldsymbol{b}) &= \lim_{t\to 0+} \frac{\mathbb{E}(f(X(t))|X(0)=\boldsymbol{x}) - f(\boldsymbol{x})}{t} \\ &= \lim_{t\to 0+} \frac{\mathbb{E}(f(X(t)) - f(\boldsymbol{x})|X(0)=\boldsymbol{x})}{t}\end{aligned}$$

To compute $\mathbb{E}_{\boldsymbol{x}}(f(X(t) - f(\boldsymbol{x})))$ we condition on every type of event that can occur in the order book and multiply by the probability of such an event occurring, and then applying the law of total expectation.

Definition

Let $(T_t : t \geq 0)$ be a strongly continuous semigroup of linear operators on a Banach space V. The *infinitesimal generator* (or simply, *generator*) of $(T_t : t \geq 0)$ is defined as the operator:

$$A(f) = \lim_{t \to 0+} \frac{T_t(f) - f}{t}, \qquad f \in V \tag{4}$$

which is defined on the linear subspace $D_A \subset V$ where this limit exists.

Proof

Sketch proof.

$$
\begin{aligned}
\mathbb{E}_{\boldsymbol{x}}(f(X(t)-f(\boldsymbol{x}))) = {} & \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|dM^{+}(t))\mathbb{P}(\text{"market buy order at time t"}) \\
& + \sum_{i=1}^{K} \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|dL_i^{+}(t))\mathbb{P}(\text{"limit sell order at level i at time t"}) \\
& + \sum_{i=1}^{K} \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|dC_i^{+}(t))\mathbb{P}(\text{"cancellation of an ask at level i at time t"}) \\
& + \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|dM^{-}(t))\mathbb{P}(\text{"market sell order at time t"}) \\
& + \sum_{i=1}^{K} \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|dL_i^{-}(t))\mathbb{P}(\text{"limit buy order at level i at time t"}) \\
& + \sum_{i=1}^{K} \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|dC_i^{-}(t))\mathbb{P}(\text{"cancellation of a bid at level i at time t"})
\end{aligned}
$$

Proof

Sketch proof.

By utilizing the dynamics given by the SDEs for the bid and ask side of the LOB, and by the definition of Poisson processes:

$$
\begin{aligned}
\mathbb{E}_{\boldsymbol{x}}(f(X(t)-f(\boldsymbol{x}))) = {} & (f([a_i(t)-(q-A_{i-1}(t))_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b}(t)))-f)\cdot\lambda^{M^+}te^{-\lambda^{M^+}t} \\
& + \sum_{i=1}^{K}(f(a_i(t)+q; J^{L_i^+}(\boldsymbol{b}(t)))-f)\cdot\lambda_i^{L^+}te^{-\lambda_i^{L^+}t} \\
& + \sum_{i=1}^{K}(f(a_i(t)-q; J^{C_i^+}(\boldsymbol{b}(t)))-f)\cdot a_i(t)\lambda_i^{C^+}te^{-a_i(t)\lambda_i^{C^+}t} \\
& + (f(J^{M^-}(\boldsymbol{a}(t); [b_i(t)+(q-B_{i-1}(t))_+]_-^{i=1,\dots,K})-f)\cdot\lambda^{M^-}te^{-\lambda^{M^-}t} \\
& + \sum_{i=1}^{K}(f(J^{L_i^-}(\boldsymbol{a}(t)); b_i-q)-f)\cdot\lambda_i^{L^-}te^{-\lambda_i^{L^-}t} \\
& + \sum_{i=1}^{K}(f(J^{C_i^-}(\boldsymbol{a}(t)); b_i+q)-f)\cdot|b_i(t)|\lambda_i^{C^-}te^{-|b_i(t)|\lambda_i^{C^-}t}
\end{aligned}
$$

Proof

Sketch proof.

Now divide the previous equation by t and take the limit $t \to 0+$

$$\boldsymbol{a}(t) = (a_1(t), ..., a_K(t)) \longrightarrow \boldsymbol{a}(0) = \boldsymbol{a} = (a_1, ..., a_K)$$

and

$$\boldsymbol{b}(t) = (b_1(t), ..., b_K(t)) \longrightarrow \boldsymbol{b}(0) = \boldsymbol{b} = (b_1, ..., b_K)$$

where we have $\boldsymbol{a}(0) = \boldsymbol{a}$ and $\boldsymbol{b}(0) = \boldsymbol{b}$ because the previous expectation is conditioned on $X(0) = (\boldsymbol{a}(0); \boldsymbol{b}(0)) = \boldsymbol{x} = (\boldsymbol{a}; \boldsymbol{b})$.

Hence, we obtain the generator of $(\boldsymbol{X}(t) : t \geq 0)$

$$\mathcal{L}f(\boldsymbol{x}) = \mathcal{L}f(\boldsymbol{a}; \boldsymbol{b}) = \lim_{t \to 0+} \frac{\mathbb{E}(f(X(t))|X(0) = \boldsymbol{x}) - f(\boldsymbol{x})}{t} \qquad \square$$

Importance and Definitions

- Desired that the process does not take widely fluctuating values as time progresses (meaning that the process is unstable)
- In one sense, the least restrictive form of stability we might require is *irreducibility* in the chain. That is, regardless of our starting point, we are always able to reach the same collection of sets
- A stronger notion of stability would be for the Markov Chain to actually be **guaranteed** to eventually reach like states from unlike starting points. This is the notion of *recurrence*.
- Establish a more "long term" version of stability in terms of the convergence of the distributions of the chain as time progresses. This is the notion of *ergodicity*

Ergodicity

- Ergodicity deals with a more important concept of stability: the concept of the chain "settling down", converging, to a stable or stationary regime.
- Interested in the question of, given the existence of an invariant measure π on $\mathscr{S}$, when do the n-step transition probabilities converge, in a suitable way, to π?

Definition

A σ-finite measure π on $\mathscr{S}$ with the property

$$\pi(A) = \int_S P(x, A)\pi(dx), \quad \text{for all } A \in \mathscr{S}$$

is called *invariant*.

Ergodicity

- We shall consider a global type of convergence of the measure P^n to π. That is, in terms of the total variation norm

Definition

Let μ be a signed measure on $\mathscr{S}$, then the <u>*total variation norm*</u> $\|\mu\|$ is given by

$$\|\mu\| := \sup_{A \in \mathscr{S}} \mu(A) - \inf_{A \in \mathscr{S}} \mu(A)$$

Ergodicity

The key limit of interest for this dissertation will be of the form:

Key limit

$$\begin{aligned}\lim_{n\to\infty} \|P^n(x,\cdot) - \pi(\cdot)\| &= \lim_{n\to\infty} [\sup_{A\in\mathscr{S}} (P^n(x,A) - \pi(A)) - \inf_{A\in\mathscr{S}} (P^n(x,A) - \pi(A))] \\ &= 2 \lim_{n\to\infty} \sup_{A\in\mathscr{S}} |P^n(x,A) - \pi(A)| \\ &= 0\end{aligned}$$

We'll adopt the term *ergodic* for processes where the above limit holds

Ergodicity

Definition

A Markov process is *ergodic* if it's aperiodic and irreducible and if there exists an invariant probability measure π such that

$$\lim_{t\to\infty} \|P^t(x,\cdot) - \pi(\cdot)\| = 0 \qquad \text{for all } x \in S$$

where P^t is the Markov transition probability function.

V-Uniform Ergodicity

Definition

Let $f : S \to \mathbb{R}$ be a measurable function. We define the <u>*f-norm*</u> as

$$\|\nu\|_f := \sup_{g:|g|\leq f} |\nu(g)| = \sup_{g:|g|\leq f} \left| \int_S g d\nu \right|$$

where ν is any sign measure on $\mathscr{S}$.

V-Uniform Ergodicity

Definition

Let P_1 and P_2 be Markov transition functions, and for a positive function $1 \leq V < \infty$, define the *V-norm distance* between P_1 and P_2 as

$$\begin{aligned} \|\!| P_1 - P_2 |\!\|_V &:= \sup_{x \in S} \frac{\|P_1(x,\cdot) - P_2(x,\cdot)\|_V}{V(x)} \\ &= \sup_{x \in S} \frac{\sup_{f:|f| \leq V} |P_1(x,f) - P_2(x,f)|}{V(x)} \end{aligned}$$

Definition

Define the *outer product* of the function 1 and the measure π by

$$[1 \otimes \pi](x, A) = \pi(A) \quad \text{for } x \in S \text{ and } A \in \mathscr{S}$$

V-Uniform Ergodicity

Definition

An ergodic Markov process $(\Phi_n)_{n\geq 0}$ is called *V-uniformly ergodic* if

$$\|\!| P^n - 1 \otimes \pi \|\!|_V = \sup_{x\in S} \frac{\| P^n(x,\cdot) - [1\otimes\pi](x,\cdot)\|_V}{V(x)} \longrightarrow 0 \quad \text{as } n \to 0$$

V-Uniform Ergodicity

Theorem

The following are equivalent:

i. *A Markov process $(\Phi_n)_{n\geq 0}$ is V-uniformly ergodic*
ii. *There exists a function $V > 1$, an invariant distribution π and constants $0 < r < 1$ and $R < \infty$ such that*

$$\|P^n(x,\cdot) - \pi(\cdot)\| \leq Rr^n V(x), \quad x \in S, n > 0 \tag{5}$$

where $\|\cdot\|$ is the total variation norm

iii. *$V > 1$ is a coercive function that satisfies the following geometric drift condition*

$$\mathcal{L}V(x) \leq -cV(x) + d \tag{6}$$

for some $c > 0$ and $d < \infty$, where $\mathcal{L}$ is the infinitesimal generator of $(\Phi_n)_{n\geq 0}$

V-Uniform Ergodicity

- We call the function V above, the *"Lyapunov test function"*
- By comparing the Definition of ergodicity from earlier with equation (7), V-uniform ergodicity means that the convergence of $\|P^n(x,\cdot) - \pi(\cdot)\|$ to 0 as $n \to \infty$ is exponentially fast. Thus, V-uniform ergodicity is a much stronger statement than just ergodicity.
- The previous Theorem tells us that V-uniform ergodicity can be characterized in terms of the infinitesimal generator of the Markov process. Therefore, proving that a Markov process is V-uniformly ergodic is equivalent to showing the existence of a coercive function $V > 1$ such that the geometric drift condition (8)

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Theorem

Under the assumption that $\underline{\lambda_C} := \min_{1 \leq i \leq K}\{\lambda_i^{C^{\pm}}\} < \infty$, $(X(t))_{t \geq 0} = (\boldsymbol{a}(t); \boldsymbol{b}(t))_{t \geq 0}$ *is an ergodic Markov process. To be more precise, it is* V*-uniformly ergodic.*

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

Let $\boldsymbol{x} := (\boldsymbol{a}; \boldsymbol{b}) = (a_1, ..., a_K; b_1, ..., b_K)$ where $a_i \in q\mathbb{Z}_+$ and $b_i \in q\mathbb{Z}_-$.

Define $V(\boldsymbol{a}; \boldsymbol{b}) := \sum_{i=1}^{K} a_i + \sum_{i=1}^{K} |b_i| + q$. As $q > 1$, it follows that $V > 1$. Moreover, V is clearly a coercive function.

We want to upper bound $\mathcal{L}V(\boldsymbol{a}; \boldsymbol{b})$ i.e. we want to upper bound the following expression:

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

Want to upper bound the first term, which is equal to

$$\lambda^{M^+}\left[\sum_{i=1}^{K}[a_i-(q-A_{i-1})_+]_+ + \sum_{i=1}^{K}|J^{M^+}(b_i)| + q - (\sum_{i=1}^{K} a_i + \sum_{i=1}^{K}|b_i| + q)\right]$$

Largest possible value the above expression can take happens when the buy market order dM^+ does not cause a shift on the bid side of the book

i.e. $J^{M^+}(b_i) = b_i$ for every $i = 1, ..., K$

This happens precisely when $a_{i_s} > q$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

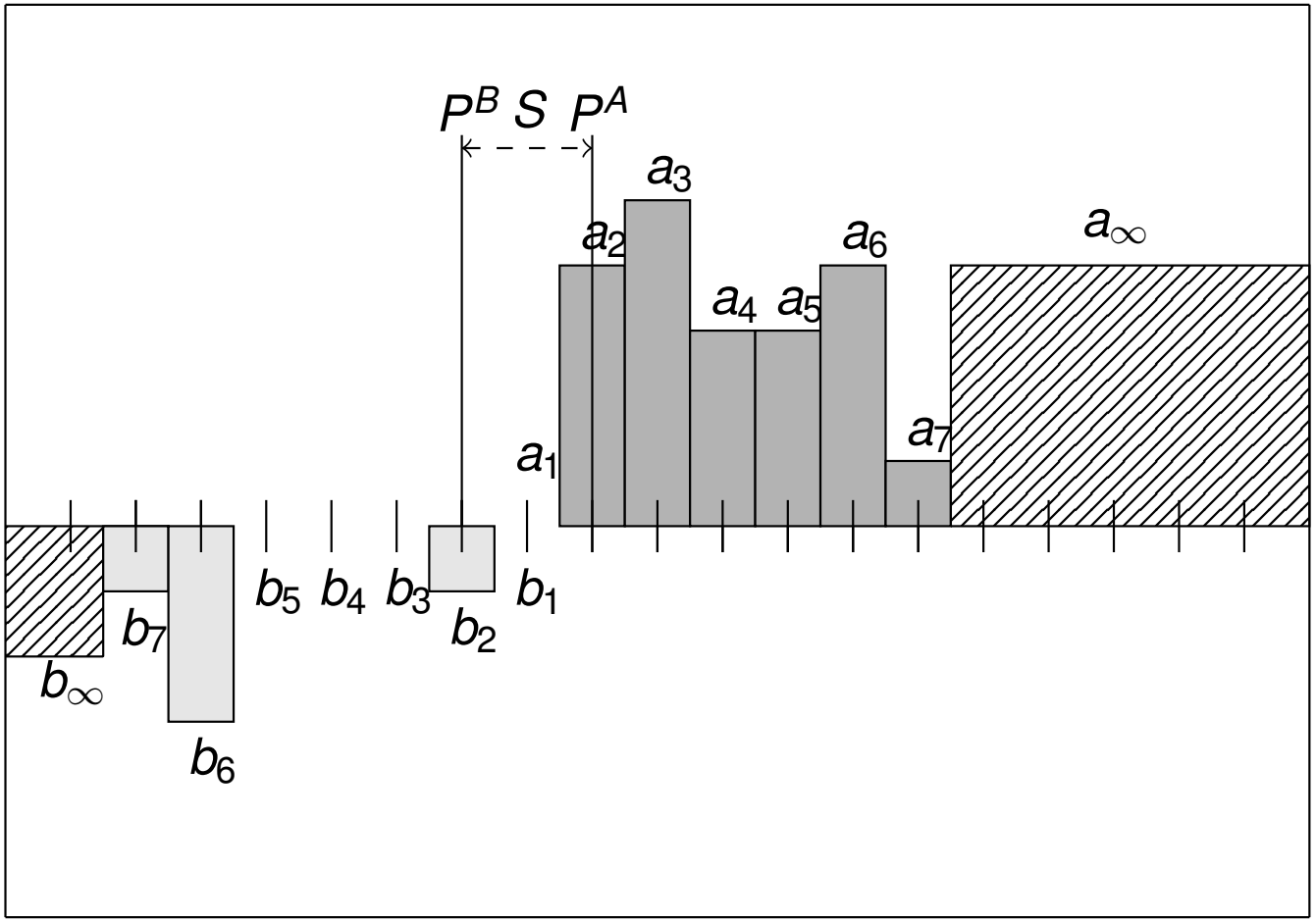

Figure: Order book with $K = 7$ and $q = 1$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

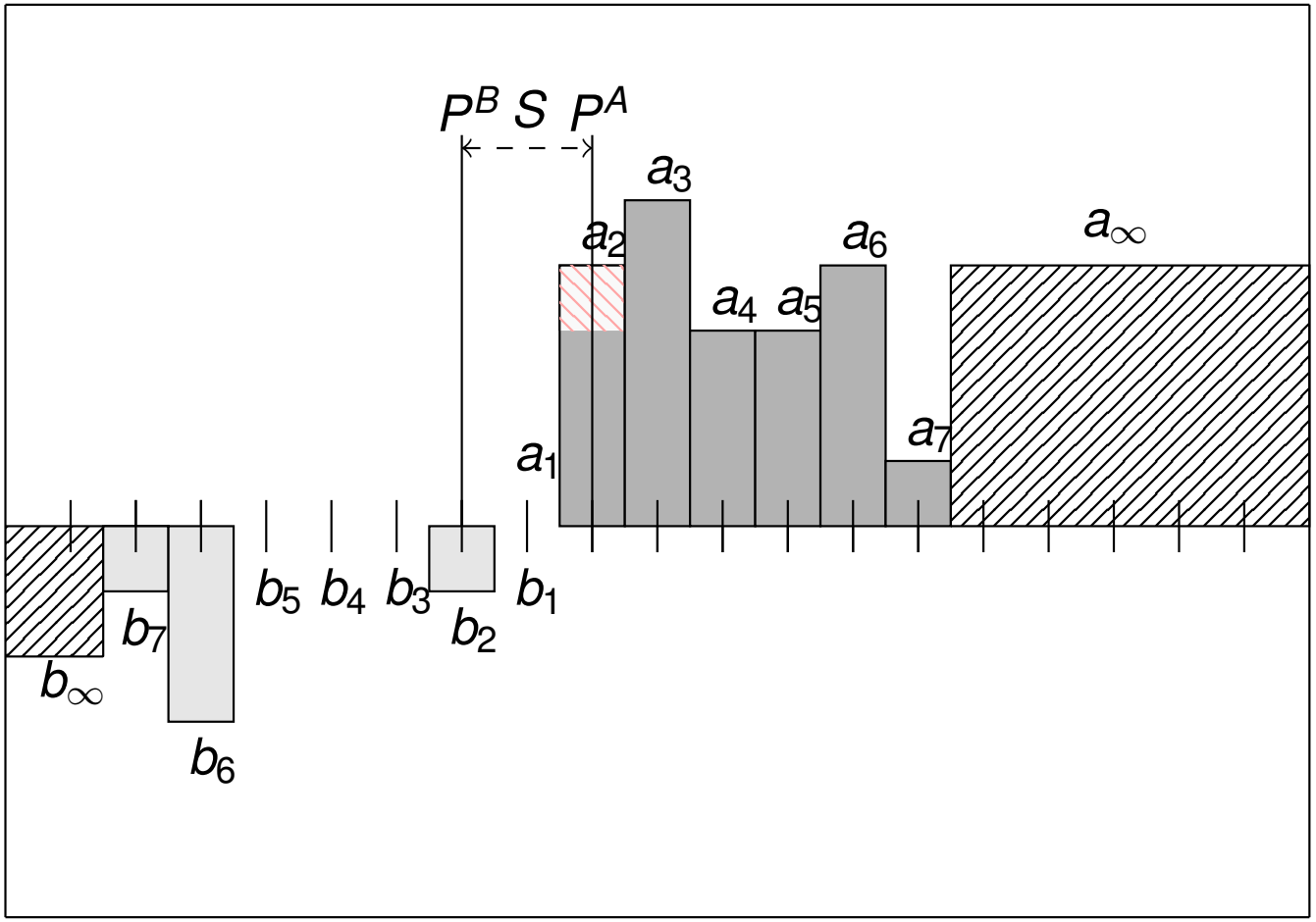

Figure: Incoming buy market order dM^+

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

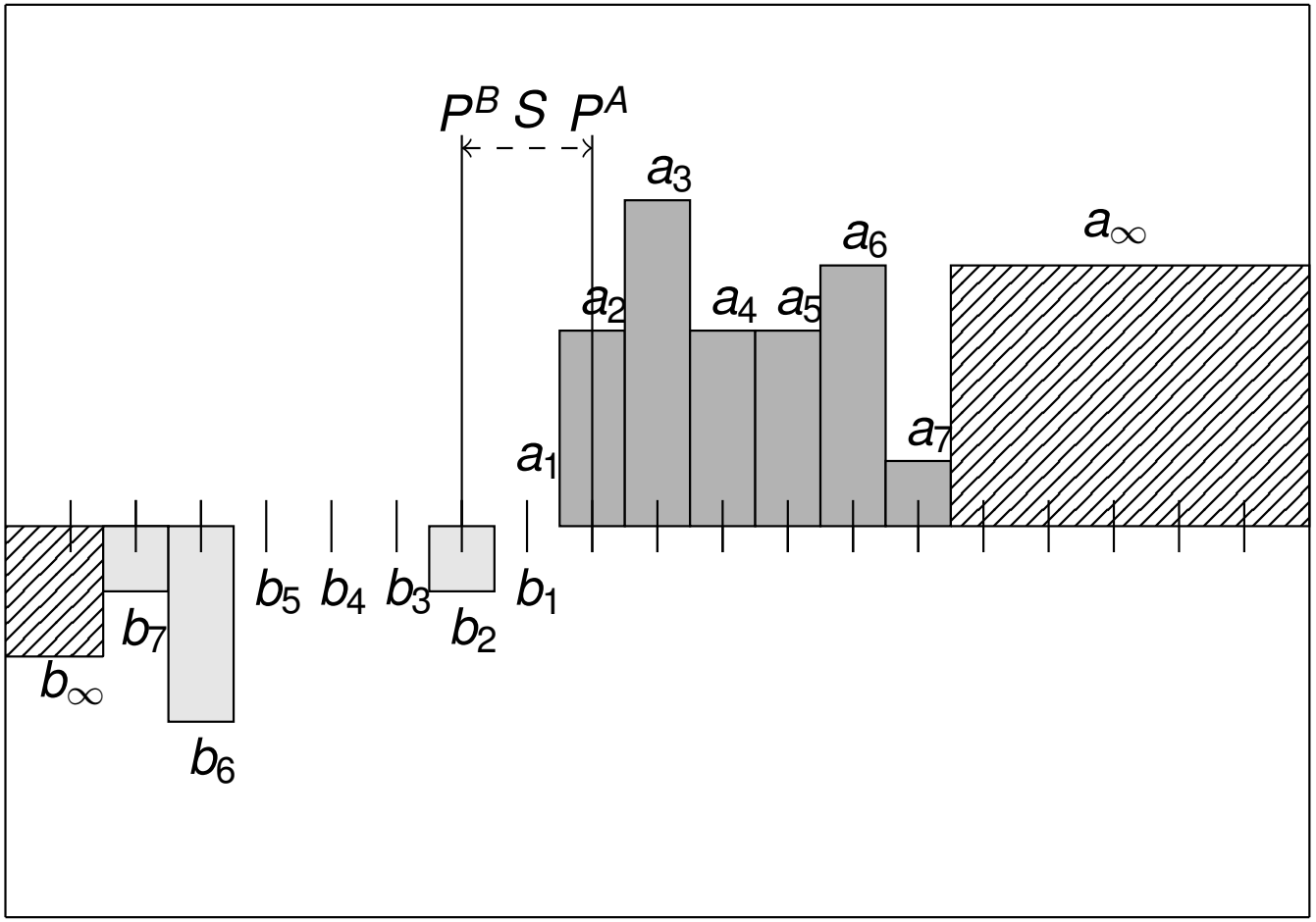

Figure: Resulting order book after dM^+

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

Largest possible value the above expression can take happens when the buy market order dM^+ does not cause a shift on the bid side of the book

i.e. $J^{M^+}(b_i) = b_i$ for every $i = 1, ..., K$

This happens precisely when $a_{i_S} > q$

If $a_{i_S} = q$, the submission of a buy market order would cause a_{i_S} to turn to 0, which in turn would shift the bid side of the order book by $A^{-1}(q) - i_S$, and so the first $A^{-1}(q) - i_S$ levels of $\boldsymbol{b}$ would turn to 0 i.e. $J^{M^+}(b_i) = 0$ for every $i = 1, 2, ..., A^{-1}(q) - i_S$.

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

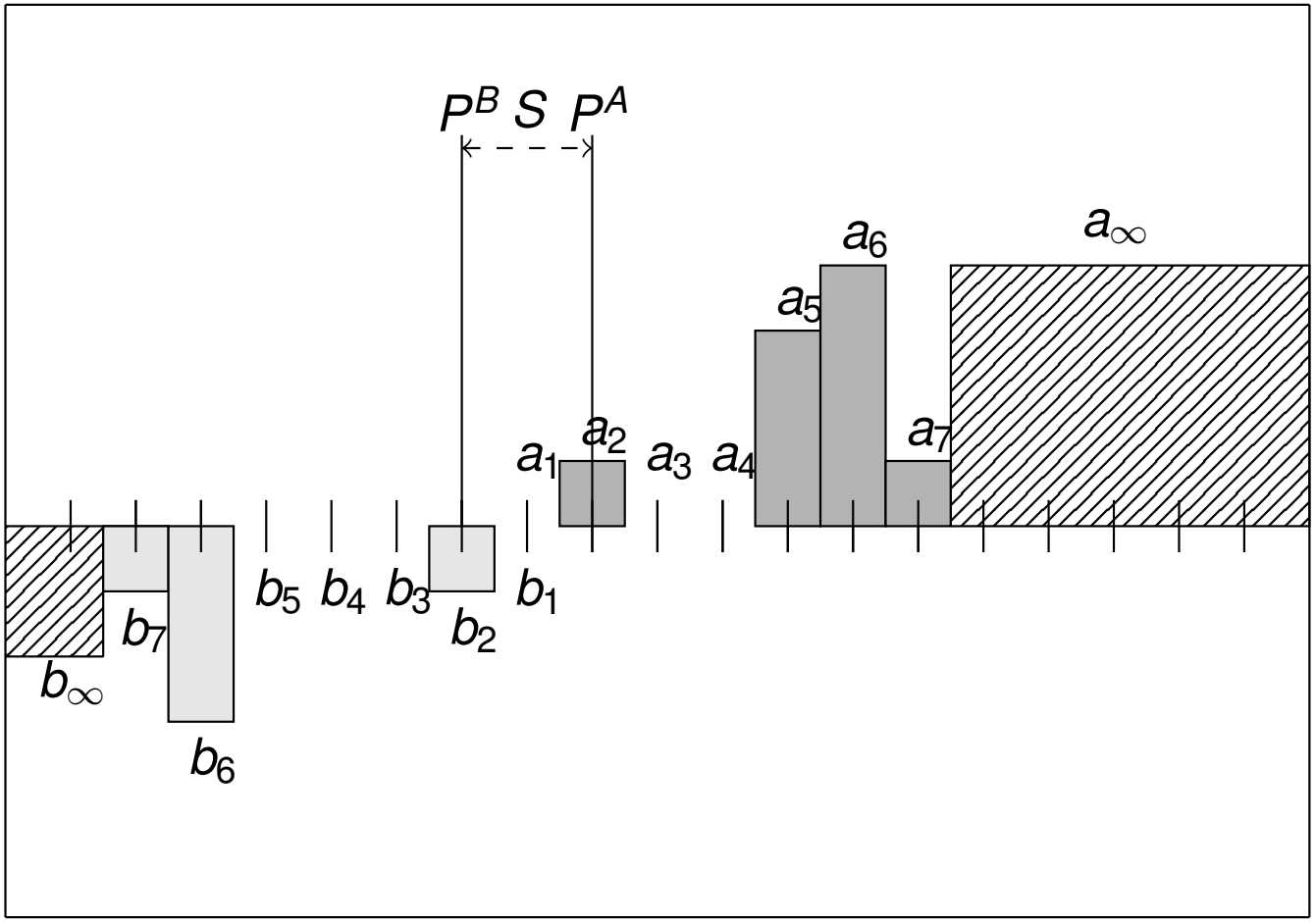

Figure: Order book with $K = 7$ and $q = 1$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

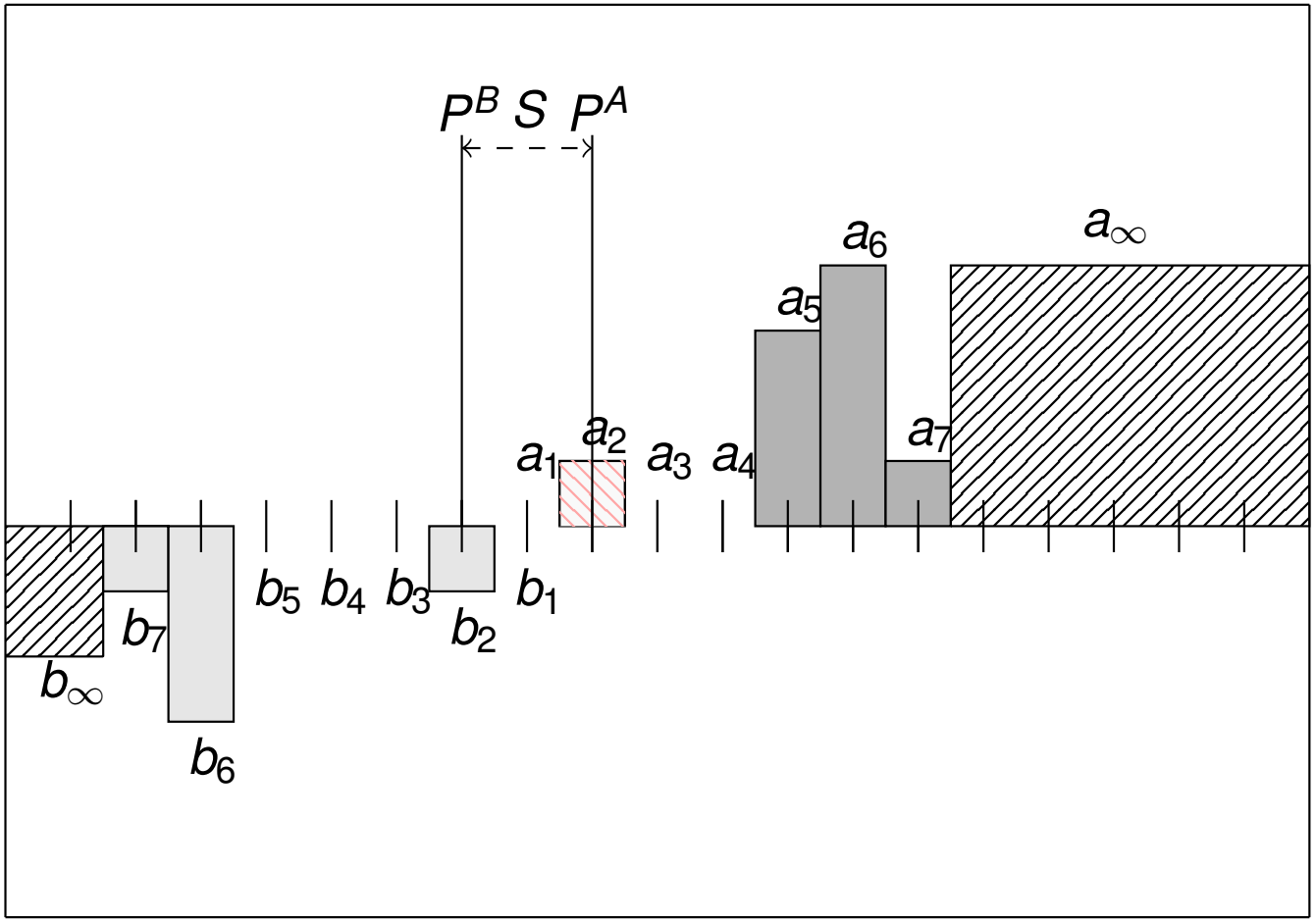

Figure: Incoming buy market order dM^+

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

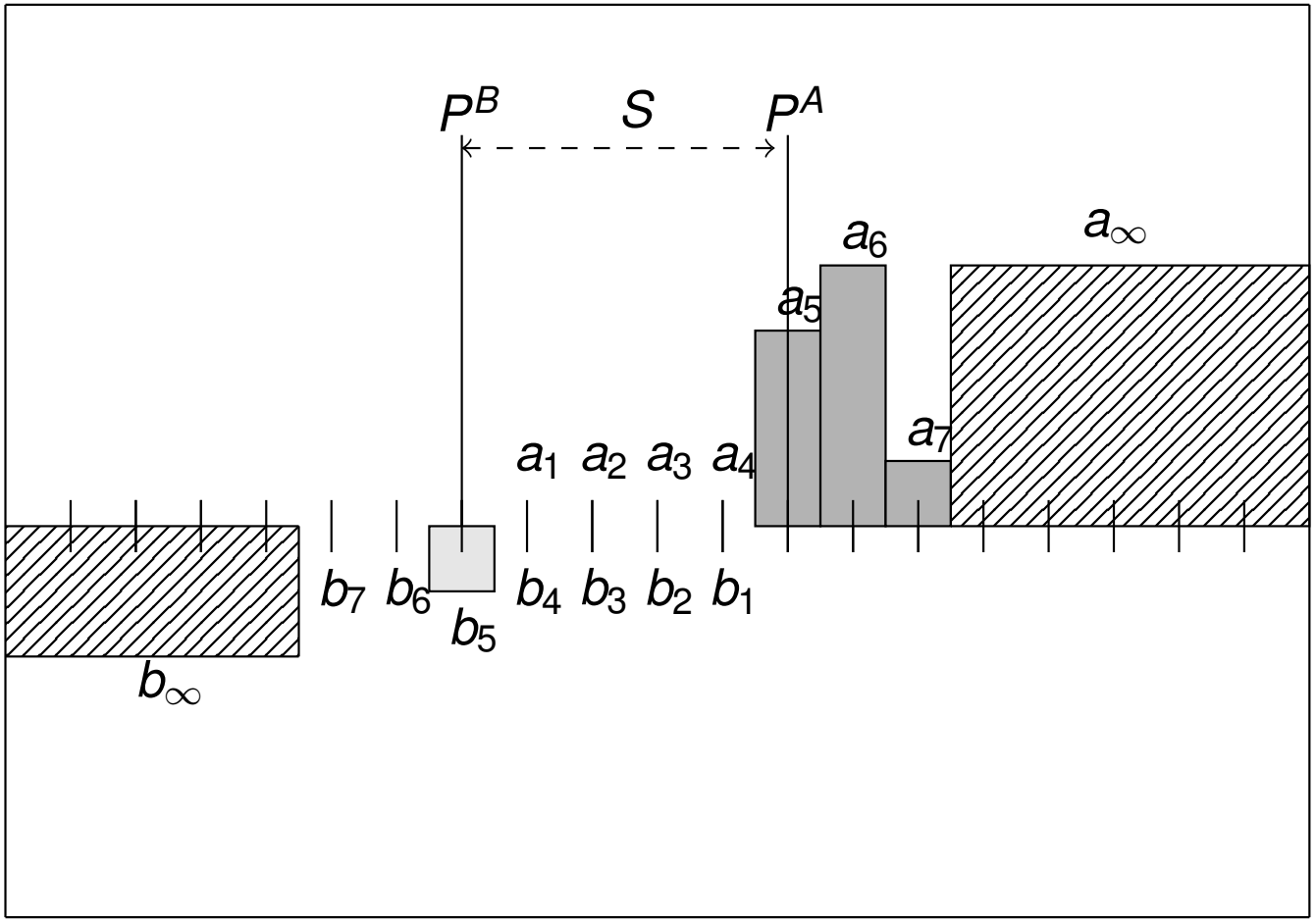

Figure: Resulting order book after dM^+

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

After a bit of careful work, we get that

$$\lambda^{M^+}\left[\sum_{i=1}^{K}[a_i-(q-A_{i-1})_+]_+ + \sum_{i=1}^{K}|J^{M^+}(b_i)| + q - (\sum_{i=1}^{K} a_i + \sum_{i=1}^{K}|b_i| + q)\right]$$

is upper bounded by $-\lambda^{M^+} q$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

By the same reasoning and symmetry of the order book, we can upper bound the fourth term by

$$-\lambda^{M^-} q$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

Want to upper bound the second term, which is equal to

$$\sum_{i=1}^{K} \lambda_i^{L^+} \left[(\cancel{a_i} + q) + \cancel{\sum_{\substack{j=1 \\ j;j \neq i}}^{K} a_j} + \sum_{j=1}^{K} |J^{L_i^+}(b_j)| + \cancel{q} - (\cancel{\sum_{i=1}^{K} a_i} + \sum_{i=1}^{K} |b_i| + \cancel{q}) \right]$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

For $i \in \{1, ..., i_S - 1\}$, a sell limit order at level i, dL_i^+, will cause a shift in the bid side of the order book (because i is in the spread). To be precise,

$$J^{L_i^+}(b_j) = b_{j+(i_S-i)} \qquad \text{for all } j \in \{1, ..., K - (i_S - i)\}$$

and

$$J^{L_i^+}(b_j) = b_\infty \qquad \text{for all } j \in \{K - (i_S - i) + 1, ..., K\}$$

For $i \in \{i_S, ..., K\}$, a sell limit order at level i, dL_i^+, will not cause a shift in the bid side of the book (because i is not in the spread) i.e. we'll have $J^{L_i^+}(b_j) = b_j$ for all $j \in \{1, ..., K\}$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

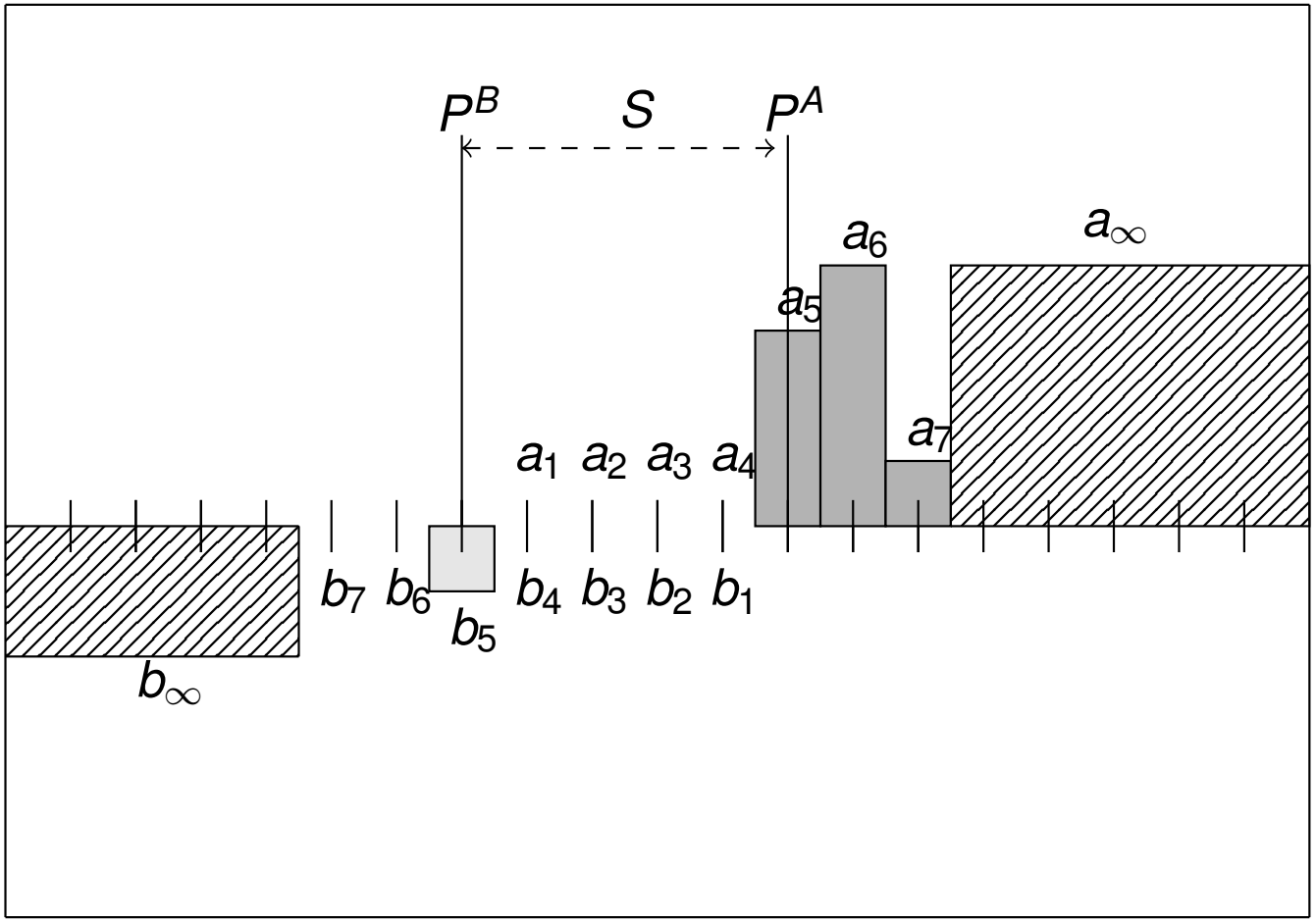

Figure: Order book with $K = 7$ and $q = 1$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

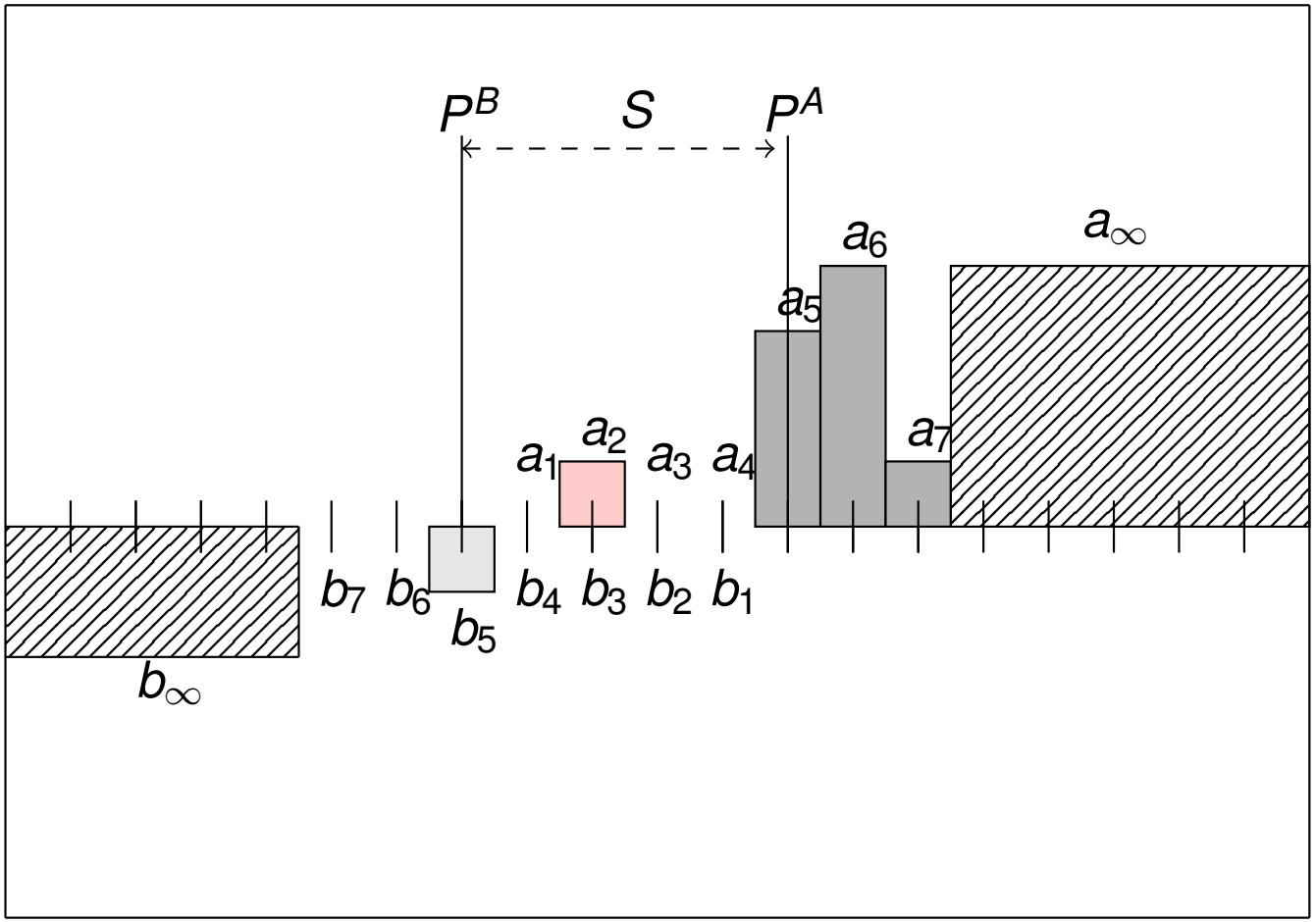

Figure: Incoming sell limit order at level $i = 2$, dL_2^+

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

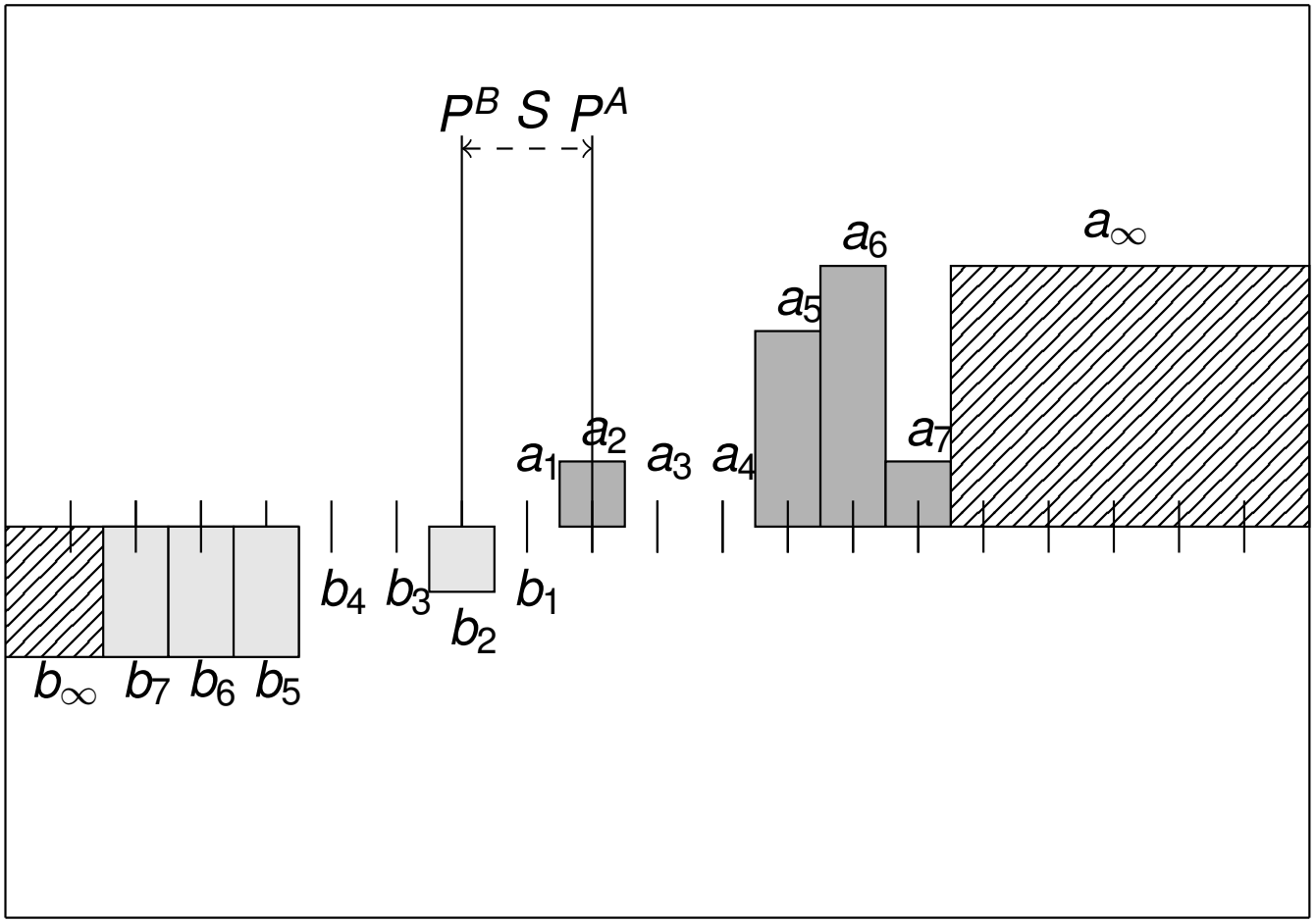

Figure: Resulting order book after dL_2^+

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

For $i \in \{1, ..., i_S - 1\}$, a sell limit order at level i, dL_i^+, will cause a shift in the bid side of the order book (because i is in the spread). To be precise,

$$J^{L_i^+}(b_j) = b_{j+(i_S-i)} \qquad \text{for all } j \in \{1, ..., K - (i_S - i)\}$$

and

$$J^{L_i^+}(b_j) = b_\infty \qquad \text{for all } j \in \{K - (i_S - i) + 1, ..., K\}$$

For $i \in \{i_S, ..., K\}$, a sell limit order at level i, dL_i^+, will not cause a shift in the bid side of the book (because i is not in the spread) i.e. we'll have $J^{L_i^+}(b_j) = b_j$ for all $j \in \{1, ..., K\}$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

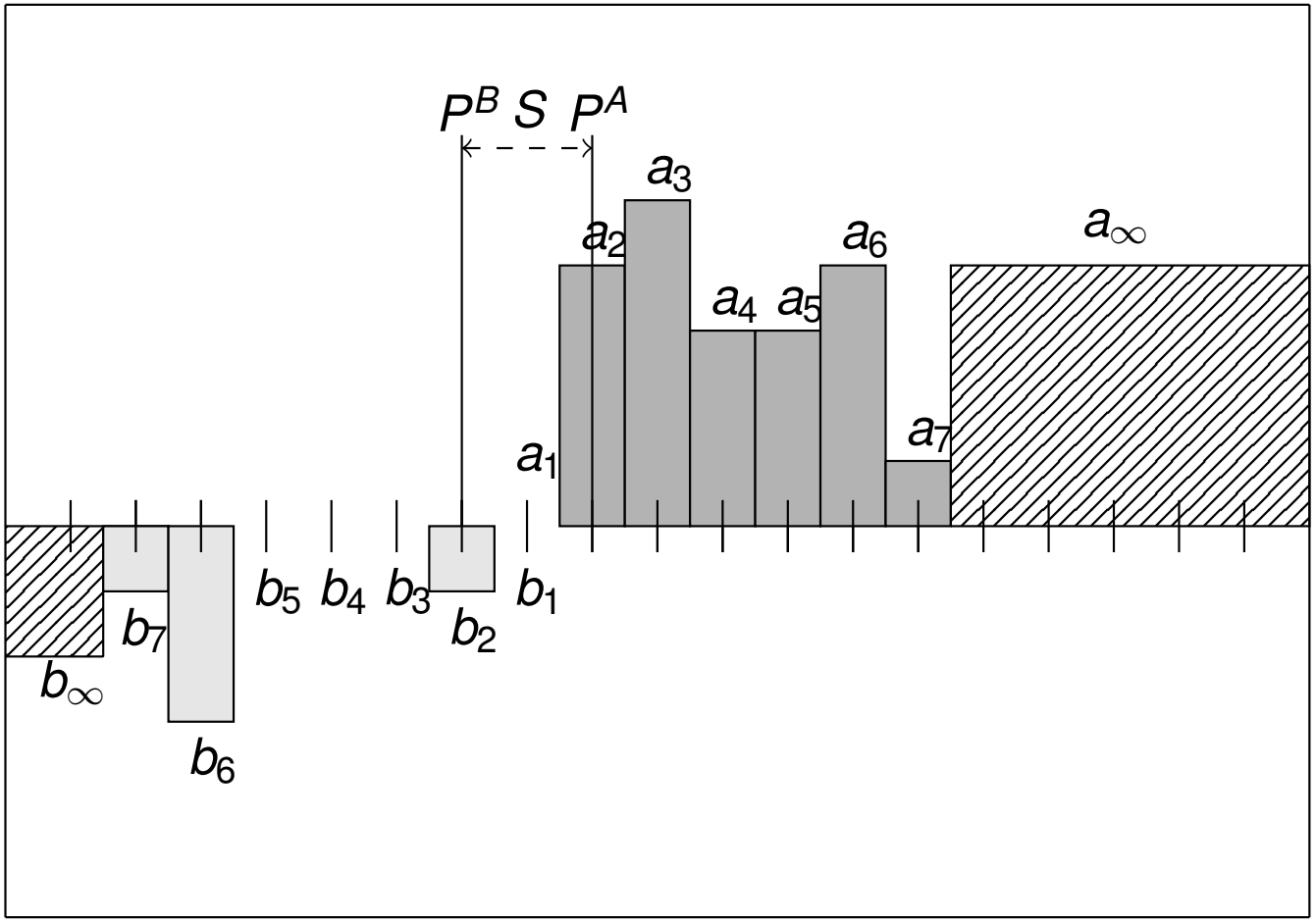

Figure: Order book with $K = 7$ and $q = 1$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

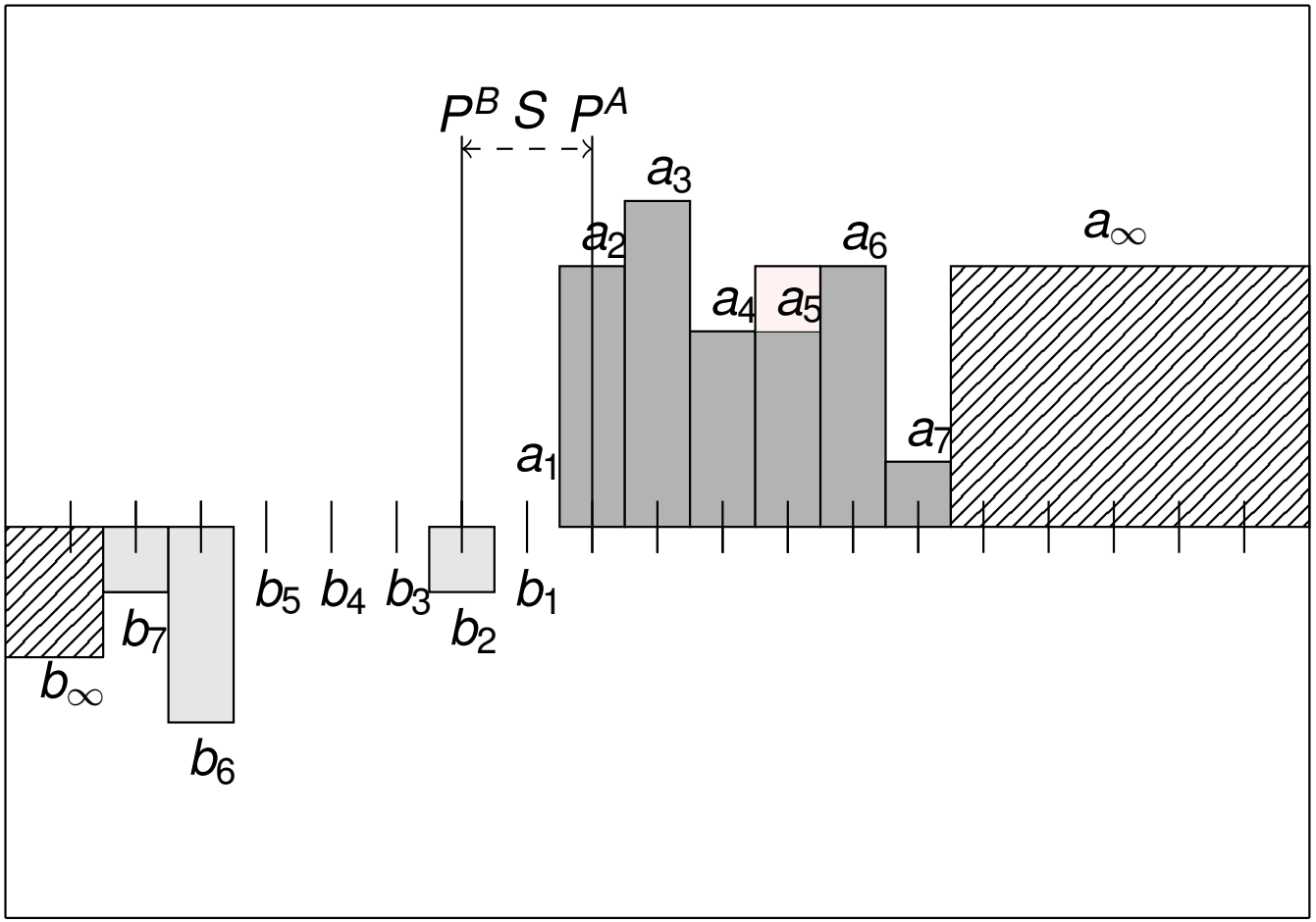

Figure: Incoming ask limit order at level $i = 5$, dL_5^+

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

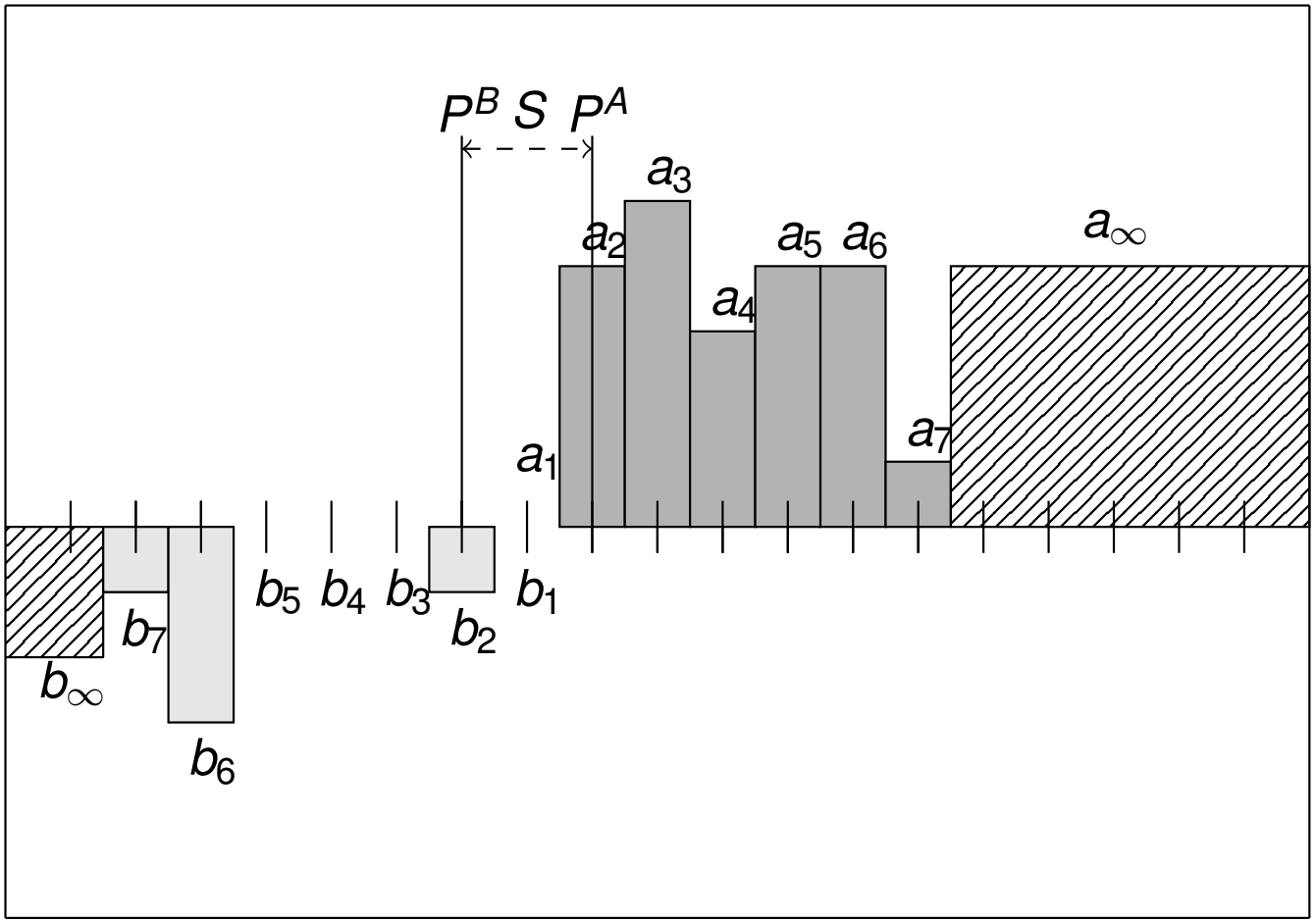

Figure: Resulting order book after dL_5^+

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

For $i \in \{1, ..., i_S - 1\}$, a sell limit order at level i, dL_i^+, will cause a shift in the bid side of the order book (because i is in the spread). To be precise,

$$J^{L_i^+}(b_j) = b_{j+(i_S-i)} \qquad \text{for all } j \in \{1, ..., K - (i_S - i)\}$$

and

$$J^{L_i^+}(b_j) = b_\infty \qquad \text{for all } j \in \{K - (i_S - i) + 1, ..., K\}$$

For $i \in \{i_S, ..., K\}$, a sell limit order at level i, dL_i^+, will not cause a shift in the bid side of the book (because i is not in the spread) i.e. we'll have $J^{L_i^+}(b_j) = b_j$ for all $j \in \{1, ..., K\}$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

Combining these two facts, we get

$$\sum_{i=1}^{i_S-1} \lambda_i^{L^+} (V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a}; \boldsymbol{b}))$$
$$\leq \sum_{i=1}^{i_S-1} \lambda_i^{L^+} q + \sum_{i=1}^{i_S-1} \lambda_i^{L^+} (i_S - i)|b_\infty|$$

$$\sum_{i=i_S}^{K} \lambda_i^{L^+} (V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a}; \boldsymbol{b}))$$
$$= \sum_{i=i_S}^{K} \lambda_i^{L^+} q$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

By combining these two inequalities and after a bit of careful work, we get that

$$\sum_{i=1}^{K} \lambda_i^{L^+} (V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a}; \boldsymbol{b}))$$

is upper bounded by

$$\sum_{i=1}^{K} \lambda_i^{L^+} q + \sum_{i=1}^{K} \lambda_i^{L^+} (i_S - i)_+ |b_\infty|$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

By the same reasoning and symmetry of the order book, we can upper bound the fifth term by

$$\sum_{i=1}^{K} \lambda_i^{L^-} q + \sum_{i=1}^{K} \lambda_i^{L^-} (i_S - i)_+ a_\infty$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

Want to upper bound the third term, which is equal to

$$\sum_{i=1}^{K} \lambda_i^{C^+} a_i \left[(\cancel{a_i} - q) + \cancel{\sum_{\substack{j=1 \\ j;j\neq i}}^{K} a_j} + \sum_{j=1}^{K} J^{C_i^+}(b_j) + \cancel{q} - (\cancel{\sum_{i=1}^{K} a_i} + \sum_{i=1}^{K} |b_i| + \cancel{q}) \right]$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

Let $i \in \{1, ..., K\}$ and consider

$$\lambda_i^{C^+} a_i \left[-q + \sum_{j=1}^{K} J^{C_i^+}(b_j) - \sum_{i=1}^{K} |b_i| \right]$$

The largest possible value that this expression can take occurs when the sell cancellation limit order at level i, dC_i^+, doesn't cause a shift in the bid side of the order book. This occurs precisely when $a_i > q$, where we get that $J^{C_i^+}(b_j) = b_j$ for all $j \in \{1, ..., K\}$.

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

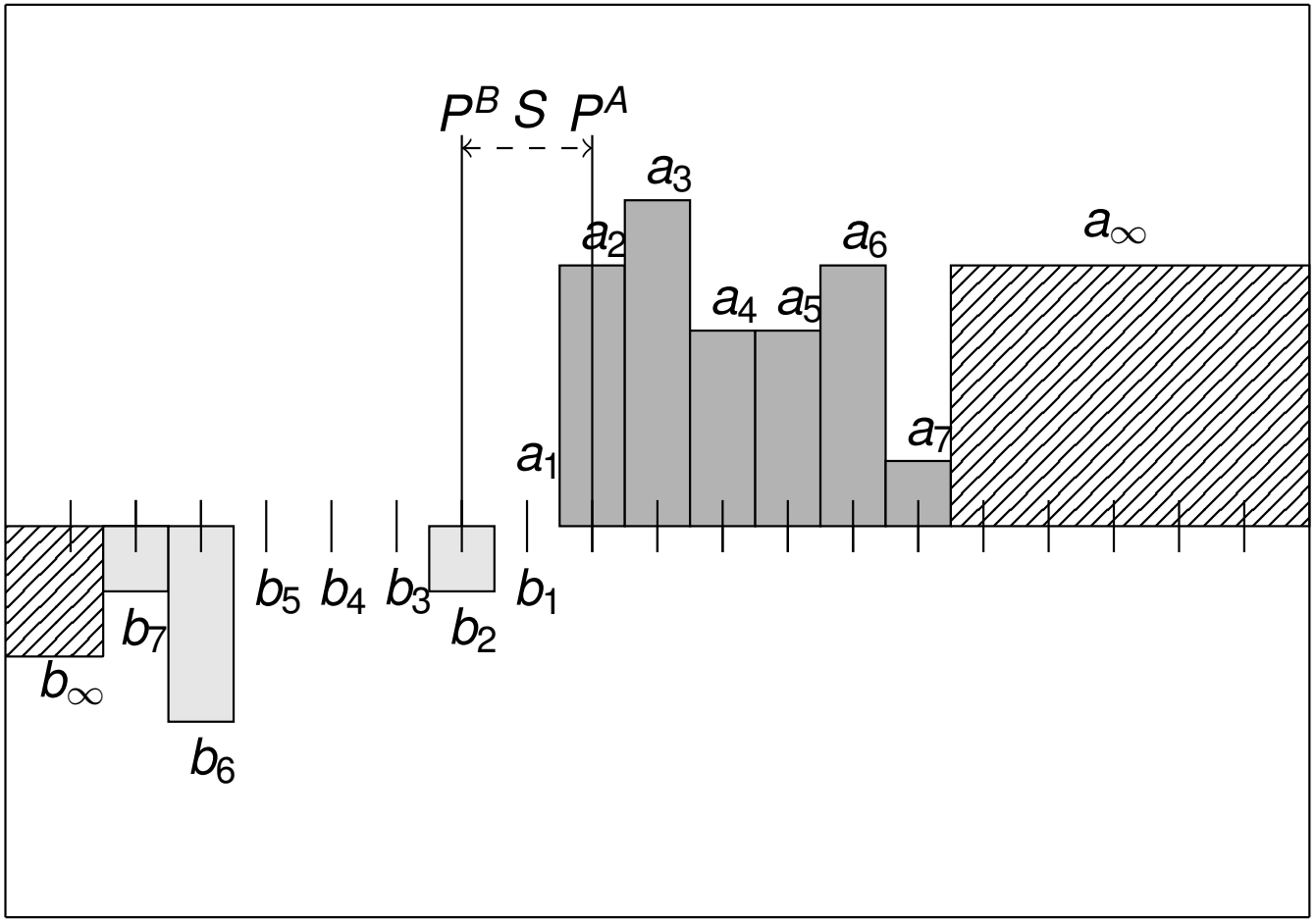

Figure: Order book with $K = 7$ and $q = 1$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

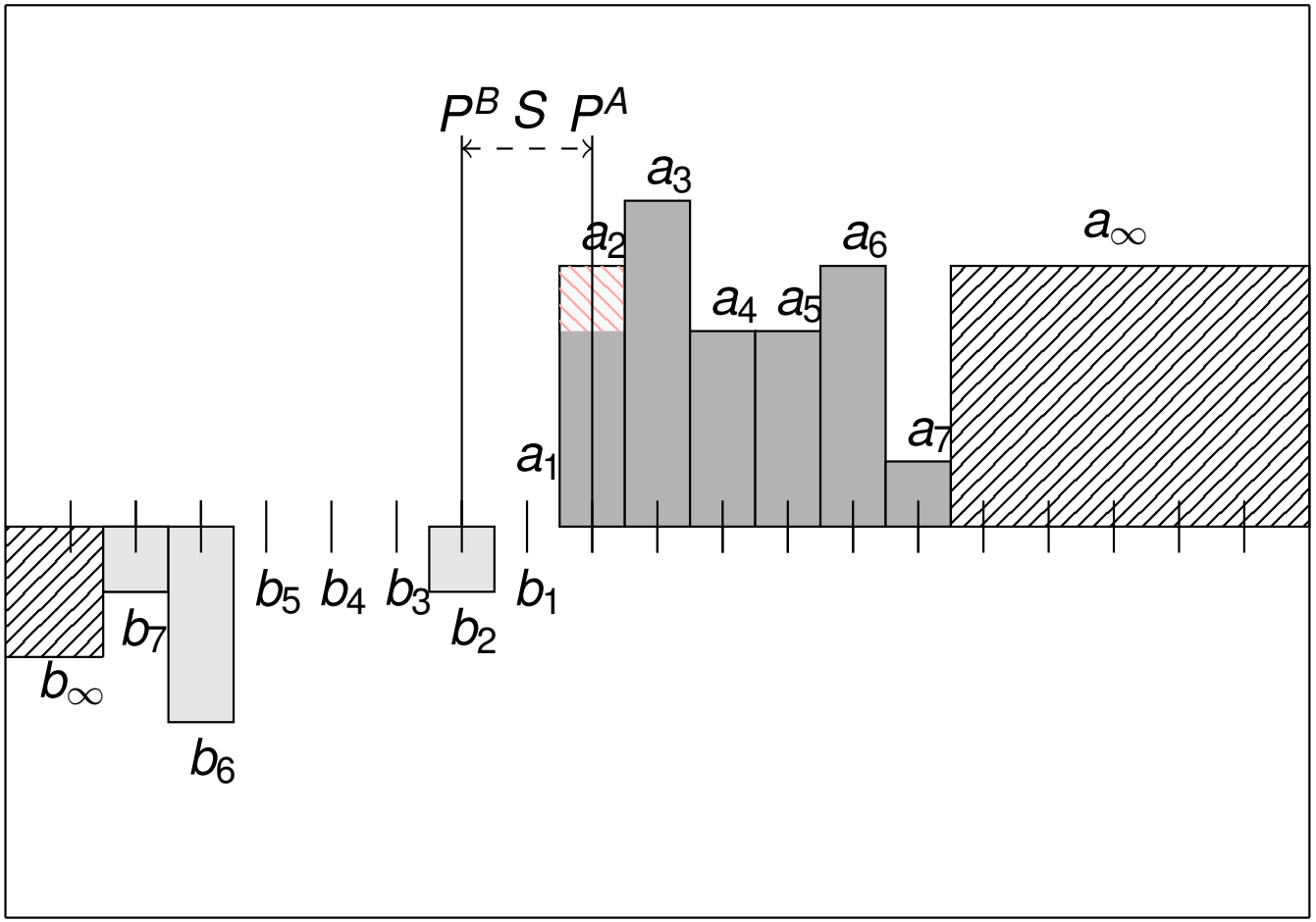

Figure: Incoming sell cancellation order at level $i = 2$, dC_2^+

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

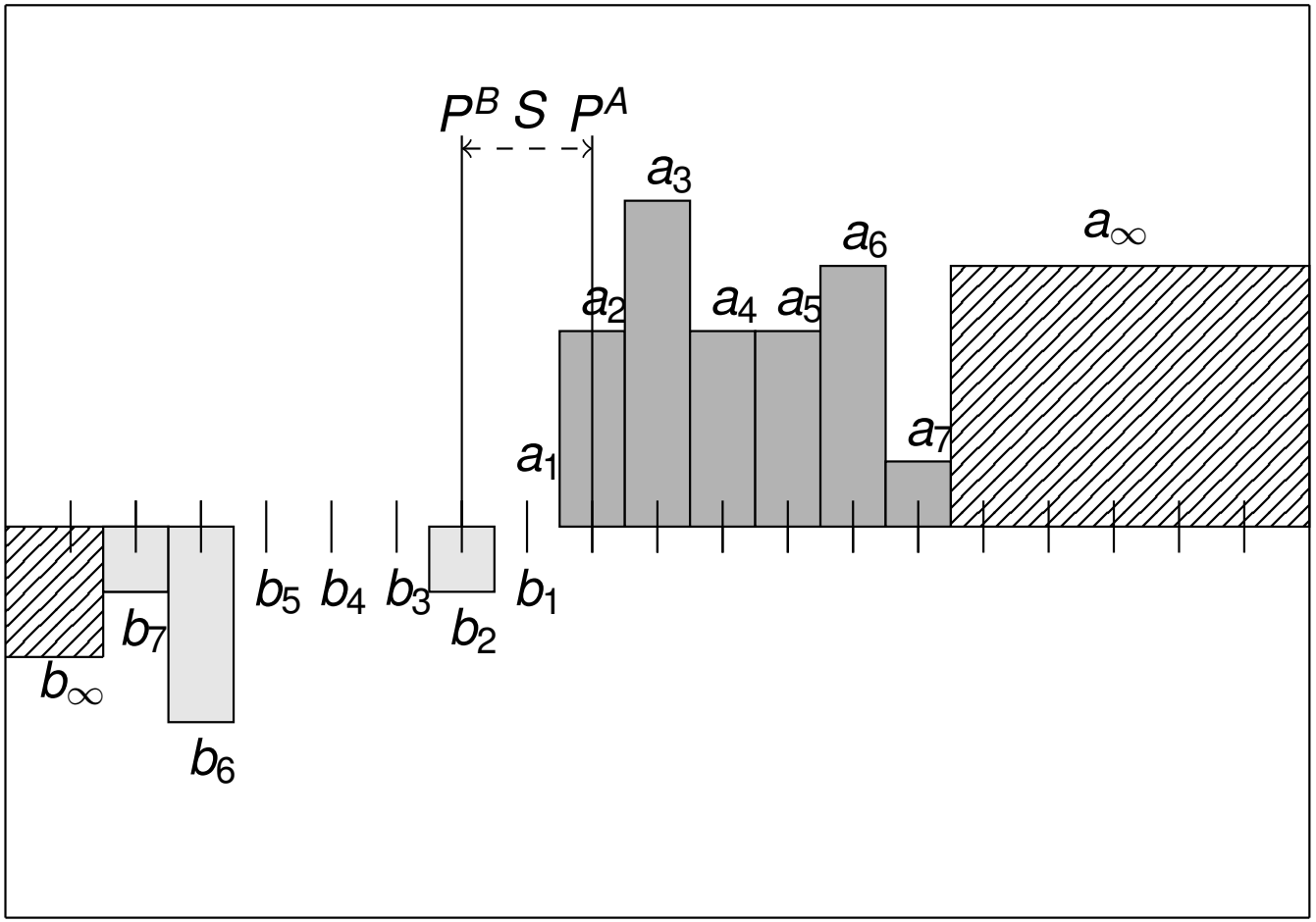

Figure: Resulting order book after dC_2^+

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

Therefore, when $i = i_S$, we have

$$\lambda_i^{C^+} a_i \left[-q + \sum_{j=1}^{K} J^{C_i^+}(b_j) - \sum_{i=1}^{K} |b_i| \right]$$
$$\leq \lambda_i^{C^+} a_i \left[-q + \sum_{j=1}^{K} |b_j| - \sum_{i=1}^{K} |b_i| \right]$$
$$= -\lambda_i^{C^+} a_i q$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

As for the case when $i \in \{1, ..., i_S - 1\}$, by the definition of i_S, $a_i = 0$. This means that no sell cancellation order can be submitted at level i (since no order even exists at that level!). This leaves the entire order book completely unchanged

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

When $i \in \{i_S + 1, ..., K\}$, no shift is caused on the bid side of the order book as a result of dC_i^+ because the cancellation does not interfere with any sell limit order at level i_S i.e. we have $J^{C_i^+}(b_j) = b_j$ for every $j \in \{1, ..., K\}$. Under the assumption that $a_i \geq q$ (when a cancellation dC_i^+ is actually possible!), we get

$$\lambda_i^{C^+} a_i \left[-q + \sum_{j=1}^{K} J^{C_i^+}(b_j) - \sum_{i=1}^{K} |b_i| \right]$$

$$= \lambda_i^{C^+} a_i \left[-q + \sum_{j=1}^{K} |b_j| - \sum_{i=1}^{K} |b_i| \right]$$

$$= -\lambda_i^{C^+} a_i q$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

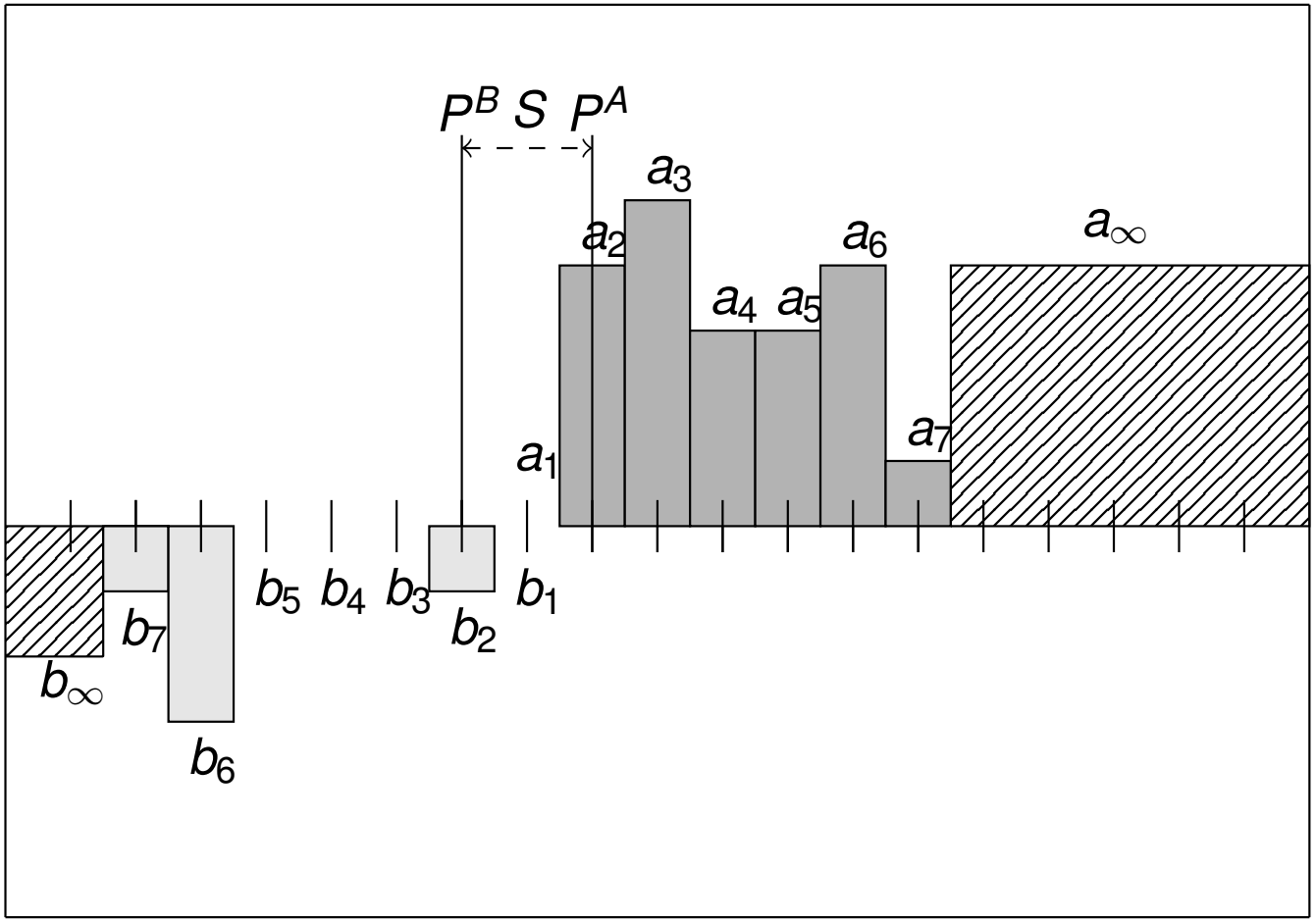

Figure: Order book with $K = 7$ and $q = 1$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

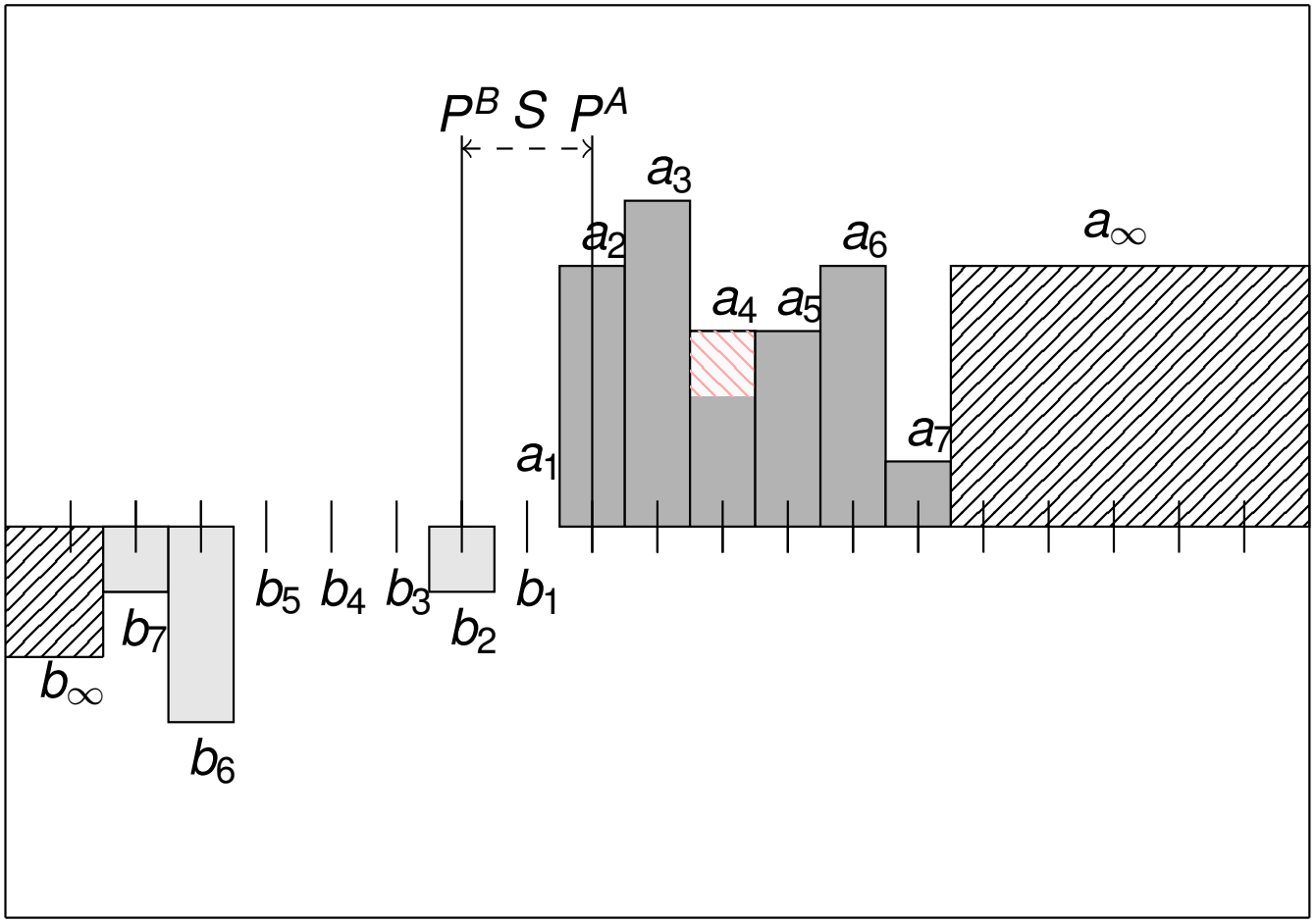

Figure: Incoming cancellation of an ask limit order at level $i = 4$, dC_4^+

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

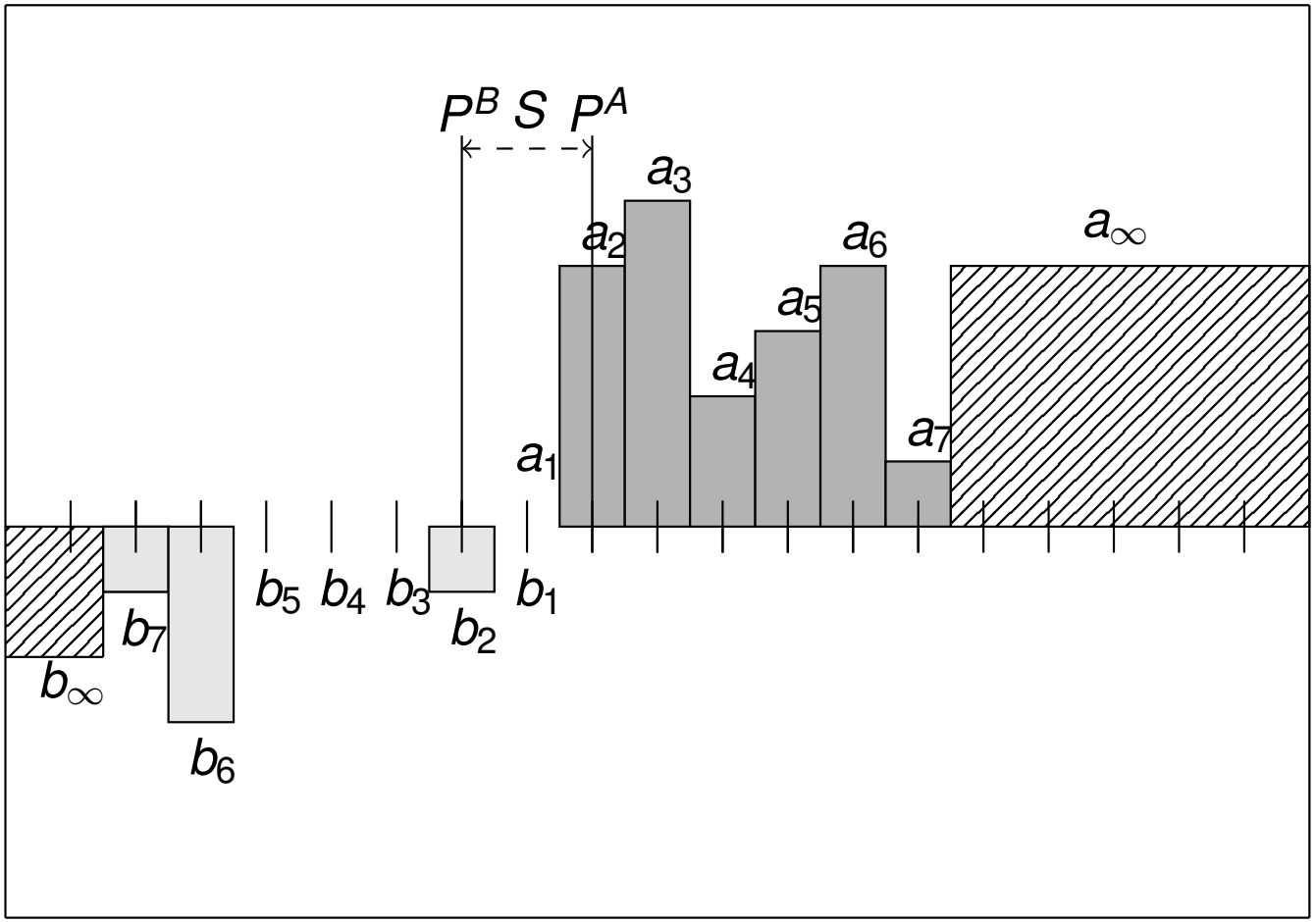

Figure: Resulting order book after dC_4^+

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

By combining the deductions of the preceding slides and after a bit of careful work, we get that

$$\sum_{i=1}^{K} \lambda_i^{C^+} a_i (V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a}; \boldsymbol{b}))$$

is upper bounded by

$$-\sum_{i=1}^{K} \lambda_i^{C^+} a_i q$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

By the same reasoning and symmetry of the order book, we can upper bound the sixth term by

$$-\sum_{i=1}^{K} \lambda_i^{C^-} |b_i| q$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

Finally, we have obtained upper bounds for all the terms in $\mathcal{L}V(\boldsymbol{a}; \boldsymbol{b})$.

The following slides will give these six upper bounds we've found for each term of the expression of $\mathcal{L}V(\boldsymbol{a}; \boldsymbol{b})$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

After a bit of careful work, we get that

$$\lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\ldots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a}; \boldsymbol{b}))$$

is upper bounded by $-\lambda^{M^+} q$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

By the same reasoning and symmetry of the order book, we get that

$$+\lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a}; \boldsymbol{b}))$$

is upper bounded by

$$-\lambda^{M^-} q$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

After a bit of careful work, we get that

$$+\sum_{i=1}^{K} \lambda_i^{L^+} (V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a}; \boldsymbol{b}))$$

is upper bounded by

$$\sum_{i=1}^{K} \lambda_i^{L^+} q + \sum_{i=1}^{K} \lambda_i^{L^+} (i_S - i)_+ |b_\infty|$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

By the same reasoning and symmetry of the order book, we get that

$$+\sum_{i=1}^{K} \lambda_i^{L^-} \left(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a}; \boldsymbol{b}) \right)$$

is upper bounded by

$$\sum_{i=1}^{K} \lambda_i^{L^-} q + \sum_{i=1}^{K} \lambda_i^{L^-} (i_S - i)_+ a_\infty$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

After a bit of careful work, we get that

$$+\sum_{i=1}^{K} \lambda_i^{C^+} a_i (V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a}; \boldsymbol{b}))$$

is upper bounded by

$$-\sum_{i=1}^{K} \lambda_i^{C^+} a_i q$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$
\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = {} & \lambda^{M^+}(V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^+}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^+} a_i(V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \\
& + \sum_{i=1}^{K} \lambda_i^{C^-} |b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b}))
\end{aligned}
$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

By the same reasoning and symmetry of the order book, we get that

$$+\sum_{i=1}^{K} \lambda_i^{C^-} |b_i| (V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a}; \boldsymbol{b}))$$

is upper bounded by

$$-\sum_{i=1}^{K} \lambda_i^{C^-} |b_i| q$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

We have thus found an upper bound for $\mathcal{L}V(\boldsymbol{a}; \boldsymbol{b})$. Before proceeding, let's introduce the following shorthand notations:

$$\Lambda^{L^+} := \sum_{i=1}^{K} \lambda_i^{L^+}$$

$$\Lambda^{L^-} := \sum_{i=1}^{K} \lambda_i^{L^-}$$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

$$\begin{aligned}
\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) &\leq -(\lambda^{M^+} + \lambda^{M^-})q + \sum_{i=1}^{K}(\lambda_i^{L^+} + \lambda_i^{L^-})q - \sum_{i=1}^{K}(\lambda_i^{C^+} a_i + \lambda_i^{C^+}|b_i|)q \\
&\quad + \sum_{i=1}^{K} \lambda_i^{L^-}(i_s - i)_+ a_\infty + \sum_{i=1}^{K} \lambda_i^{L^+}(i_s - i)_+ |b_\infty| \\
&\leq -(\lambda^{M^+} + \lambda^{M^-})q + (\Lambda^{L^-} + \Lambda^{L^+})q - \sum_{i=1}^{K}(\underline{\lambda_C} a_i + \underline{\lambda_C}|b_i|)q \\
&\quad + \sum_{i=1}^{K} \lambda_i^{L^-}(i_s - i)_+ a_\infty + \sum_{i=1}^{K} \lambda_i^{L^+}(i_s - i)_+ |b_\infty| \\
&\qquad \text{(by our assumption that } \underline{\lambda_C} < \infty) \\
&\leq -(\lambda^{M^+} + \lambda^{M^-})q + (\Lambda^{L^-} + \Lambda^{L^+})q + \underline{\lambda_C} q^2 - \underline{\lambda_C} q V(\boldsymbol{a};\boldsymbol{b}) \\
&\quad + K(\Lambda^{L^-} a_\infty + \Lambda^{L^+}|b_\infty|)
\end{aligned}$$

Recall

Theorem

The following are equivalent:

i. *A Markov process $(\Phi_n)_{n\in\mathbb{N}_0}$ is V-uniformly ergodic*
ii. *There exists a function $V > 1$, an invariant distribution π and constants $0 < r < 1$ and $R < \infty$ such that*

$$\|P^n(x,\cdot) - \pi(\cdot)\| \leq Rr^n V(x), \quad x \in S, n > 0 \tag{7}$$

where $\|\cdot\|$ is the total variation norm

iii. *$V > 1$ is a coercive function that satisfies the following geometric drift condition*

$$\mathcal{L}V(x) \leq -cV(x) + d \tag{8}$$

for some $c > 0$ and $d < \infty$, where $\mathcal{L}$ is the infinitesimal generator of $(\Phi_n)_{n\in\mathbb{N}_0}$

$(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic

Sketch proof.

Finally, let's set

$$c := \underline{\lambda_C} q > 0$$

and

$$d := -(\lambda^{M^+} + \lambda^{M^-})q + (\Lambda^{L^-} + \Lambda^{L^+})q + \underline{\lambda_C} q^2 + K(\Lambda^{L^-} a_\infty + \Lambda^{L^+} |b_\infty|) < \infty$$

Therefore, we have $\mathcal{L}V(\boldsymbol{a}; \boldsymbol{b}) \leq -cV(\boldsymbol{a}; \boldsymbol{b}) + d$ with $c > 0$ and $d < \infty$. And so the geometric drift condition (8) is satisfied. Thus, by the Theorem, $(X(t) : t \geq 0)$ is V-uniformly ergodic and hence converges to its stationary state exponentially fast. $\square$

Bibliography

- Abergel, F. and Anane, M. and Chakraborti, A. and Jedidi, A. and Toke, I.M., "Limit Order Books", PHYSICS OF SOCIETY: ECONOPHYSI., 2016
- S. Meyn, R.L. Tweedie, and P.W. Glynn., "Markov Chains and Stochastic Stability", Cambridge Mathematical Library, 2009
- Sean P. Meyn and R. L. Tweedie., "Stability of markovian processes iii: Foster-lyapunov criteria for continuous-time processes", Advances in Applied Probability, 1993
- V.N. Kolokol'cov., "Markov Processes, Semigroups, and Generators", De Gruyter studies in mathematics, 2011