Learning seminar on Euler systems

§1 Introduction to the seminar (28th July)

The first talk will be an introduction to the learning seminar. We will briefly explain what Euler systems are and why they are useful in studying special values of L-functions. We will also give an introduction to three key examples that we will study in future talks: cyclotomic units and their application to Iwasawa theory (due to Thaine, Kolyvagin and Rubin), elliptic units and their application to BSD in analytic rank zero for CM elliptic curves (due to Coates and Wiles), and Heegner points and their application to proving BSD and finiteness of the Tate—Shafarevich group in anayltic rank <=1 (due to Gross-Zagier and Kolyvagin).

§2 Cyclotomic units and applications to Iwasawa theory (4th August)

In this talk we will introduce cyclotomic units and explain some of their properties. We will then introduce the statement of the Iwasawa main conjecture and briefly sketch how cyclotomic units are used to prove one inclusion in the Iwasawa main conjecture.
Reference: Section 1 of [Lei22].

§3 Background on Selmer groups (11th August)

We first state the Coates–Wiles theorem and explain the strategy of the proof. We then introduce the π^n-Selmer group of an elliptic curve E with complex multiplication by an imaginary quadratic field K and give a condition for this Selmer group to be trivial in terms of the ideal class group of $K(E[\mathfrak{p}])$ (Corollary 3.4.12 of [Ros19]).
Reference: Chapter 3 of [Ros19].

§4 The Euler system of elliptic units (18th August)

In this talk, we will introduce elliptic units and explain their basic properties.
Reference: Chapters 4 and 5 of [Ros19].

§5 Bounding the ideal class group of $K(E[\mathfrak{p}])$ (25th August)

We will apply the the Euler system of elliptic units in the previous talk to bound the ideal class group of $K(E[\mathfrak{p}])$; the precise result is Theorem 6.2.4 of [Ros19].
Reference: Chapter 6 of [Ros19].

§6 Elliptic units & L-functions of elliptic curves (1st September)

If E is an elliptic curve with CM, we explain why the L-function of E equals the Hecke L-function attached to a Grossencharacter. We use this result to relate elliptic units to special values of L-functions of elliptic curves with CM.
Reference: Chapter 7 of [Ros19].

§7 Proof of the Coates–Wiles theorem (8th September)

In this talk, we will combine the results proved in the previous talks to give a proof of the Coates–Wiles theorem.
Reference: Chapter 8 of [Ros19].

§8 Heegner points (15th September)

In this talk, we will state Gross-Zagier and Kolyvagin's results on the Birch and Swinnerton–Dyer conjecture in analytic rank $<=1$. We will also introduce the Heegner points and explain their basic properties.
Reference: Lectures 1 and 2 of [Cas21] and the article [Gro91].

§9 Kolyvagin's derived classes (22nd September)

In this talk we will explain how to use Heegner points to construct "Kolyvagin's derived classes", which are Galois cohomology classes in $H^1(K, E[p])$, and explain some of their properties.
Reference: Lectures 3 [Cas21] and the article [Gro91].

§10 Proof of Kolyvagin's theorem (29th September)

We will combine the results of the previous talks to give a proof of Kolyvagin's theorem.
Reference: Lecture 4 of [Cas21] and the article [Gro91].

References

[Cas21]   F. Castella, Heegner Points, Lecture notes (2021) at the ICTS summer school on *Elliptic Curves and Special Values of L-functions.*

[Gro91]   B. Gross, Koylvagin's work on modular elliptic curves, chapter in *L-functions and arithmetic.*

[Lei22]   A. Lei, Euler systems and Beilison –Flach elements, Lecture notes (2022) at the ICTS summer school on *Elliptic Curves and Special Values of L-functions*

[Ros19]   M. Roset, CM elliptic curves and Coates–Wiles theorem, Masters thesis (2019).

Besides the references mentioned above, the following may be useful as supplementary references:

- Videos of Castella and Lei's lectures at the ICTS summer school, which can be found here and here.

- Rubin's expository article on Kolyvagin's work on the BSD conjecture.

- Daoud's essay on the Coates–Wiles theorem

- The survey article of Bertolini et.al explaining special value formulae relating to cyclotomic units, elliptic units and and Heegner points to p-adic L-functions (as opposed to complex L-functions).

- Loeffler and Zerbes' lecture notes on Euler systems for a more modern account on certain aspects of the theory.