Some notes on Poisson processes

Compiled from [3], [2], [1]

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1 Poisson processes

What are Poisson processes?

Definition 1.0.1. A process $X = (X_t : t \geq 0)$ in $\mathbb{R}$ has *independent increments* if for all $0 \leq t_1 < ... < t_n$, for all $n \in \mathbb{N}$, the random variables $X_{t_{j+1}} - X_{t_j}, j = 1, ..., n$ are pairwise independent.
The process X has *stationary increments* if $X_t - X_s$ has the same distribution as $X_{t-s} - X_0$ (shorthand notation: $X_t - X_s \sim X_{t-s} - X_0$).
The process X is *stochastically continuous* if $X_t \xrightarrow{\mathsf{P}} X_s$ as $t \to s$ for all $s \geq 0$ (i.e. $\forall \epsilon > 0 \forall s \geq 0, \lim_{t \to s} \mathsf{P}(|X_t - X_s| > \epsilon) = 0$).

Definition 1.0.2. The process X is called a *Lévy process* if

1. $X_0 = 0$ almost surely.
2. X has stationary and independent increments.
3. X is stochastically continuous.

Definition 1.0.3. A *Poisson process of intensity $c > 0$* is an example of a right-continuous Lévy process $(N_t)_{t \geq 0}$ such that $N_t \sim \text{Poisson}(tc)$ for all $t \geq 0$.

The existence of such processes:

We begin by giving some definitions.

Definition 1.0.4. Let μ be a σ-finite measure on $\mathbb{R}$ and let $S \in \mathscr{B}(\mathbb{R})$. A *random measure on S* is a collection of random variables $(\phi(B) : B \in \mathscr{B}(S))$ parameterized by Borel subsets of S, and such that $\phi(B)$ is a measure as a function of B.

Definition 1.0.5. A random measure on S is called a *Poisson random measure with intensity measure μ* if for all $B \in \mathscr{B}(S)$ such that $\mu(B) < \infty$, $\phi(B) \sim \text{Poisson}(\mu(B))$ and $\phi(B_1), ..., \phi(B_n)$ are independent for all $B_1, ..., B_n \in \mathscr{B}(S)$ disjoint.

We then prove the existence of Poisson random measures.

Proposition 1.0.1. *For all σ-finite measures μ on S, there exists a Poisson random measure with intensity μ.*

Proof. First, let's assume the weaker condition that μ is finite.
Let $\xi_1, \xi_2, ...$ be a sequence of iid random variables with common law $\frac{\mu}{\|\mu\|}$ (where $\|\cdot\|$ here is the measure of the whole space i.e. $\|\mu\| = \mu(S)$), and let $N \sim \text{Poisson}(\|\mu\|)$ be independent of $(\xi_j)_{j \in \mathbb{N}}$.
Consider the random measure

$$\nu = \sum_{j=1}^{N} \delta_{\xi_j}$$

Notice how, for $B \in \mathscr{B}(S)$, $\nu(B)$ counts the number of points ξ_j lying in B, and its characteristic function is given by:

$$\begin{aligned}
\phi_{\nu(B)}(t) &:= \mathbb{E}(e^{it\nu(B)}) = \sum_{n=0}^{\infty} \mathbb{E}(e^{it\nu(B)}|N=n)\mathsf{P}(N=n) && \text{(by the law of total probability)} \\
&= \sum_{n=0}^{\infty} \mathbb{E}(e^{it(\mathbb{1}_B(\xi_1)+\ldots+\mathbb{1}_B(\xi_n))})\mathsf{P}(N=n) \\
&= \sum_{n=0}^{\infty} \prod_{j=1}^{n} \mathbb{E}(e^{it\mathbb{1}_B(\xi_j)})\mathsf{P}(N=n) && \text{(by independence of the } \xi_j\text{'s)} \\
&= \sum_{n=0}^{\infty} [\mathbb{E}(e^{it\mathbb{1}_B(\xi_1)})]^n \mathsf{P}(N=n) && \text{(since the } \xi_j\text{'s are identically distributed)} \\
&= \sum_{n=0}^{\infty} [e^{it\cdot 1}\mathsf{P}(\xi_1 \in B) + e^{it\cdot 0}\mathsf{P}(\xi_1 \notin B)]^n \mathsf{P}(N=n) \\
&= \sum_{n=0}^{\infty} [e^{it}\frac{\mu(B)}{\|\mu\|} + (1-\mathsf{P}(\xi_1 \in B))]^n \mathsf{P}(N=n) && \left(\xi_1 \text{ has law } \frac{\mu}{\|\mu\|} \implies \frac{\mu(B)}{\|\mu\|} = \mathsf{P}(\xi_1 \in B)\right) \\
&= \sum_{n=0}^{\infty} [e^{it}\frac{\mu(B)}{\|\mu\|} + 1 - \frac{\mu(B)}{\|\mu\|}]^n \frac{\|\mu\|^n}{n!} e^{-\|\mu\|} && (N \sim \text{Poisson}(\|\mu\|)) \\
&= \sum_{n=0}^{\infty} \frac{[e^{it}\mu(B) + \|\mu\| - \mu(B)]^n}{n!} e^{-\|\mu\|} \\
&= \sum_{n=0}^{\infty} \frac{[e^{it}\mu(B) + \|\mu\| - \mu(B)]^n}{n!} e^{-(e^{it}\mu(B)+\|\mu\|-\mu(B))} e^{(e^{it}-1)\mu(B)} \\
&= e^{(e^{it}-1)\mu(B)} \sum_{n=0}^{\infty} \frac{[e^{it}\mu(B) + \|\mu\| - \mu(B)]^n}{n!} e^{-(e^{it}\mu(B)+\|\mu\|-\mu(B))} \\
&= e^{(e^{it}-1)\mu(B)}
\end{aligned}$$

(by the def. of the p.m.f. of a Poisson random variable with parameter $e^{it}\mu(B) + \|\mu\| - \mu(B)$, the above sum equals 1)

Now, $e^{(e^{it}-1)\mu(B)}$ is precisely the characteristic function of a Poisson random variable with parameter $\mu(B)$.

$$\therefore \nu(B) \sim \text{Poisson}(\mu(B))$$

Moreover, it's clear that $\nu : \mathscr{B}(S) \to [0, \infty)$ is a measure. So, it follows from Definition 1.0.4 that ν is in fact a random measure.

In order to show that ν is a Poisson random measure, it remains to prove that for all $B_1, ..., B_n \in \mathscr{B}(S)$ disjoint, $\nu(B_1), ..., \nu(B_n)$ are independent. To do this, we shall use the criteria of independence using characteristic functions (Theorem 2.1 (Kac's Theorem) from [4]):

So, take $B_1, ..., B_n \in \mathscr{B}(s)$ disjoint. Fix $a_1, ..., a_n \in \mathbb{R}$.

$$
\begin{aligned}
\mathbb{E}(e^{ia_1\nu(B_1)+\ldots+ia_2\nu(B_n)}) &= \sum_{k=0}^{\infty} \mathbb{E}(e^{ia_1\nu(B_1)+\ldots+ia_2\nu(B_n)}|N=k)\mathsf{P}(N=k) \\
&\quad \text{(by law of total expectation)} \\
&= \sum_{k=0}^{\infty} \mathbb{E}(e^{ia_1\sum_{j=1}^{k}\mathbb{1}_{B_1}(\xi_j)+\ldots+ia_n\sum_{j=1}^{k}\mathbb{1}_{B_n}(\xi_j)})\mathsf{P}(N=k) \\
&= \sum_{k=0}^{\infty} \mathbb{E}(e^{i\sum_{m=1}^{n}\sum_{j=1}^{k}a_m\mathbb{1}_{B_m}(\xi_j)})\mathsf{P}(N=k) \\
&= \sum_{k=0}^{\infty} \mathbb{E}(e^{i\sum_{j=1}^{k}\sum_{m=1}^{n}a_m\mathbb{1}_{B_m}(\xi_j)})\mathsf{P}(N=k) \\
&= \sum_{k=0}^{\infty} \prod_{j=1}^{k} \mathbb{E}(e^{i\sum_{m=1}^{n}a_m\mathbb{1}_{B_m}(\xi_j)})\mathsf{P}(N=k) \\
&\quad \text{(by indep. of } \xi_j\text{'s)} \\
&= \sum_{k=0}^{\infty} \Big[\mathbb{E}(e^{i\sum_{m=1}^{n}a_m\mathbb{1}_{B_m}(\xi_1)})\Big]^k \mathsf{P}(N=k) \\
&\quad (\xi_j\text{'s are identically distributed)} \\
&= \sum_{k=0}^{\infty} \Bigg[\sum_{l=1}^{n}(\mathbb{E}(e^{i\sum_{m=1}^{n}a_m\mathbb{1}_{B_m}(\xi_j)}|\xi_1\in B_l)\mathsf{P}(\xi_1\in B_l)) \\
&\qquad + \mathbb{E}(e^{i\sum_{m=1}^{n}a_m\mathbb{1}_{B_m}(\xi_1)}|\xi_1\notin\bigcup_{j=1}^{n}B_j)\mathsf{P}(\xi_1\notin\bigcup_{j=1}^{n}B_j)\Bigg]^k \mathsf{P}(N=k) \\
&\quad \text{(by law of total expectation and the fact that the } B_j\text{'s are disjoint)} \\
&= \sum_{k=0}^{\infty} \Big[\sum_{l=1}^{n}\mathbb{E}(e^{ia_l})\frac{\mu(B_l)}{\|\mu\|} + 1\cdot(1-\mathsf{P}(\xi_1\in\bigcup_{j=1}^{n}B_j))\Big]^k \mathsf{P}(N=k)
\end{aligned}
$$

(by the facts that $\frac{\mu}{\|\mu\|}$ is the law of ξ_1 and $\xi_1 \notin \bigcup_{j=1}^{n} B_j \implies \mathbb{1}_{B_m}(\xi_1)=0 \ \forall m=1,..,n$ and by the fact that the B_j's are disjoint $\implies \forall l\in\{1,..,n\},\ \xi_1\in B_l \implies \xi\notin B_k \ \forall k\neq l$)

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \Big[\sum_{l=1}^{n} e^{ia_l}\frac{\mu(B_l)}{\|\mu\|} + 1 - \frac{\mu(\bigcup_{j=1}^{n}B_j)}{\|\mu\|}\Big]^k \frac{\|\mu\|^k}{k!}e^{-\|\mu\|} \\
&= \sum_{k=0}^{\infty} \frac{\Big[\sum_{l=1}^{n} e^{ia_l}\mu(B_l) + \|\mu\| - \mu(\bigcup_{j=1}^{n}B_j)\Big]^k}{k!}e^{-\|\mu\|} \\
&= \sum_{k=0}^{\infty} \frac{\Big[\sum_{l=1}^{n} e^{ia_l}\mu(B_l) + \|\mu\| - \mu(\bigcup_{j=1}^{n}B_j)\Big]^k}{k!}e^{-(\sum_{l=1}^{n} e^{ia_l}\mu(B_l)+\|\mu\|-\mu(\bigcup_{j=1}^{n}B_j))} \\
&\qquad\qquad \cdot e^{\sum_{l=1}^{n}\mu(B_l)-\mu(\bigcup_{j=1}^{n}B_j)}
\end{aligned}
$$

$$= e^{\sum_{l=1}^n \mu(B_l) - \sum_{j=1}^n \mu(B_j)}$$

$$\cdot \sum_{k=0}^{\infty} \frac{\left[\sum_{l=1}^n e^{ia_l}\mu(B_l) + \|\mu\| - \mu(\bigcup_{j=1}^n B_j)\right]^k}{k!} e^{-(\sum_{l=1}^n e^{ia_l}\mu(B_l) + \|\mu\| - \mu(\bigcup_{j=1}^n B_j))}$$

$$\Big(\text{as } \mu \text{ is a measure, } B_j\text{'s disjoint } \implies \mu(\bigcup_{j=1}^n B_j) = \sum_{j=1}^n \mu(B_j)\Big)$$

$$= e^{\sum_{l=1}^n (e^{ia_l}\mu(B_l) - \mu(B_l))}$$

(by the def. of the p.m.f. of a Poisson random variable with parameter $\sum_{l=1}^n e^{ia_l}\mu(B_l) + \|\mu\| - \mu(\bigcup_{j=1}^n B_j)$, the above sum equals 1)

$$= e^{\sum_{l=1}^n \mu(B_l)(e^{ia_l}-1)}$$

$$= \prod_{l=1}^n e^{(e^{ia_l}-1)\mu(B_l)}$$

$$= \prod_{l=1}^n \mathbb{E}(e^{ia_l\nu(B_l)})$$

(characteristic functions of $\nu(B_l) \sim \text{Poisson}(\mu(B_l))$)

So, we just showed that for any $B_1, ..., B_n \in \mathscr{B}(S)$ disjoint,

$$\mathbb{E}(e^{ia_1\nu(B_1)+...+ia_2\nu(B_n)}) = \prod_{l=1}^n \mathbb{E}(e^{ia_l\nu(B_l)})$$

$\implies \nu(B_1), ..., \nu(B_n)$ are independent as required.

So by Definition 1.0.5, ν is in fact a Poisson random measure with intensity μ. And so we proved the Proposition for the case when ν is a finite measure.

Assume now that μ is σ-finite. This means that there exist disjoint sets $\{A_j\}_{j=1}^\infty$ such that $S = \bigcup_{j=1}^\infty A_j$, and $\mu(A_j) < \infty$ for all $j = 1, 2, 3, ...$

As $\mu(A_j) < \infty$ for all $j \in \mathbb{N}$, from what we've proved before for the finite measure case, we can find a sequence $(\phi_j)_{j\geq 1}$ of independent Poisson random measures with the intensities $\mathbb{1}_{A_j}\mu$.

This implies that $\phi = \sum_{j=1}^\infty \phi_j$ is a Poisson random measure with intensity μ. □

We can now prove the existence of Poisson processes:

Corollary 1.0.1. *A Poisson process of any given intensity $c > 0$ exists.*

Proof. By Proposition 1.0.1, there exist a ϕ Poisson random measure on $\mathbb{R}_+$ with intensity $m \cdot c$, where m is the Lebesgue measure.

This means that for all $B \in \mathscr{B}(\mathbb{R}_+)$ such that $c \cdot m(B) < \infty$, $\phi(B) \sim \text{Poisson}(c \cdot m(B))$ and

$\phi(B_1), ..., \phi(B_n)$ are independent for all $B_1, ..., B_n \in \mathscr{B}(\mathbb{R}_+)$ disjoint.

In particular, this implies that $\phi([0,t]) \sim \text{Poisson}(ct)$ (since $c \cdot m([0,t]) = c(t-0) = ct$) for all $t \geq 0$.

Let's set $N_t \coloneqq \phi([0,t])$ for $t \geq 0$.

We then have $N_0 = \phi([0,0]) \sim \text{Poisson}(0)$, since $c \cdot m([0,0]) = 0$. Moreover, $\mathsf{P}(N_0 = 0) = \frac{0^0 e^{-0}}{0!} = 1$ $\therefore N_0 = 0$ almost surely.

We would like to show that $(N_t)_{t \geq 0}$ has independent increments:

Take $0 \leq t_1 < ... < t_n$ and $i, j \in \{1, 2, ..., n\}$ with $i < j$.

Want to show that $N_{t_i} - N_{t_{i-1}}$ and $N_{t_j} - N_{t_{j-1}}$ are independent.

Now,

$$\begin{aligned} N_{t_i} - N_{t_{i-1}} &= \phi([0,t_i]) - \phi([0,t_{i-1}]) \\ &= \phi([0,t_i] \backslash [0,t_{i-1}]) && \text{(by the excision property of measures)} \\ &= \phi((t_{i-1}, t_i]) \end{aligned}$$

and

$$\begin{aligned} N_{t_j} - N_{t_{j-1}} &= \phi([0,t_j]) - \phi([0,t_{j-1}]) \\ &= \phi([0,t_j] \backslash [0,t_{j-1}]) && \text{(by the excision property of measures)} \\ &= \phi((t_{j-1}, t_j]). \end{aligned}$$

The fact that $i < j$ implies that $(t_{i-1}, t_i]$ and $(t_{j-1}, t_j]$ are disjoint.

This implies that $\phi((t_{i-1}, t_i]) = N_{t_i} - N_{t_{i-1}}$ and $\phi((t_{j-1}, t_j]) = N_{t_j} - N_{t_{j-1}}$ are independent, by the definition of ϕ being a Poisson random measure.

$\therefore (N_t)_{t \geq 0}$ has independent increments.

Now we want to show stationarity of the increments of $(N_t)_{t \geq 0}$:

Take $t > s$, then by the excision property of measures and the construction of ϕ,

$$N_t - N_s = \phi([0,t]) - \phi([0,s]) = \phi((s,t]) \sim \text{Poisson}(c(t-s)).$$

We also know that $N_{t-s} = \phi([0, t-s]) \sim \text{Poisson}(c(t-s))$, by the definition of $(N_t)_{t \geq 0}$.

$\therefore (N_t)_{t \geq 0}$ has stationary increments.

Finally, we want to show that $(N_t)_{t \geq 0}$ is stochastically continuous i.e. we want to show that $N_t \xrightarrow{\mathsf{P}} N_s$ as $t \to s$ for all $s \geq 0$.

So fix $a > 0$ and $s \geq 0$,

$$\begin{aligned}
\mathsf{P}(|N_t - N_s| > a) &= \mathsf{P}(N_{t-s} > a) && \text{(by stationarity of increments)} \\
&= 1 - \mathsf{P}(N_{t-s} \leq a) \\
&= 1 - \sum_{i=0}^{a} \mathsf{P}(N_{t-s} = i) && \text{(by discreteness of the Poisson distribution)} \\
&= 1 - \sum_{i=0}^{a} [c(t-s)]^i \frac{e^{-c(t-s)}}{i!} && \text{(since } N_{t-s} \sim \text{Poisson}(c(t-s))) \\
&= 1 - e^{-c(t-s)} - \sum_{i=1}^{a} [c(t-s)]^i \frac{e^{-c(t-s)}}{i!}
\end{aligned}$$

$$\begin{aligned}
\therefore \lim_{t \to s} \mathsf{P}(|N_t - N_s| > a) &= 1 - \lim_{t \to s} e^{-c(t-s)} - \lim_{t \to s} \sum_{i=1}^{a} [c(t-s)]^i \frac{e^{-c(t-s)}}{i!} \\
&= 1 - 1 - \sum_{i=1}^{a} \lim_{t \to s} [c(t-s)]^i \frac{e^{-c(t-s)}}{i!} && \text{(by continuity of } f(t) = e^{-c(t-s)}) \\
&= 0 && \text{(by algebra of limits)}
\end{aligned}$$

$\therefore (N_t)_{t \geq 0}$ is stochastically continuous.

It follows by Definition 1.0.3 that $(N_t)_{t \geq 0}$ is a Poisson process with intensity c. □

Now, to give a direct proof of the Markov property for Poisson processes, we make use of an equivalent definition for Poisson processes. For the proof of the equivalence of the following definition with Definition 1.0.3, we refer to the proof of Theorem 2.4.3 in [3].

Definition 1.0.6 (Equivalent definition of Poisson processes)**.** Let $(X_t)_{t \geq 0}$ be an increasing, right-continuous, integer-valued process starting from 0. Let $c > 0$.
Then, $(X_t)_{t \geq 0}$ is a *Poisson process of intensity c* if the *holding times* $S_1, S_2, ...$ of $(X_t)_{t \geq 0}$ are independent random variables such that $S_i \sim \text{Exp}(c)$ for all $i \in \mathbb{N}$ and the *jump chain* is given by $Y_n = n$ for all $n \in \mathbb{N}$.

So, a simple way to construct a Poisson process is to take a sequence $S_1, S_2, ...$ of iid Exp(c)-distributed random variables, then set $J_0 = 0$ and $J_n = S_1 + ... + S_n$ for $n \in \mathbb{N}$ and finally set

$$X_t = n \quad \text{if} \quad J_n \leq t < J_{n+1}$$

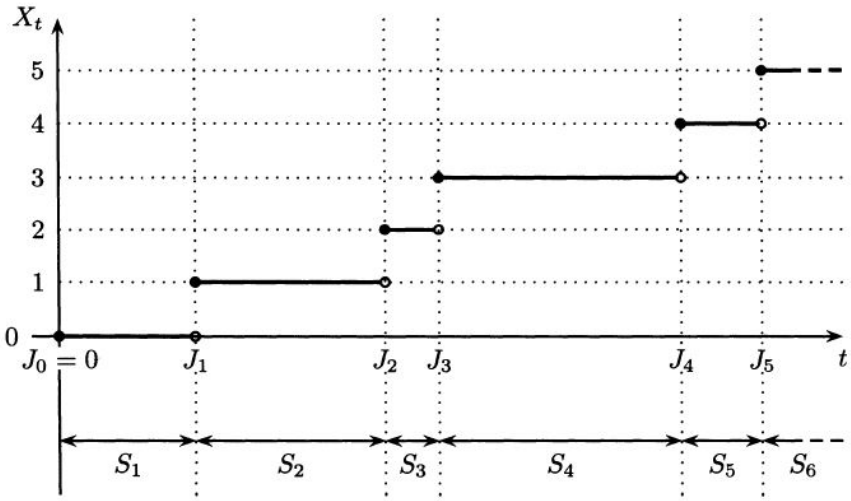

Figure 1: Taken from [3] to illustrate the construction of a Poisson process with the above notation.

In order to proceed, we need a theorem - the memoryless property of the exponential distribution:

Theorem 1.0.1 (Memoryless property of Exponential distribution)**.**

$T \sim Exp(c) \iff$ The positive random variable T satisfies following memoryless property:

$$\mathsf{P}(T > s + t | T > s) = \mathsf{P}(T > t) \quad \textit{for all } s, t \geq 0.$$

Proof. $\underline{\implies:}$
Suppose $T \sim \text{Exp}(c)$.
$\implies \mathsf{P}(T > s + t | T > s) = \frac{\mathsf{P}(T > s+t)}{\mathsf{P}(T > s)} = \frac{e^{-c(s+t)}}{e^{-cs}} = e^{-ct} = \mathsf{P}(T > t)$.

$\underline{\impliedby:}$
Suppose T has the memoryless property whenever $\mathsf{P}(T > s) > 0$ (otherwise, the conditional probability would be undefined).

Then $g(t) := \mathsf{P}(T > t)$ satisfies:

$$\begin{aligned}
g(s+t) = \mathsf{P}(T > s+t) &= \frac{\mathsf{P}(T > s+t, T > s)}{\mathsf{P}(T > s)} \cdot \mathsf{P}(T > s) && (1)\\
&= \mathsf{P}(T > s+t | T > s)\mathsf{P}(T > s) && (2)\\
&= \mathsf{P}(T > t)\mathsf{P}(T > s) \qquad \text{(by the memoryless property of T)} && (3)\\
&= g(s)g(t) \qquad \text{for all } s, t \geq 0. && (4)
\end{aligned}$$

As T is a positive random variable, we have $g(\frac{1}{n}) > 0$ for some $n > 0$.

Then, by (21) and induction,

$$g(1) = g(\underbrace{\frac{1}{n} + ... + \frac{1}{n}}_{n \text{ times}}) = \Big[g(\frac{1}{n})\Big]^n > 0.$$

This implies that $g(1) = e^{-c}$ for some $c > 0$ and for any integers $p, q \geq 1$,

$$g\Big(\frac{p}{q}\Big) = g\Big(\frac{1}{q}\Big)^p = g\Big(1 \cdot \frac{1}{q}\Big)^p = g(1)^{\frac{p}{q}}$$
$$\implies g(r) = e^{-cr} \qquad \text{for any } r \in \mathbb{Q}_{>0}$$

Now, for a real number $t > 0$, choose rationals $r, s > 0$ such that $r \leq t \leq s$.

As g is a decreasing function (with respect to t),

$$e^{-cr} = g(r) \geq g(t) \geq g(s) = e^{-cs},$$

and since we can choose r and s arbitrarily close to t (due to the density of the rationals on the reals), this forces, by the continuity of the exponential function, $g(t) = e^{-ct}$. So in fact, $T \sim \text{Exp}(c)$ as required. □

We finally give a direct proof of the Markov property for Poisson processes:

Theorem 1.0.2 (Markov Property for Poisson processes)**.** *Let $(X_t)_{t\geq 0}$ be a Poisson process with intensity $c > 0$.*
Then, for all $s \geq 0$, $(X_{s+t} - X_s)_{t\geq 0}$ is also a Poisson process of rate c, independent of $(X_r : r \leq s)$.

Proof. Suffices to prove the theorem conditional on the event $X_s = i$ for all $i \in \mathbb{N}_0$.

Using the same notations that we had used in the construction of a Poisson process on Page 32, let $\widetilde{X}_t = X_{s+t} - X_s$.

We have $\{X_s = i\} = \{J_i \leq s < J_{i+1}\} = \{J_i \leq s\} \cap \{S_{i+1} > s - J_i\}$ (since $S_{i+1} = J_{i+1} - J_i$).

On this event, $X_r = \sum_{j=1}^{i} \mathbb{1}_{\{J_j \leq r\}} = \sum_{j=1}^{i} \mathbb{1}_{\{\sum_{k=1}^{j} S_k \leq r\}}$ for $r \leq s$; and the holding times $\widetilde{S}_1, \widetilde{S}_2, ...$ of $(\widetilde{X}_t)_{t\geq 0}$ are given by:

$\widetilde{S}_1 = S_{i+1} - (s - J_i), \qquad \widetilde{S}_n = S_{i+n}$ for $n \geq 2$.

Recall that, by definition, the holding times $S_1, S_2, ...$ are independent $\text{Exp}(c)$ random variables.

We shall condition $\widetilde{S}_1$ on $S_1, ..., S_i$ and $\{X_s = i\}$. Recall that $S_{i+1} \sim \text{Exp(c)}$. By Theorem 1.0.1, $\mathsf{P}(S_{i+1} > t + s | S_{i+1} > t) = \mathsf{P}(S_{i+1} > s)$.

By definition of $\widetilde{S}_1$ and by the analogous Proposition **??** for non-negative, continuous random variables (like $\widetilde{S}_1$),

$$\mathbb{E}(\widetilde{S}_1 | S_1, ..., S_i, \{X_s = i\}) = \int_0^\infty \mathsf{P}(\widetilde{S}_1 > t | S_1, ..., S_i, \{X_s = i\}) dt$$

$$
\begin{aligned}
&= \int_0^\infty \mathsf{P}(S_{i+1} - (s - J_i) > t | S_1, ..., S_i, \{X_s = i\}) dt \\
&= \int_0^\infty \mathsf{P}(S_{i+1} > t + (s - \sum_{k=1}^{i} S_k) | S_1, ..., S_i, \{X_s = i\}) dt \qquad \text{(by the definition of } J_i) \\
&= \int_0^\infty \mathsf{P}\left(S_{i+1} > t + (s - \sum_{k=1}^{i} S_k) \middle| S_{i+1} > (s - \sum_{k=1}^{i} S_k) \right) dt \\
&= \int_0^\infty \mathsf{P}(S_{i+1} > t) dt \qquad \text{(by Theorem 1.0.1)} \\
&= \mathbb{E}(S_{i+1}) = \frac{1}{c}.
\end{aligned}
$$

And so, conditional on $S_1, ..., S_i$ and $\{X_s = i\}$, $\widetilde{S}_1 \sim \text{Exp}(c)$.

And by definition, $\widetilde{S}_n = S_{i+n} \sim \text{Exp}(c)$ for all $n \geq 2$.

Now, since $\{X_s = i\} = \{J_i \leq s\} \cap \{S_{i+1} > s - J_i\}$, by conditioning on $\{X_s = i\}$, from the above equations we get the independence of the sequence of random variables $\widetilde{S}_1, \widetilde{S}_2, ...$ (by the independence of $S_1, S_2, ...$). Moreover, we've shown that $\widetilde{S}_n \sim \text{Exp}(c)$ for every $n \geq 1$. And so by Definition 1.0.6, $(\widetilde{X}_t)_{t \geq 0}$ is a Poisson process with intensity c as required.

Moreover, the equation at the top of the page implies the independence of $(\widetilde{S}_n)_{n \geq 1}$ and $S_1, ..., S_i$. And as $X_r = \sum_{j=1}^{i} \mathbb{1}_{\{\sum_{k=1}^{j} S_k \leq r\}}$ for $r \leq s$, conditional on $\{X_s = i\}$, $(\widetilde{X}_t)_{t \geq 0}$ is independent of $(X_r)_{r \leq s}$. This follows because, conditionally on $\{X_s = i\}$, the holding times $(\widetilde{S}_n)_{n \geq 1}$ are independent of $S_1, ..., S_i$, but X_r, as we saw, is fully dependent on $S_1, ..., S_i$ for every $r \leq s$, and thus the jump chains of $(\widetilde{X}_t)_{t \geq 0}$ and those of $(X_r)_{r \leq s}$ are independent. And since we're dealing with right-continuous processes, independence of the respective jump chains determines the independence of the actual processes.

And so, conditional on $\{X_s = i\}$, $(\widetilde{X}_t)_{t \geq 0}$ is a Poisson process with intensity c, that is independent of $(X_r)_{r \leq s}$, as required. □

Furthermore, there is also quite a nice property that Poisson processes possess. This is, that the jump times of the Poisson processes, conditional on the value of the process at some time $t \geq 0$, have the same distribution as an ordered sample of the size of that value, from the uniform distribution on the interval $[0, t]$. We prove this property next:

Theorem 1.0.3. *Let $(X_t)_{t \geq 0}$ be a Poisson process with intensity c.*

Then, conditionally on the event $\{X_t = n\}$, the jump times $J_1, ..., J_n$ have joint density function

$$f(t_1, ..., t_n) = n! \mathbb{1}_{\{0 \leq t_1 \leq ... \leq t\}}.$$

Thus, conditional on $\{X_t = n\}$, the jump times $J_1, ..., J_n$ have the same distribution as an ordered sample of size n from the uniform distribution on $[0, t]$.

Proof. The holding times $S_1, ..., S_{n+1}$, as they are independent $\text{Exp}(c)$ random variables, their joint density is the product of their individual densities:

$$c^{n+1}e^{-c(S_1+...+S_{n+1})}\mathbb{1}_{\{S_1,...,S_{n+1}\geq 0\}}. \tag{5}$$

We want to find the joint density of the jump times $J_1, ..., J_{n+1}$. We shall make use of the multivariate transformation of random variables.

Let $g(s_1, ..., s_{n+1}) = c^{n+1}e^{-c(s_1+...+s_{n+1})}$ (the joint density of $S_1, ..., S_{n+1}$).

We have $J_i = h_i(S_1, ..., S_{n+1})$ where $h_i(x_1, ..., x_{n+1}) = \sum_{k=1}^{i} x_k$ for $i = 1, ..., n+1$.

Let $f(j_1, ..., j_{n+1})$ denote the joint density of $J_1, ..., J_{n+1}$.

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial j_1} & \cdots & \frac{\partial h_1^{-1}}{\partial j_{n+1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_{n+1}^{-1}}{\partial j_1} & \cdots & \frac{\partial h_{n+1}^{-1}}{\partial j_{n+1}} \end{vmatrix} \qquad \text{is the Jacobian}$$

where

$$s_1 = j_1 = h_1^{-1}(j_1, ..., J_{n+1})$$

$$s_i = \sum_{k=1}^{i} s_k - \sum_{k=1}^{i-1} s_k = j_i - j_{i-1} = h_i^{-1}(j_1, ..., j_{n+1}) \quad \text{for } i = 2, ..., n+1$$

$$\therefore J = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 1 \end{vmatrix} = 1$$

$$\begin{aligned} \implies f(j_1, ..., j_{n+1}) &= g(h_1^{-1}(j_1, ..., j_{n+1}), ..., h_{n+1}^{-1}(j_1, ..., j_{n+1}))|J| \\ &= c^{n+1}e^{-c(j_1+\sum_{i=2}^{n+1}(j_i-j_{i-1}))} \\ &= c^{n+1}e^{-cj_{n+1}} && \text{(telescoping sum)} \end{aligned}$$

$\therefore$ The joint density of $J_1, ..., J_{n+1}$ is

$$c^{n+1}e^{-ct_{n+1}}\mathbb{1}_{\{0\leq t_1\leq ...\leq t_{n+1}\}}.$$

And so, for $A \subset \mathbb{R}^n$,

$$\begin{aligned} \mathsf{P}((J_1, ..., J_n) \in A, X_t = n) &= \mathsf{P}((J_1, ..., J_n) \in A, J_n \leq t < J_{n+1}) \\ &= \int\limits_{(t_1,...,t_n)\in A} e^{-ct}c^n\mathbb{1}_{\{0\leq t_1\leq ...\leq t_n\leq t\}}dt_1 \cdots dt_n \\ &= e^{-ct}c^n \int\limits_{(t_1,...,t_n)\in A} \mathbb{1}_{\{0\leq t_1\leq ...\leq t_n\leq t\}}dt_1 \cdots dt_n. \end{aligned}$$

And since $\mathsf{P}(X_t = n) = e^{-ct}\frac{c^n}{n!}$, we get

$$\begin{aligned}\mathsf{P}((J_1, ..., J_n) \in A | X_t = n) &= \frac{\mathsf{P}((J_1, ..., J_n) \in A, X_t = n)}{\mathsf{P}(X_t = n)} \\ &= \int\limits_{(t_1,...,t_n)\in A} n! \mathbb{1}_{\{0 \le t_1 \le ... \le t_n \le t\}} dt_1 \cdots dt_n.\end{aligned}$$

And so the result follows. $\square$

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