Classification of Transformations of Equivalent Kernels of Some Determinantal Point Processes, arXiv:2302.02471

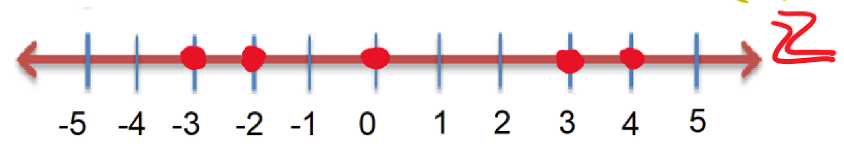
1. Introduction to Determinantal Point Processes (abbr., DPPs)

1 General DPPs

DPP X with kernel $K:\Lambda^2\to\mathbb{F}$

Discrete setting: $(\Lambda=\mathbb{Z})$

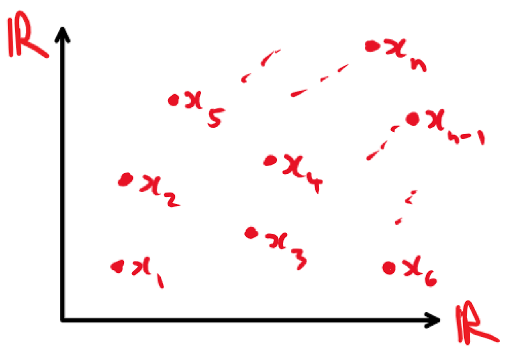
$A=\{-3,-2,0,3,4\}$



$$\boxed{\mathbb{P}(\text{particles in } A) = \det(K_A)}$$

$K_A = (K(x,y))_{x,y\in A}$

Continuum: $(\Lambda=\mathbb{R}^2)$



$\mathbb{E}(X(B_1)\cdots X(B_n)) = \int_{B_1\times\cdots\times B_n} \det(K(x_i,x_j))_{i,j=1}^n \, \nu(dx_1)\cdots\nu(dx_n)$, $B_i\subseteq\Lambda$.

$$\boxed{\text{“}\mathbb{P}(\text{particles at } x_1,\dots,x_n) = \det(K(x_i,x_j))_{i,j=1}^n \, \nu(dx_1)\cdots\nu(dx_n)\text{”}}$$

2. The Setup of the Research Problem

2 Equivalent Kernels

The set-up

X is a DPP with kernel $K:\Lambda^2\to\mathbb{F}$

> Λ, a set;

> $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$.

Def.: $Q:\Lambda^2\to\mathbb{F}$ is an equivalent kernel of K if

$Q\equiv K$

$$\boxed{\det(K(x_i,x_j))_{i,j=1}^n \overset{(*)}{=} \det(Q(x_i,x_j))_{i,j=1}^n}$$

$\forall n>0 \quad \forall (x_1,\dots,x_n)\in\Lambda^n.$

3. Examples of Equivalent Kernels

3 Examples of equivalent kernels

Examples of equivalent kernels

- $Q:\Lambda^2\to\mathbb{F}$ is a transposition transformation of K if
$$\boxed{Q(x,y)=K(y,x)} \quad \forall x,y\in\Lambda.$$

$\det(A)=\det(A^T)$

- $Q:\Lambda^2\to\mathbb{F}$ is a conjugation transformation of K if
$$\boxed{Q(x,y)=\frac{g(x)}{g(y)}K(x,y)} \quad \forall x,y\in\Lambda,$$
for some (conjugation) function $g:\Lambda\to\mathbb{F}\setminus\{0\}$.

$\det(A)=\det(MAM^{-1})$

$M=\begin{pmatrix} g(x_1) & & 0\\ & \ddots & \\ 0 & & g(x_n)\end{pmatrix}$

4. Conjecture & Solution for Symmetric Kernels

4 Classification #1

Classification of equivalent kernels

Conj. (Bufetov, '17): If K and Q are equivalent kernels, then they can be transformed into one another by transposition and conjugation transformations.

Thm. (Stevens, '19): If $K\equiv Q$ are both symmetric kernels (i.e., $K(x,y)=K(y,x)$ and $Q(x,y)=Q(y,x)$ $\forall x,y\in\Lambda$), then Q is a conjugation transformation of K, i.e., $\exists\, g:\Lambda\to\mathbb{F}\setminus\{0\}$ such that
$$Q(x,y)=\frac{g(x)}{g(y)}K(x,y) \quad \forall x,y\in\Lambda.$$

5. Counterexample to Conjecture with Relaxed Symmetry Assumptions

5 Counterexample

A counterexample

non-symmetric

$$B=\left[\begin{array}{c|c} C & 0\\ \hline 0 & D\end{array}\right] \xrightarrow[\text{transformation}]{\text{“Partial” transposition}} B'=\left[\begin{array}{c|c} C^T & 0\\ \hline 0 & D\end{array}\right]$$

non-symmetric

$=(K(x,y)) \qquad\qquad =(Q(x,y))$

$\Rightarrow \det(B)=\det(B')$. ✓ TRUE

But, • “$K(x,y)=Q(y,x) \quad \forall x,y$” ✗ FALSE

• “$B=MB'M^{-1}$ for some invertible M” ✗ FALSE

6. Solution to Conjecture with Relaxed Symmetry Assumptions

6 Classification #2

Classification of equivalent kernels

Conj. (Bufetov, '17): If K and Q are equivalent kernels, then they can be transformed into one another by transposition (i) and conjugation (ii) transformations.

Thm. (Sapranidis '23): Under the conditions ① $K\neq 0$; ② $\begin{vmatrix} K(x,y) & K(x,z)\\ K(w,y) & K(w,z)\end{vmatrix}\neq 0$ $\forall x,y,z,w\in\Lambda$ distinct,

$Q\equiv K \Rightarrow Q(x,y)=g(x)K(x,y)g(y)^{-1} \quad \forall x,y\in\Lambda$ (ii)

OR

$Q(x,y)=g(x)K(y,x)g(y)^{-1} \quad \forall x,y\in\Lambda.$ (i)

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