Projections of self-affine fractals

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joint work with Ian D. Morris

What is in this talk?

1. Falconer's theorem and its extension to projections
2. Constructions 1: Strongly irreducible self-affine sets with exceptional projections
3. Constructions 2: Small sumsets with no arithmetic resonance
4. Constructions 3: Non-exact dimensional projections

Affine Iterated function systems

Theorem (Moran–Hutchinson)

Let X be a complete metric space and $(\varphi_1, \ldots, \varphi_N)$ and N-tuple of contracting maps $X \to X$ (called an Iterated Function System, IFS). Then there exists a unique compact K satisfying $K = \cup_{i=1}^{N} \varphi_i(K)$, called the attractor of the IFS $(\varphi_1, \ldots, \varphi_N)$. More constructively, K is the image of the coding map

$$\begin{aligned} \{1, \ldots, N\}^{\mathbb{N}} &\to X \\ (i_1, i_2, \ldots) &\mapsto \lim_{n \to \infty} \varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_n}(x). \end{aligned} \tag{1.1}$$

The limit exists and does not depend on $x \in X$.

Self-affine fractals

When $X = \mathbb{R}^d$ and the contractions φ_i's are similarities (i.e. scalings of Euclidean rigid motions, $x \mapsto \alpha O x + v$ for $\alpha \in \mathbb{R}^*$, $O \in O_d(\mathbb{R})$, and $v \in \mathbb{R}^d$), the attractor K is called a *self-similar set*.

More generally, when the contractions are affine maps $x \mapsto Ax + v$ ($A \in \mathrm{GL}_d(\mathbb{R})$ and $v \in \mathbb{R}^d$), K is called a *self-affine set*.

Already the class of self-similar sets contain many familiar fractal sets (The middle-third Cantor set, Sierpiński triangle, Peano curve, von Koch curve, Minkowski sausage, Menger sponge etc.).

Dimensions of self-affine fractals

How does a self-affine set look like? How large it is?

These questions have by-now a long history dating back to early '80s.

A foundational result is due to Falconer (1988). I would now like to describe this.

Let $N \in \mathbb{N}$ and $\mathrm{A} = (A_1, \ldots, A_N)$ be an N-tuple of matrices in $\mathrm{GL}_d(\mathbb{R})$. Given an N-tuple vectors $\mathrm{v} = (v_1, \ldots, v_N)$ in $\mathbb{R}^d$, let T_i denote the affine map $x \mapsto A_i x + v_i$.

We will denote by A^{v} the tuple $(T_1, \ldots, T_N)$. These are contractions if $\|A_i\| < 1$ for every $i = 1, \ldots, N$. In this case, denote by $K^{\mathrm{v}} \subset \mathbb{R}^d$ the attractor of A^v.

Hausdorff dimension

Let's start by recalling the Hausdorff dimension: for a subset $K \in \mathbb{R}^d$, $\dim_H(K)$, the Hausdorff dimension of K is defined as:

$$\inf\{s > 0 : \lim_{\delta \to 0} \inf_{(S_i):\ \delta-\text{cover}} \sum_{i=1}^{\infty} \text{diam}(S_i)^s = 0\}$$

To bound the dimension from above, one only needs a collection of δ-covers with $\delta \to 0$.
Bounding from the Hausdorff dimension from below is far more complicated.

Singular values

For $M \in \mathrm{GL}_d(\mathbb{R})$, denote by $\sigma_1(M) \geqslant \cdots \geqslant \sigma_d(M) > 0$ its singular values in decreasing order. These are the lengths of the semi-axes of the ellipsoid $M(S_1)$. (S_1 =unit sphere in $\mathbb{R}^d$).

Let $\mathsf{A}^{\mathsf{v}} = (T_1, \ldots, T_N)$ be a contracting affine IFS (recall $\mathsf{A} = (A_1, \ldots, A_N)$ and $\mathsf{v} = (v_1, \ldots, v_N)$ and $T_i x = A_i x + v_i$). Recall that $K^{\mathsf{v}} \subset \mathbb{R}^d$ denotes its attractor.

For a finite word $i = i_1 \ldots i_n$, write $|i| = n$ its length, and extend $i_1 \to A_{i_1}$ as semigroup morphism, i.e. $A_i = A_{i_1} \ldots A_{i_n}$.

An algorithm to upper-bound the Hausdorff dimension of self-affine sets

1. Check if $\dim_H(K^v) \leqslant 1$: is there $s \leqslant 1$ such that

$\lim_{n\to\infty} \frac{1}{n} \log \sum_{|i|=n} \sigma_1(A_i)^s \leqslant 0$? If yes, this $s \leqslant 1$ is an upper bound. Also the infimum of such $s \in (0, 1]$. If not, continue:

2. Check if $\dim_H(K^v) \leqslant 2$: is there $s \leqslant 2$ such that ...

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{|i|=n} \frac{\sigma_1(A_i)}{\sigma_2(A_i)} \sigma_2(A_i)^s = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|i|=n} \sigma_1(A_i)\sigma_2(A_i)^{s-1} \leqslant 0?$$

(there is a more efficient covering at this scale!)

3. etc.

Singular value potential, affinity dimension

Given a matrix $M \in \mathrm{GL}_d(\mathbb{R})$, set
$\varphi^s(M) := \sigma_1(M) \cdots \sigma_{\lfloor s \rfloor}(M) \sigma_{\lceil s \rceil}^{s - \lfloor s \rfloor}(M)$

This is called *the singular value potential* (Falconer, Douady—Oesterlé, Kaplan–Yorke).

The argument we have seen implies that
The zero $s_0(\mathrm{A})$ of the map $P_{\mathrm{A}}(s) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{|i|=n} \varphi^s(A_i)$
is an upper bound for the Hausdorff dimension of the attractor K^{v} of A^{v}.

Where does the translation parts $v = (v_1, \ldots, v_N)$ appear in the above construction of upper bound?

Nowhere! The upper bound $s_0(\mathrm{A})$ only depends on the linear parts A of the affine IFS A^{v}. It is called *the affinity dimension* of A.

Falconer's 1988 result

Theorem (Falconer)

Let $\mathsf{A} = (A_1, \ldots, A_N)$ *be* N*-tuple of contracting matrices in* $\mathrm{GL}_d(\mathbb{R})$.

1) For every $\mathsf{v} \in (\mathbb{R}^d)^N$, *we have* $\dim_H(K^{\mathsf{v}}) \leqslant s_0(\mathsf{A})$.

2) If, moreover $\|A_i\| < \frac{1}{2}$ *for every* $i = 1, \ldots, N$,
then for Lebesgue almost every $\mathsf{v} \in (\mathbb{R}^d)^N$, $\dim_H(K^{\mathsf{v}}) = s_0(\mathsf{A})$.

The constant $1/2$ here is due to Solomyak.

In other words, the covering of K^{v} that we discussed is the optimal one (at least) for Lebesgue almost every $\mathsf{v} \in (\mathbb{R}^d)^N$!

This foundational result has been guiding over three decades of research: how to get rid of almost every? (this is *exact overlaps conjecture* in self-similar context)

Projections

Let A^v be a contracting affine IFS and K^v its attractor. And let $P \in \mathrm{Gr}_k(\mathbb{R}^d)$ be a k-plane in $\mathbb{R}^d$ and $Q_P \in \mathrm{End}(\mathbb{R}^d)$ be the orthogonal projection onto P.

What can be said of the size $Q_P(K^\mathsf{v})$? (e.g. Hausdorff dimension.)

Theorem (Marstand 1954, Mattila 1975)

Let B be a Borel set in $\mathbb{R}^d$. Then for every $k = 1, \ldots, d$, for Lebesgue almost every k-plane in $\mathrm{Gr}_k(\mathbb{R}^d)$,

$$\dim_H(Q_P(B)) = \min\{k, \dim_H B\}.$$

Almost every is with respect to $k(d-k)$-dimensional Lebesgue measure.

The inequality $\leqslant$ is trivial. Those planes for which it is strict $<$ are called *exceptional projections* for B. They constitute a measure zero set.

Our result

Theorem

Let $\mathsf{A} = (A_1, \ldots, A_N)$ *be an N-tuple of contracting matrices in* $\mathrm{GL}_d(\mathbb{R})$*. Then, for every* $k = 0, \ldots, d$ *there exist an integer* $m \geqslant 1$*, a finite filtration* $\emptyset = \mathcal{W}_{m+1} \subset \mathcal{W}_m \subset \cdots \subset \mathcal{W}_0 = \mathrm{Gr}(k, d)$ *of algebraic varieties each invariant under the linear algebraic group generated by* A*, and real numbers* $s_m < \cdots < s_0 \leqslant k$ *with the following properties:*

1)For every $\mathsf{v} \in (\mathbb{R}^d)^N$*, for every* $P \in \mathcal{W}_j \setminus \mathcal{W}_{j+1}$*, we have* $\dim_H Q_P K^{\mathsf{v}} \leqslant s_j$*.*

2) If, moreover $\|A_i\| < \frac{1}{2}$ *for every* $i = 1, \ldots, N$*, then for every* $P \in \mathcal{W}_j \setminus \mathcal{W}_{j+1}$*, for Lebesgue almost every* $\mathsf{v} \in (\mathbb{R}^d)^N$*,* $\dim_H Q_P K^{\mathsf{v}} = s_j$*.*

The case $k = d$ is precisely Falconer's 1988 theorem.

An immediate consequence

The G-invariant varieties $\mathcal{W}_j$ are explicit. They are given by level sets of some pressure function. They are Schubert-type varieties. This allows us to deduce for example

Corollary

Suppose the semi-group $\langle \mathrm{A} \rangle < \mathrm{GL}_d(\mathbb{R})$ acts irreducibly on all $\bigwedge^k \mathbb{R}^d$ (irreducible: no proper non-trivial invariant subspace). Then, the filtration is trivial $m = 0$ and therefore for any ℓ-plane P, $\dim_H(K^{\mathrm{v}}) = \min\{s_0, \ell\}$ for Lebesgue almost every $\mathrm{v} \in (\mathbb{R}^d)^N$.

Corollary

Suppose $\langle A \rangle$ lies in $\mathbb{R}^* O_d(\mathbb{R})$ (the setting of similarities). The same conclusion holds.

Even these two consequences seem to be new.

A stratified Marstand–Falconer type consequence

Corollary (Stratified Marstand-type projection theorem for self-affine sets)

Let $\mathsf{A} = (A_1, \ldots, A_N)$ *be an* N*-tuple of matrices in* $\mathrm{GL}_d(\mathbb{R})$ *such that* $\|A_i\| < \frac{1}{2}$ *for every* $i = 1, \ldots, N$*. For every* $k = 0, \ldots, d$ *let* $\emptyset = \mathcal{W}_{m+1} \subset \mathcal{W}_m \subset \cdots \subset \mathcal{W}_0 = \mathrm{Gr}(k, d)$ *and* $s_m < \cdots < s_0 \leqslant k$ *be as in the previous theorem. Then, Lebesgue a.e.* $\mathsf{v} \in (\mathbb{R}^d)^N$ *we have* $\dim_H Q_P K^{\mathsf{v}} = s_j$ *for Lebesgue a.e.* $P \in \mathcal{W}_j$*. .*

This follows from the previous result by Fubini and the fact that $\dim W_j > \dim W_{j+1}$.

Strongly irreducible self-affine sets with exceptional projections

It is expected that a stronger irreducibility assumption on all exterior powers $\bigwedge^k \mathbb{R}^d$ and a separation assumption imply that the there are no exceptional projections for any $\mathrm{v} \in (\mathbb{R}^d)^N$.

It was not clear whether requiring this only for $\mathbb{R}^d$ action is sufficient for such a conclusion (in small dimensions $d \leqslant 3$, $\langle \mathrm{A} \rangle$ acts irreducibly on $\mathbb{R}^d$ implies it acts irreducibly on $\bigwedge^k \mathbb{R}^d$).

We construct several classes of examples of A that are strongly irreducible in $\mathbb{R}^d$ but the filtration $(\mathcal{W}_j)$ is non-trivial. Therefore for such A, for almost every v, the attractor K^{v} has a positive dimensional subvariety of planes consisting of exceptional projections.

Theorem (Strongly irreducible self-affine sets with exceptional projections)

If $d = 2k$ *and* $H = \mathbb{R}^* \mathrm{SO}(k, k)$, *and if* A *is* k*-dominated (which contains a non-empty open subset of* H^N*), then there exists a* $k(k-1)/2$*-dimensional subvariety* $\mathcal{V}$ *of the Grassmannian* $\mathrm{Gr}(k, 2k)$ *such that for almost every* $\mathsf{v} \in (\mathbb{R}^d)^N$, $\mathcal{V}$ *is contained in the set of exceptional projections of* K^{v}*.*

Theorem (Some more strongly irreducible self-affine sets with exceptional projections)

If H *is given by the tensor product representation of* $\mathrm{GL}_{d_1}(\mathbb{R}) \times \mathrm{GL}_{d_2}(\mathbb{R})$ *on* $\mathbb{R}^{d_1 d_2}$ *and* A *is* 2*-dominated (again, non-empty open in* H^N*), then there exists either a* $(d_1 + 2d_2 - 5)$ *or* $(2d_1 + d_2 - 5)$*-dimensional subvariety* $\mathcal{V}$ *of the Grassmannian* $\mathrm{Gr}(d_1 d_2 - 2, d_1 d_2)$ *such that for almost every* $\mathsf{v} \in (\mathbb{R}^d)^N$, $\mathcal{V}$ *is contained in the set of exceptional projections of* K^{v}*.*

Furstenberg predicted the following statement, now a result of Hochman–Shmerkin (after Peres–Shmerkin):

Theorem (Hochman–Shmerkin)

Let $p, q \in \mathbb{N}$ and X, Y be two non-empty closed subsets of $[0,1]$ invariant under $\times p$ and $\times \mathfrak{q}$ (mod 1), respectively. Then,

$$\dim_H(X+Y) < \min\{1, \dim_H X + \dim_H Y\} \implies \frac{\log p}{\log q} \in \mathbb{Q}.$$

Arithmetic independence $\implies$ geometric independence.
Equivalently, geometric resonance $\implies$ arithmetic resonance.

A result of Pyörälä

More recently, building on work of Hochman, Pyörälä has proven a result in similar spirit for self-affine sets in dimension 2:

He showed the following **theorem**: Let A and B be two contracting N-tuples in $\mathrm{GL}_2(\mathbb{R})$, $\mathrm{v}, \mathrm{w} \in (\mathbb{R}^d)^N$ and denote by X and Y the corresponding self-affine sets. Then, under some assumptions on A and B and translations parts v, w (irreducibility, domination, separation), if

$$\dim_H(X + Y) < \min\{2, \dim_H X + \dim_H Y\}$$

then the log-eigenvalues of matrices in the tuples A and B belong to an arithmetic set/lattice $\alpha\mathbb{Z}$.

This result indicates that the arithmetic-geometric phenomenon predicted by Furstenberg on the circle is still valid in a non-commutative setting in dimension 2.

Small sumsets with no arithmetic resonance

Theorem

Let $G < \mathrm{GL}_{d_1 d_2}(\mathbb{R})$ be the image of $\mathrm{GL}_{d_1}(\mathbb{R}) \times \mathrm{GL}_{d_2}(\mathbb{R})$ via the tensor product representation. Then, for $N, M \geqslant 1$ large enough, there exists open sets of tuples $(\mathrm{A}, \mathrm{B}) \in G^N \times G^M$ each generating a Zariski-dense semigroup in G and with the following property. For Lebesgue almost every $\mathrm{v} \in (\mathbb{R}^d)^N$ and $\mathrm{w} \in (\mathbb{R}^d)^M$, the associated fractals X^{v} and Y^{w} satisfy

$$\dim_H(X^{\mathrm{v}} + Y^{\mathrm{w}}) < \dim_H(X^{\mathrm{v}}) + \dim_H(Y^{w}) < d_1 d_2.$$

Note that the tuples can come from an open set in G therefore, their log-eigenvalues do not live in a finite-rank $\mathbb{Z}$-module.

This construction shows that in higher dimensions ($\geqslant 4$), Furstenberg's phenomenon (arithmetic vs geometric resonance) should take into account further aspects (it does not hold with a naive generalization).

Exact dimensionality of measures

Let μ be a Borel measure on a complete metric space. The local dimension of μ at $x \in X$ is

$$\dim_{\mathrm{loc}}(\mu, x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}$$

if it exists (otherwise consider upper loc dim and lower loc dim).
If this limit exists μ-a.e., μ is called *exact-dimensional*.

Popularized by Young in early 80's and soon recognized to play important role in dimension theory of dynamical systems, fractal geometry etc. (e.g. Ledrappier–Young formula).

Eckmann–Ruelle conjecture and self-affine version

Recall that for a C^1-diffeomorphism of a compact manifold, a long line of works by Young, Ledrappier, and finally Barreira–Pesin–Schmeling (99) showed that any hyperbolic ergodic invariant measure is exact-dimensional [Eckmann–Ruelle conjecture].

In fractal geometry, the analogous result was proven by Feng (2019) following Hutchinson, Mcmullen, Gatzouras–Lalley, Kenyon–Peres, Feng–Hu:

Theorem (Feng)

Given a contracting affine IFS A^{v}*, the image* $c_*\mu$ *of any ergodic shift-invariant measure* μ *on* $\{1,\ldots,N\}^{\mathbb{N}}$ *by the coding map*

$$\begin{aligned} c : \{1,\ldots,N\}^{\mathbb{N}} &\to X \\ (i_1, i_2, \ldots) &\mapsto \lim_{n\to\infty} \varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_n}(x). \end{aligned} \tag{4.1}$$

is exact-dimensional.

Non-exact dimensional projections

This result is very general: it is valid for any contracting affine IFS A^{v} and any ergodic-shift invariant measure μ on $\{1, \ldots, N\}^{\mathbb{N}}$.

How about exact-dimensionality of the image of $c_*\mu$ by an orthogonal projection Q?

Whereas it can be shown that if μ is a Bernoulli measure, for any Q, $Q_* c_* \mu$ is exact-dimensional, we show that this is not true in general, and construct the following examples of non-exact-dimensional projections.

Last theorem

Theorem

For every $d \geqslant 2$ there exists an irreducible affine iterated function system A^{v} *on* $\mathbb{R}^d$ *which admits a unique and ergodic invariant measure* μ *such that the dimension of* $c_*\mu$ *equals that of the attractor, and such that there exist projections* Q *with the property that* $Q_*c_*\mu$ *is not exact-dimensional.*

In every even dimension $d := 2k \geqslant 4$ one may construct examples in which the set of rank-k orthogonal projections such that $Q_*c_*\mu$ is not exact-dimensional includes an algebraic variety of dimension $\frac{1}{2}k(k-1)$, A is additionally *strongly irreducible*.

Thank you

Thanks for your attention!