The Absolute Galois Group of $\mathbb{C}(t)$

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This paper will be divided into three parts: in the first part, we will review some equivalences of categories that we have previously seen in the seminar; in the second part, we will prove a key theorem which gives us an equality of a certain Galois group and (after taking profinite completion) a certain fundamental group; in the third part, we will use the key theorem to give two applications: we first solve the inverse Galois problem for $\mathbb{C}(t)$ and then given an explicit description of the absolute Galois Group of $\mathbb{C}(t)$. The exposition given here is a detailed version of the material appearing in Section 4 of Chapter 3 of [1].

1 Review

In this section, we recall some key equivalences of categories from [1] (slightly modified for our purposes) that will be used repeatedly through the paper. We adopt the following convention: all maps between Riemann surfaces are assumed to be non-constant.

Throughout this paper, we let X denote a compact connected Riemann surface with field of meromorphic functions $\mathcal{M}(X)$. Let $S \subseteq X$ be a discrete set and denote by $\mathrm{Hol}_{X/S}$ the category of compact Riemann surfaces Y equipped with a holomorphic map $Y \to X$ whose branch points all lie over S. A morphism in this category is a holomorphic map compatible with projections onto X.

Theorem 1.1. *We have an equivalence of categories*

$$Hol_{X/S} \xleftrightarrow{1:1} \{\textit{finite topological covers of } X \setminus S\}$$

given by mapping $\phi : Y \to X$ to $Y \setminus \phi^{-1}(S) \to X \setminus S$.

Proof. See [1, Theorem 3.2.7]. $\square$

If Y is a compact Riemann surface, a holomorphic map $\phi : Y \to X$ induces a map $\phi^* : \mathcal{M}(X) \to \mathcal{M}(Y)$ given by $f \mapsto f \circ \phi$ and turns $\mathcal{M}(Y)$ into a finite etale algebra over $\mathcal{M}(X)$. Let $\mathcal{A}$ denote the category of compact Riemann surfaces Y mapping holomorphically onto X.

Theorem 1.2. *The functor given by*

$$Y \mapsto \mathcal{M}(Y).$$

is an anti-equivalence of categories

$$\mathcal{A} \overset{1:1}{\longleftrightarrow} \{\textit{finite etale algebras of } \mathcal{M}(X)\}$$

which restricts to a degree preserving anti-equivalence

$$\{\textit{finite Galois branched covers of } X\} \overset{1:1}{\longleftrightarrow} \{\textit{finite Galois extnensions of } \mathcal{M}(X)\}$$

Proof. See [1, Theorem 3.3.7]. □

2 The Key Theorem

Definition 2.1. Let X' be the complement of a finite set of points of X. Let $K_{X'}$ be the composite in a fixed algebraic closure $\overline{\mathcal{M}(X)}$ of $\mathcal{M}(X)$ of all finite subextensions arising from holomorphic maps of connected compact Riemann surfaces $Y \to X$ that restrict to a cover over X'.

For any such finite subextension L appearing in Definition 2.1, any Galois conjugate of L will come from an automorphism in $\mathrm{Aut}(Y/X)$; in particular, the Galois conjuagte also satisfies Definition 2.1. Thus, $K_{X'}$ is Galois over $\mathcal{M}(X)$. Our goal is to describe the Galois group of $K_{X'}/\mathcal{M}(X)$.

Lemma 2.2. *Given two extensions L_1 and L_2 of $\mathcal{M}(X)$ that come from compact connected Riemann surfaces $Y_i \to X$ that restrict to covers $Y_i' \to X'$ for $i \in \{1, 2\}$, their compositum $L_1 \cdot L_2$ in $\overline{\mathcal{M}(X)}$ comes from a connected compact Riemann surface that restricts to a cover over X'.*

Proof. Consider the fiber product $Y_1' \times_{X'} Y_2'$ and recall that $Y_1' \times_{X'} Y_2' \to X'$ is a cover. We get a compact Riemann surface $Y_{12} \to X$ restricting to $Y_1' \times_{X'} Y_2' \to X'$. Recall that

$$\begin{aligned}\{\text{finite covers over } X'\} &\overset{1:1}{\leftrightarrow} \{\text{compact Riemann Surfaces } Y \to X \text{ which are covers over } X'\}\\ &\subseteq \{\text{compact Riemann surfaces } Y \to X\}\\ &\overset{1:1}{\leftrightarrow} \{\text{finite etale algebras of } \mathcal{M}(X)\}.\end{aligned}$$

Since a functor that induces an anti-equivalence of categories maps products to coproducts, we have that $\mathcal{M}(Y_{12}) = L_1 \otimes_{\mathcal{M}(X)} L_2$.
Write $L_1 \otimes_{\mathcal{M}(X)} L_2 = K_1 \times K_2 \times \cdots \times K_N$, where each K_i is a finite separable extension of $\mathcal{M}(X)$. Consider the surjective map $\varphi : L_1 \otimes_{\mathcal{M}(X)} L_2 \twoheadrightarrow L_1 \cdot L_2$ given by $a \otimes b \mapsto a \cdot b$. Then $\ker \varphi$ is maximal and moreover $(L_1 \otimes_{\mathcal{M}(X)} L_2)/\ker \varphi \cong L_1 \cdot L_2$. Since maximal ideals of $L_1 \otimes_{\mathcal{M}(X)} L_2$ are of the form $K_1 \times \cdots \times \{0\} \times \cdots \times K_N$, we conclude that there exists $i \in \{1, \ldots, N\}$ such that $K_i = L_1 \cdot L_2$. Now, since Y_{12} is compact, let $Y_{12} = Y_1' \sqcup \cdots \sqcup Y_M'$ denote a finite decomposition of Y_{12} into its connected components. Then $\mathcal{M}(Y_{12}) = \mathcal{M}(Y_1') \times \cdots \times \mathcal{M}(Y_M')$. Using a standard fact from algebra that the decomposition of a finite etale algebra

into a finite product of finite separable field extensions is unique up to permutation, we conclude that $N = M$. Moreover, we may assume that $\mathcal{M}(Y_j') = K_j$ for all $j \in \{1, \ldots, N\}$. In particular, $\mathcal{M}(Y_i) = K_i = L_1 \cdot L_2$ as desired. $\square$

Lemma 2.3. *Every finite subextension F of $K_{X'}$ comes from a connected compact Riemann surface that restricts to a cover over X'.*

Proof. We can write $F = \mathcal{M}(Y)$, corresponding to a compact connected Riemann surface $Y \to X$. It remains to show that Y restricts to a cover over X'. By the previous lemma, $K_{X'} = \bigcup L$, where the union runs though all finite extensions of $\mathcal{M}(X)$ that come from connected compact Riemann surfaces that restrict to a cover over X'. By the primitive element theorem, $F = \mathcal{M}(X)(\alpha)$ for some $\alpha \in F \subseteq K_{X'}$. Hence, there exists a finite extension of $\mathcal{M}(X)$, say L', which comes from connected compact Riemann surface $Z \to X$ which restricts to a cover over X'.
Thus, each point of X' has $[L' : \mathcal{M}(X)]$ preimages in Z and atmost $[F : \mathcal{M}(X)]$ preimages in Y. On the other hand, each point of Y has atmost $[L' : F]$ preimages in Z. This forces equality everywhere: each point of X has exactly $[F : \mathcal{M}(X)]$ preimages in Y and each point of Y has exactly $[L' : F]$ preimages in Z. In particular, Y restricts to a cover over X'. $\square$

Theorem 2.4. *The Galois group of the field extension $K_{X'}|\mathcal{M}(X)$ is isomorphic the profinite completion of $\pi_1(X', x)$, where $x \in X'$ is a basepoint. i.e.*

$$Gal(K_{X'}/\mathcal{M}(X)) = \pi_1\widehat{(X', x)}.$$

Proof. We have that

$$\begin{aligned}
\{\text{finite quotients of } \pi_1(X', x)\} &\overset{1:1}{\leftrightarrow} \{\text{finite Galois covers of } X'\} \\
&\overset{1:1}{\leftrightarrow} \{\text{finite Galois branched covers of } X \text{ w.r.t } X'\} \\
&\overset{1:1}{\leftrightarrow} \{\text{finite Galois extensions of } \mathcal{M}(X) \text{ contained in } K_{X'}\},
\end{aligned}$$

where surjectivity in the last equivalence follows from Lemma 2.3. The bijections are also compatible and so

$$\begin{aligned}
\mathrm{Gal}(K_{X'}/\mathcal{M}(X)) &\cong \varprojlim_{L \subseteq K_{X'},\ \text{Gal.}} \mathrm{Gal}(L/\mathcal{M}(X)) \cong \varprojlim_{\text{Gal.}} \mathrm{Aut}(Y/X) \\
&\cong \varprojlim_{\text{Gal.}} \mathrm{Aut}(Y'/X') \cong \varprojlim_{N \trianglelefteq \pi_1(X', x)\ \text{fin.}} \pi_1(X', x)/N \\
&\cong \pi_1\widehat{(X', x)}.
\end{aligned}$$

$\square$

3 Applications

As a first application, we will solve a problem that falls under the umbrella of problems called 'Inverse Galois Problems'. The fundamental question is the following: Let K be a field. What finite groups occur as the Galois groups of finite field extensions over K?

Example 3.1.
- Let $K = \mathbb{C}$. Then only the trivial group as the Galois groups of finite field extensions over K.
- Let $K = \mathbb{F}_p$. Then the groups which occur as the Galois groups of finite field extensions over K are precisely $\mathbb{Z}/N\mathbb{Z}$ for all $N \in \mathbb{N}_{\geq 1}$.

It is much harder to answer this question for fields coming from geometry or number theory. For example, it is an open problem whether every finite group occurs as the Galois group of a field extension of $\mathbb{Q}$. In this report, we will solve the problem when $K = \mathbb{C}(t) = \mathcal{M}(\mathbb{P}^1(\mathbb{C}))$.

Theorem 3.2. *Every finite group occurs as the Galois group of some finite Galois extension $L|\mathbb{C}(t)$.*

Proof. Let G be a finite group and let $N := \#G$. Note that we have a surjection $F_N \twoheadrightarrow G$, where F_N is the free group generated by N elements. Now for any $m \in \mathbb{N}$, we get from algebraic topology that $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{x_1, \ldots, x_m\}, x) = \langle \gamma_1, \ldots, \gamma_m | \gamma_1 \cdots \gamma_m = 1 \rangle$, where each γ_i can be represented by a loop at x passing around x_i. By sending γ_i to a free generator f_i of F_{m-1} for $i < m$ and sending γ_m to $(f_1 \cdots f_{m-1})^{-1}$, we see that $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{x_1, \ldots, x_m\}, x)$ is isomorphic to F_{m-1}.

Thus, we have a surjection $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{x_1, \ldots, x_{N+1}\}, x) \twoheadrightarrow G$ with kernel $:= H$ and so

$$G \cong \frac{\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{x_1, \ldots, x_{N+1}\}, x)}{H}.$$

Let $X' := \mathbb{P}^1(\mathbb{C}) \setminus \{x_1, \ldots, x_{N+1}\}$ and so by the Key Theorem,

$$\mathrm{Gal}(K_{X'}/\mathcal{M}(X)) = \mathrm{Gal}(K_{X'}/\mathbb{C}(t)) = \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \widehat{\{x_1, \ldots}, x_{N+1}\}, x) := R.$$

Note that if H' denotes the kernel of the projection map

$$R \twoheadrightarrow \frac{\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{x_1, \ldots, x_{N+1}\}, x)}{H} \cong G$$

then

$$\mathrm{Gal}(K_{X'}/\mathbb{C}(t))/H' = R/H' \cong G.$$

Set $L := K_{X'}^H$, then by Galois theory,

$$\mathrm{Gal}(L/\mathbb{C}(t)) \cong \mathrm{Gal}(K_{X'}/\mathbb{C}(t))/H' \cong G.$$

$\square$

Our second application is about describing the absolute Galois group of a field K. For example:

Example 3.3.
- Let $K = \mathbb{C}$. Then $\mathrm{Gal}(\overline{\mathbb{C}}/\mathbb{C})$ is trivial.
- Let $K = \mathbb{F}_p$. Then

$$\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) = \varprojlim_{N \in \mathbb{N}_{\geq 1}} \mathrm{Gal}(\mathbb{F}_{p^N}/\mathbb{F}_p) = \varprojlim_{N \in \mathbb{N}_{\geq 1}} \mathbb{Z}/N\mathbb{Z} = \widehat{\mathbb{Z}}.$$

Again, it is much harder to answer this question for fields coming from geometry or number theory. For example, it extremely hard to describe $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$: it is an uncountable group, but apart from the identity map and complex conjugation, we cannot write down any other elements of this group.

We now proceed to solve this problem, when $K = \mathbb{C}(t)$. To do this, we first introduce the notion of a free profinite group:

Definition 3.4 (Free profinite group)**.** Let X be a set and let $F(X)$ be the free group with basis X. The free profinite group $\widehat{F}(X)$ with basis X is defined as the inverse limit formed by the natural system of quotients $F(X)/U$, where $U \subseteq F(X)$ is a normal subgroup of finite index containing all but finitely many elements of X.

Example 3.5. If X is finite, then $\widehat{F}(X)$ is the profinite completion of $F(X)$.

We now arrive at an important result concerning free profinite groups. Since the proof of this proposition is rather long and also a bit unrelated to the theme of this paper, we will state this result without proof.

Theorem 3.6. *Let X be a set and $\mathcal{S}$ the system of finite subsets $S \subseteq X$ partially ordered by inclusion. Let (G_S, λ_{ST}) be an inverse system of profinite groups indexed by $\mathcal{S}$ satisfying the following conditions:*

1. *The maps λ_{ST} are surjective for all $S \subseteq T$.*
2. *Each G_S has a system $\{g_x : x \in S\}$ of elements so that the map $\widehat{F}(S) \to G_S$ induced by the inclusion $S \to G_S$ is an isomorphism and moreover for every $S \subseteq T$, we have $\lambda_{ST}(g_x) = 1$ for $x \notin S$.*

Then

$$\varprojlim G_S \cong \widehat{F}(X)$$

.

Proof. See [1, Proposition 3.4.9]. □

We can now describe the absolute Galois group of $\mathbb{C}(t)$.

Theorem 3.7. *There is an isomorphism of profinite groups*

$$Gal(\overline{\mathbb{C}(t)}/\mathbb{C}(t)) \equiv \widehat{F}(\mathbb{C})$$

of the absolute Galois group of $\mathbb{C}(t)$ *with the free profinite group on the set of complex numbers.*

Proof. Let $S \subseteq \mathbb{C}$ be a finite set of m points. Let $X_S := \mathbb{P}^1(\mathbb{C}) \setminus \{S \cup \{\infty\}\}$. By the key theorem, $\mathrm{Gal}(K_{X_S}/\mathbb{C}(t)) = \pi_1(X_S, x_0) = \widehat{F_m}$. Let T is a finite subset of $\mathbb{C}$ of size n with $S \subseteq T$. Let $X_T := \mathbb{P}^1(\mathbb{C}) \setminus \{T \cup \{\infty\}\}$; then we have an inclusion $K_{X_S} \subseteq K_{X_T}$ and hence by Galois theory, a surjection $\lambda_{ST} : \mathrm{Gal}(K_{X_T}/\mathbb{C}(t)) \twoheadrightarrow \mathrm{Gal}(K_{X_S}/\mathbb{C}(t))$. The groups $\mathrm{Gal}(K_{X_S}/\mathbb{C}(t))$ together with the maps λ_{ST} form an inverse system indexed by the system of finite subsets of $\mathbb{C}$ partially ordered by inclusion.

The inclusion $X_T \hookrightarrow X_S$ induces a map $\pi_1(X_T, x_0) \to \pi_1(X_S, x_0)$ on fundamental groups, where x_0 is an arbitrarily chosen basepoint. This gives us an induced map $\pi_1\widehat{(X_T, x_0)} \to \pi_1\widehat{(X_S, x_0)}$ on the profinite completions; in particular, note that γ_x gets mapped to the identity for all $x \in T \setminus S$. By identifying $\pi_1\widehat{(X_T, x_0)}$ with $\mathrm{Gal}(K_{X_T}/\mathbb{C}(t))$ and $\pi_1\widehat{(X_S, x_0)}$ with $\mathrm{Gal}(K_{X_S}/\mathbb{C}(t))$ and tracing through these isomorphisms, one can check that the map on profinite completions is precisely λ_{ST}. Now note that all conditions in Theorem 3.6 are satisfied; applying this theorem yields an isomorphism

$$\varprojlim \mathrm{Gal}(K_{X_S}/\mathbb{C}(t)) \cong \widehat{F}(\mathbb{C}).$$

However, recall that

$$\mathrm{Gal}(\overline{\mathbb{C}(t)}/\mathbb{C}(t)) \cong \varprojlim \mathrm{Gal}(L/\mathbb{C}(t)),$$

where the limit runs through all finite extensions of $\mathbb{C}(t)$. Note that every finite subextension of $\mathbb{C}(t)$ is contained in K_{X_S} for a sufficinetly large S and so

$$\varprojlim \mathrm{Gal}(K_{X_S}/\mathbb{C}(t)) \cong \varprojlim \mathrm{Gal}(L/\mathbb{C}(t)).$$

In conclusion,

$$\mathrm{Gal}(\overline{\mathbb{C}(t)}/\mathbb{C}(t)) \cong \varprojlim \mathrm{Gal}(L/\mathbb{C}(t)) \cong \varprojlim \mathrm{Gal}(K_{X_S}/\mathbb{C}(t)) \cong \widehat{F}(\mathbb{C}).$$

as desired. □

We end this paper by remarking that for any compact connected Riemann surface X, the same reasoning as in the previous proof shows that

$$\mathrm{Gal}(\overline{\mathcal{M}(X))}/\mathcal{M}(X))) \cong \varprojlim_{S\subseteq X \text{ fin.}} \mathrm{Gal}(K_{X_S}/\mathbb{C}(t)) \cong \varprojlim_{S\subseteq X \text{ fin.}} \pi_1\widehat{(X_S, x)}.$$

However, in this general setting, is much more difficult to calculate $\mathcal{M}(X)$ and the various $\pi_1\widehat{(X_S, x)}$'s; hence, obtaining an explicit result in the spirit of Theorem 3.7 is harder.

References

[1] Tamas Szamuely. *Galois Groups and Fundamental Groups*. Cambridge University Press, Cambridge, United Kingdom, 2015.