Elliptic Units

§1. Basic Def^{n}s

K - imag quad field, class # 1.

E/K - ell curve with CM by $\mathcal{O}_K$.

elliptic units = units in abelian ext^{n}s of K that are "generated" by a certain rational fn on E.

Def^n: $\mathfrak{a} \subseteq \mathcal{O}_K$ integral ideal coprime to 6
$= (\gamma)$ for some $\gamma \in \mathcal{O}_K$.

$$\Theta_{E,\mathfrak{a}}(Q) = \gamma^{-12} \underbrace{\Delta(E)}_{\text{discriminant of } E}{}^{N\mathfrak{a}-1} \cdot \prod (x(Q) - x(P))^{-6}.$$

$P \in E[\mathfrak{a}] \setminus 0$

$Q \in E(\mathbb{C})$.

Properties:

1) $\Theta_{E,\mathfrak{a}}$ is a rational fn on E defined over K.

i.e: $\Theta_{E,\mathfrak{a}} \in K(E)$.

Prop: $\mathfrak{m} \subseteq \mathcal{O}_K$ ideal coprime to $\mathfrak{a}$.

$Q \in E[\mathfrak{m}]$ point of exact order $\mathfrak{m}$.

(a) Suppose $\mathfrak{f}$ is not a prime power.
Then $\Theta_{E,\mathfrak{a}}(\mathfrak{f})$ is a unit in
$K(\mathfrak{f})$. ray class field of K modulo $\mathfrak{f}$.

(b) Suppose $\mathfrak{f} = \mathfrak{p}^k$. Then

$\Theta_{E,\mathfrak{a}}(\mathfrak{f})$ belongs to $K(\mathfrak{f})$,
and it is a local unit at all primes away from $\mathfrak{f}$.

To define elliptic units, it's not enough to use $\Theta_{E,\mathfrak{a}}$. We need to define an additional rat'l fn $\Lambda_{E,\mathfrak{a}}$.

<u>Defn</u>: E/K - ell curve with CM by $\mathcal{O}_K$.

$$\psi_E : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$$

Grossenchar of E/K, conductor $\mathfrak{f} \subseteq \mathcal{O}_K$.

$$\Lambda_{E,\mathfrak{a}} := \prod_{\sigma \in \mathrm{Gal}\left(\frac{K(\mathfrak{f})}{K}\right)} \Theta_{E,\mathfrak{a}} \circ \tau_{S^\sigma},$$

where $S \in E[\mathfrak{f}]$ is an $\mathcal{O}_K$-

generator and $\mathfrak{f}' \mid \mathcal{O}_K \cdot S$

$$\tau_{S^\sigma}(P) = P + S^\sigma.$$

<u>Prop:</u> $\mathfrak{f} \subseteq \mathcal{O}_K$ ideal coprime to $\mathfrak{F}_{\mathfrak{a}}$, $Q \in E[\mathfrak{f}]$. Then

$\Lambda_{E,\mathfrak{a}}(Q)$ is a (global) unit in $K(E[\mathfrak{f}])$.

$$\begin{array}{ccc} L & = \text{lattice associated to } E = & \Omega\,\mathcal{O}_K. \\ \cap\!\!\shortmid & & \cap\!\!\shortmid \\ \mathbb{C} & & \mathbb{C}^\times \end{array}$$

Def^n: $\mathfrak{r} \subseteq \mathcal{O}_K$ ideal coprime to $\mathfrak{f}\mathfrak{a}$.

n^{th} elliptic unit w.r.t $\mathfrak{r}$

$$\eta_n(\mathfrak{r}) = \underline{\underline{\Theta}}_{E,\mathfrak{a}}\left(\frac{\Omega}{\underbrace{\Psi_E(\mathfrak{p}^n\mathfrak{r})}}\right)$$

Fact: $\Psi_E(\mathfrak{p}^n\mathfrak{r})$ generates $\mathfrak{p}^n\mathfrak{r}$.

$\Downarrow$

Then $\Omega / \Psi_E(\mathfrak{p}^n\mathfrak{r})$ has exact order $\mathfrak{p}^n\mathfrak{r}$ in $\mathbb{C}/L$.

$$\mathbb{C}/L \cong E(\mathbb{C})$$

Prop: $\mathfrak{r}$ coprime $6\mathfrak{f}\mathfrak{a}\mathfrak{p}$, squarefree

$$K_n := K(E[p^n])$$

$$K_n(\tau) := K(E[p^n \tau])$$

(a) $\eta_n(\tau)$ is a unit in $K_n(\tau)$.

<u>Norm relations:</u>

(a) If q is a prime ideal of K,

$$\mathrm{Norm}_{K_n(\tau q)/K_n(\tau)}(\eta_n(q\tau)) =$$

$$\eta_n(\tau)^{1 - \mathrm{Frob}_q^{-1}}.$$

(b) $\mathrm{Norm}_{K_{n+1}(\tau)/K_n(\tau)}(\eta_{n+1}(\tau))$

$$= \eta_n(\tau).$$

§2. Applications to IMC

Goal: given an elliptic unit $\eta_n(\tau)$, we want to produce a principal ideal in $K_n(\tau)^\times$.

$$K_n(\tau) = K(E[p^n\tau]).$$

this will yield an upper bound for the class gp.

Strategy: Given $\eta_n(\tau)$, we construct a map

$$\alpha : \{\text{ideals of } \mathcal{O}_K\} \longrightarrow \overline{K}^\times$$

s.t. for all ideals $\mathfrak{a} \subseteq \mathcal{O}_K$, all primes $\ell \mid \mathfrak{a}$:

(i) $\alpha(\mathfrak{a}) \in K(\mathfrak{a})^\times$ ← ray class field of K mod $\mathfrak{a}$.

(ii) $\alpha(\mathfrak{a})$ is a global unit if $\mathfrak{a} \neq 1$,

(iii) $N_{K(\mathfrak{a})/K(\mathfrak{a}/\ell)}(\alpha(\mathfrak{a})) = \alpha\left(\frac{\mathfrak{a}}{\ell}\right)^{\mathrm{Frob}_\ell - 1}$,

(iv) $\alpha(\mathfrak{a}) \equiv \alpha(\mathfrak{a}/\ell)^{(N\ell-1)/\ell} \mod \ell$.

We associate to α a map

$$K_\alpha : \left\{ \begin{array}{c} \text{ideals of} \\ \mathcal{O}_K \end{array} \right\} \longrightarrow K_n^\times / K_n^{\times p^n}$$

s.t.

$$K_\alpha(\mathfrak{a}) \equiv \alpha(\mathfrak{a})^{D_{\mathfrak{a}}} \mod K_n^{\times p^n},$$

where

$$D_{\mathfrak{a}} = \prod_{\ell \mid \mathfrak{a}} D_\ell \, , \quad \text{and}$$

for every prime ℓ of $\mathcal{O}_K$

put $\quad G_\ell := \mathrm{Gal}(K(\ell)/K).$

$$\underset{\langle \sigma_\ell \rangle .}{\overset{\text{cyclic of order } M}{\|}}$$

$$\underline{D_\ell} := \sum_{i=0}^{M-1} i\sigma_\ell^{\,i} \in \mathbb{Z}[G_\ell].$$

<u>Upshot</u>: each $\kappa_\alpha(\mathfrak{a})$ gives a principal ideal of $K_n := K(E[p^n])$.

↳ we can explicitly study how this principal ideal factors into primes.

↳ <u>next talk</u>: use this factorization to give an upper bd on the class gp of K_n.