# Chaos & Order

Cagri Sert

University of Warwick

Taster lecture, Years 12-13th
25.02.2026

# What is in this talk?

1. General thoughts and a mathematical formalism
2. A simple example: simple pendulum
3. Let's add one more weight: double pendulum
4. Transition from order to chaos: the logistic map
5. Back to order, and so forth

— I note that these slides provide an outline of the lecture and do not capture everything that was discussed. —

# Randomness and chaos: what are they?

I think the notion of randomness originates in our experience of lack of ability to predict.

Think of throwing a dice or tossing a coin. Nothing is random about them. Yet they are the basis of our idea of randomness.

Chaos, not a rigorous mathematical term, is suggestive word to describe systems that are very hard, if not impossible, to predict in near future - we refer to these as chaotic systems (again not necessarily a mathematical term).

Hard to predict... Near future..? Let's take it slowly. What does prediction of a system mean? But first, what is a system?

# A mathematical formulation of systems and their evolutions

Let $X$ be a set and $T : X \to X$ a function from $X$ to $X$. We think of $X$ the set of all possible states of a system and $T$ mapping a state $x \in X$ to the next one, after one unit of time. So if $x$ is an actual state, $T(x)$ represents the next state and $T \circ T(x) = T^2(x)$ is the state two units of time after $x$.

One can also model the transformation in continuous time considering a family of maps $T_t : X \to X$ parametrized by $t \in [0, \infty)$ such that $T_0 = \mathrm{id}$ and $T_{t+s} = T_t \circ T_s$.

# Simple pendulum

Let us consider the simple pendulum (picture).

An angle $\theta \in [0, 2\pi)$: in other words a circle $\mathbb{S}^1$, and an angular velocity: in other words a real number (we fix an orientation to make sense of $+$ (clockwise) and $-$ (counter-clockwise)).

So the set of all possible states of simple pendulum: $\mathbb{S}^1 \times \mathbb{R} =$ a cylinder. it is a two dimensional object.

A not-so-complicated differential equation (derived from laws..hmm.. assumptions of physics) reigns the motion. we will just assume this is how the nature works in our system.

And gives us the transformations $T_t : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ (not very difficult to write down - believe me). So we can describe whole motion starting from a point.

# Now: double pendulum

Let us now consider the double pendulum picture.

The state space is $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}$ - a four dimensional space.

This time it is slightly more complicated, but one can write down the differential equation that governs the motion.

Double pendulum video

# Towards a description of chaos

What is going on here? Why does it look so chaotic? Why do we talk about chaos (or sensitive dependence to initial conditions) for double pendulum and not for simple pendulum? After all simple pendulum trajectories also diverge in some sense.

The answer is a combination of two things: one aspect is exponential (very very quick) divergence of trajectories. The other is less precise (but can be made mathematically precise): possibility of reaching very different states from a given initial state.

These two aspects characterize the chaos.

In popular culture: Butterfly effect.

# Transition from order (absence of chaos) to chaos: example of the logistic map

Is there a blackboard?

For each $r \in (0,4]$, consider the map $T_r : [0,1] \to [0,1]$ given by $T_r(x) = rx(1-x)$.

Logistic map wiki

Observe: We have $T_r(x) \geqslant 0$ for every $x \in [0,1]$ and $r \in (0,4]$. We also have $T_r(0) = 0 = T_r(1)$.

We have $T_r'(x) = r - 2rx = r(1-2x)$. So for every $r \in (0,4]$, $T_r$ is strictly increasing on $[0, \frac{1}{2}]$ and strictly decreasing on $[\frac{1}{2}, 1]$. In particular, $T_r$ reaches to its unique maximum at $\frac{1}{2}$. $T_r(\frac{1}{2}) = \frac{r}{4}$ and hence for every $r \in (0,4]$, $T_r$ maps $[0,1]$ to itself.

# Logistic map 2

- For $r < 1$, the map $T_r$ is contracting: $|T_r(x)| \leqslant rx$ so all trajectories $(T_r^n(x))$ converge to 0 as $n \to \infty$ (except $x = 1$). No dependence on initial conditions.

Let us consider $r \in (1, 4]$. Then there are two fixed points $x_0$ and $x_1$ of $T_r$: solving $T_r(x) = x$, we easily find them to be $x_0 = 0$ and $x_1 = 1 - \frac{1}{r}$. $T_r x_0 = x_0$ and $T_r x_1 = x_1$. These are the steady states. Starting from them nothing changes ever.

One question:
What happens near them? This can be understood by looking at the absolute value of the derivatives at the fixed points. Let us look at $x_0 = 0$ for some fixed $r \in (1, 3)$. We have $T_r'(x) = r - 2rx$ and hence $|T_r'(0)| = r > 1$. For points very close to 0, the map $T_r$ expands them – i.e. sends away further away from 0. We say that the fixed point 0 is a *repelling or unstable fixed point*.

# Logistic map 3

How about the other fixed point?
$T_r'(x_1) = r - 2rx_1 = r - 2r(1 - \frac{1}{r}) = 2 - r$. So $|T_r'(x_1)| < 1$. Locally, it is contracting. – i.e. the map sends points close to $x_1$ even closer. We say that the fixed point $x_1$ is an *attracting or stable fixed point*.

• It follows from a theorem (not directly from the above argument) that actually for $r \in (1, 3)$, all trajectories converge to $x_1$ (except of course $x_0 = 0$ and 1).in other words, for every $x \in (0, 1)$, we have $T_r^n(x) \to_{n\to\infty} x_1$.

For $r > 3$ we have $|T_r'(x_1)| = |2 - r| > 1$ so the fixed point $x_1$ becomes unstable. Nearby points can no longer converge to $x_1$.

Something interesting happens here. There exist two fixed points of $T_r \circ T_r$, call them $y_1$ and $y_2$ so that all trajectories oscillated between $y_1$ and $y_2$ getting closer and closer to these two points.

# Logistic map 4

This survives until some $4 > r_1 > 3$. Something else happens when $r > r_1$: there are four fixed points of $T_r^3$, call them $y_1, y_2, y_3, y_4$ and all trajectories are attracted these 4 points by every oscillating.

This continues as $r$ grows: there exists $3 = r_0 < r_1 < r_2 < r_3 < ...$ such that for every $r \in (r_n, r_{n+1})$, $T_r^{n+1}$ has $2^n$-fixed points and all trajectories of $T_r$ get attracted to these fixed points.

What next?
This continues until a limit point
$3 = r_0 < r_1 < r_2 < r_3 < ... < r_\infty = 3.569...$. Most points (in a sense that can be made mathematically precise) are attracted under iterations by $T_{r_\infty}$ to a fractal subset of $[0, 1]$ again by ever oscillating.

# Transitions between chaotic and orderly regimes

Next? For $r > r_\infty$ we see the appearance of chaotic behaviour (some may say we transitioned to chaos): namely, there are arbitrarily close points that exponentially separates for a finite amount of time.

But this is still not the end of the story: as $r$ keeps approaching 4 for some values above $r_\infty$ we see re-appearence of stable=non-chaotic=order behaviour!

Feigenbaum tree

It turns out that the same transition from chaos to order back and forth also appears for the double pendulum trajectories.

Back to double pendulum video

Stability regions

# Concluding remarks

We need to note that the above behaviours that I described reflect the content of mathematical theorems. They are not result of mere computer simulations. Numerical simulations cannot make these assertions since we are talking about behaviour at infinity of infinitely many points. Language of mathematics provides a way to formulate and rigorously demonstrate these statements.