REAL GEOMETRIC TRANSCENDENCE FOR THE GAMMA FUNCTION

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Abstract. We show that the x-axis is the only real algebraic curve in $\mathbb{R}^2$ whose image via the Gamma function is contained in an algebraic curve. Our proof employs an elegant base-change argument due to Tamiozzo (2023) to deduce the result from the corresponding complex geometric transcendence result of Eterović, Padgett and Zhao (2025). As an application, we then use the complex and real geometric transcendence results to study analogues of the Manin–Mumford conjecture for the Gamma function.

1. Introduction

If $\Omega : X \to Y$ is a transcendental map between two complex algebraic varieties, the image of a generic algebraic subvariety of X will usually not be an algebraic subvariety of Y; it is thus often of special interest to characterise the relevant *bialgebraic varieties* for the map Ω, *i.e.*, those varieties $V \subseteq X$ such that $\Omega(V)$ is also algebraic. For instance, letting $\exp(z) := e^{2\pi i z}$, the irreducible algebraic subvarieties of $\mathbb{C}^n$ whose image via the map

$$\begin{aligned} \mathbb{C}^n &\to (\mathbb{C}^\times)^n \\ (x_1, \ldots, x_n) &\mapsto (\exp(x_1), \ldots, \exp(x_n)) \end{aligned}$$

is algebraic, are precisely translates of linear subspaces of $\mathbb{C}^n$ defined over $\mathbb{Q}$. Similar results are known, for instance, for the modular j-function and for Weierstrass elliptic functions. The problem of determining the relevant bialgebraic varieties in a given situation can be thought of as a problem of understanding how the algebraic structures on both the domain and the codomain interact under the map Ω; this problem can also be regarded as a geometric analogue of a fundamental problem in transcendental number theory where, given a holomorphic transcendental function f, one is interested in determining all the *bialgebraic numbers* with respect to f, *i.e.*, all algebraic numbers whose values under f remain algebraic.

1.1. Complex geometric transcendence for the Γ-function. The focus of the present paper is to explore the geometric transcendence properties of the celebrated Gamma function, which is defined by $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad$ for $\mathrm{Re}(z) > 0$ and admits a meromorphic continuation to the entire complex plane with simple poles at the non-positive integers $\mathbb{Z}_{\leq 0}$.

Theorem 1.1 (Eterović–Padgett–Zhao [EPZ25]). *Consider the function*

$$\begin{aligned} \Gamma^2 : (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^2 &\longrightarrow \mathbb{C}^2 \\ (z_1, z_2) &\mapsto (\Gamma(z_1), \Gamma(z_2)). \end{aligned}$$

Suppose $C \subseteq \mathbb{C}^2$ is an irreducible algebraic curve such that $\Gamma^2(C \cap (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^2)$ is contained in an algebraic curve. Then C is defined by one of the following equations:

(1) $X = Y$;
(2) $X = w$ for some $w \in \mathbb{C}$;
(3) $Y = w$ for some $w \in \mathbb{C}$.

The situation of geometric transcendence for the Γ function differs from those of the exponential function, the j-function and the Weierstrass elliptic functions, since unlike the latter three functions, Γ does not satisfy an algebraic differential equation by Hölder's theorem. Indeed, the proof of Theorem 1.1 does not use o-minimal methods, but rather proceeds by complex analytic computations describing the behavior of the fibers of Γ. We refer to [DVP25, EP25] for further progress on the topic of geometric and functional transcendence for the Gamma function.

Remark 1.2. *The varieties appearing in Theorem 1.1 (and their natural generalisations to higher dimensions) are called trivially bialgebraic in the notation of [EPZ25]; indeed, they are always bialgebraic for the n-fold product of any set-theoretic function. Theorem 1.1 is a special case of the main theorem of [EPZ25], which deals with the map*

$$\begin{aligned}\Gamma^n : (\mathbb{C} \smallsetminus \mathbb{Z}_{\leq 0})^n &\longrightarrow \mathbb{C}^n \\ (z_1, \ldots, z_n) &\mapsto (\Gamma(z_1), \ldots, \Gamma(z_n))\end{aligned}$$

for any $n \geq 2$. If $V \subseteq \mathbb{C}^n$ is an irreducible algebraic subvariety such that the dimension of the Zariski closure of $\Gamma^n(V \cap (\mathbb{C} \smallsetminus \mathbb{Z}_{\leq 0})^n)$ equals the dimension of V, the authors of [EPZ25] show that V is trivially bialgebraic.

1.2. Real geometric transcendence. One can investigate analogues of all of the above results in the setting of real algebraic geometry , *i.e.*, by regarding the relevant spaces as real algebraic varieties. The study of real analogues of complex geometric transcendence results was initiated by Tamiozzo in [Tam23], where he proved real geometric transcendence results for the exponential and the j-function. The results obtained in these cases have interesting connections to various arithmetic objects, such as class numbers of real quadratic fields and special geodesics in the upper half-plane. The techniques of [Tam23] were extended by the author and Tamiozzo in [ST25] to deal with the case of the Weierstrass elliptic functions and, in current work in progress, for the uniformisation maps of higher genus curves. In this paper, we prove the real analogue of Theorem 1.1. Let S denote the subset $\mathbb{Z}_{\leq 0} \times \{0\}$ of $\mathbb{R}^2$; by making the natural identification $\mathbb{R}^2 \simeq \mathbb{C}$, we are led to consider the function

$$\begin{aligned}G : \mathbb{R}^2 \smallsetminus S &\to \mathbb{R}^2 \\ (x, y) &\mapsto (\mathrm{Re}(\Gamma(x + iy)), \mathrm{Im}(\Gamma(x + iy))).\end{aligned}$$

Definition 1.3. *A non-empty subset $\mathcal{V} \subsetneq \mathbb{R}^2$ is called weakly bialgebraic for G if*

(1) there exist algebraic subvarieties $V \subseteq \mathbb{A}^2_{\mathbb{R}}$ and $W \subsetneq \mathbb{A}^2_{\mathbb{R}}$ such that $\mathcal{V} = V(\mathbb{R})$ and $G(\mathcal{V} \cap (\mathbb{R}^2 \smallsetminus S)) \subseteq W(\mathbb{R})$;

(2) the set $\mathcal{V}$ cannot be written in the form $V_1(\mathbb{R}) \cup V_2(\mathbb{R})$, where $V_1, V_2 \subseteq \mathbb{A}^2_{\mathbb{R}}$ are algebraic subvarieties and the inclusions $V_i(\mathbb{R}) \subseteq \mathcal{V}$ are proper for $i = 1, 2$.

Theorem 1.4. *Suppose $\mathcal{V} \subseteq \mathbb{R}^2$ is weakly bialgebraic for G. Then $\mathcal{V}$ must be the x-axis.*

The fact that the x-axis is indeed weakly bialgebraic follows from the relation

$$\overline{\Gamma(z)} = \Gamma(\overline{z}), \tag{1.1}$$

and we see that the image of the x-axis under G is contained in the x-axis. In Section 4, we use Theorem 1.1 and Theorem 1.4 to study complex and real analogues of the Manin–Mumford conjecture in this setting. To the best of our knowledge, this is the first instance in the literature where such analogues have been studied for the Gamma function.

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2. Proof of Theorem 1.4

2.1. Preliminaries. Let $\mathcal{V}$ be a weakly bialgebraic set which is not a singleton. By considering sums of squares of polynomials, we can write $\mathcal{V}$ as the zero locus of a single polynomial $R \in \mathbb{R}[X,Y]$. By Definition 1.3 (2), the set $\mathcal{V}$ must be the vanishing locus of some irreducible factor P of R. Furthermore, since $\mathcal{V}$ is infinite, the polynomial P is also irreducible in $\mathbb{C}[X,Y]$ (otherwise, we could write $P = S\overline{S}$ for an irreducible polynomial $S \in \mathbb{C}[X,Y]$ that is not a multiple of a real polynomial, and both S and $\overline{S}$ would vanish on $\mathcal{V}$, contradicting Bézout's theorem).

2.2. The base-change diagram. By Equation (1.1),

$$\operatorname{Re}(\Gamma(x+iy)) = \frac{\Gamma(x+iy)+\Gamma(x-iy)}{2}, \quad \operatorname{Im}(\Gamma(x+iy)) = \frac{\Gamma(x+iy)-\Gamma(x-iy)}{2i}.$$

We define maps

$$f\colon \mathbb{C}^2 \to \mathbb{C}^2 \qquad\qquad g\colon \mathbb{C}^2 \to \mathbb{C}^2$$
$$(v,w) \mapsto (v+iw, v-iw) \qquad\qquad (a,b) \mapsto \left(\frac{a+b}{2}, \frac{a-b}{2i}\right),$$

and thus obtain a commutative diagram

$$\begin{tikzcd} \mathbb{R}^2 \smallsetminus S \arrow[r, "G"] \arrow[d, hook] & \mathbb{R}^2 \arrow[r, hook] & \mathbb{C}^2 \arrow[d, "\iota\circ g^{-1}"] \\ \mathbb{C}^2 \arrow[r, "f"] & \mathbb{C}^2 \arrow[r, "\Gamma^2"] & \mathbb{P}^1(\mathbb{C})^2, \end{tikzcd} \tag{2.1}$$

where the left vertical and the top right horizontal map above are induced by the inclusion $\mathbb{R} \subseteq \mathbb{C}$, and the map $\iota : \mathbb{C}^2 \to \mathbb{P}^1(\mathbb{C})^2$ is given by the inclusion $\mathbb{C} \subseteq \mathbb{P}^1(\mathbb{C})$ on each component. Let $\widetilde{\mathcal{V}} = \mathcal{V} \cap (\mathbb{R}^2 \smallsetminus S)$. Since $\mathcal{V}$ is weakly bialgebraic, there exists a non-zero polynomial $Q \in \mathbb{R}[X,Y]$ such that $G(\widetilde{\mathcal{V}})$ is contained in the real algebraic curve with equation $Q = 0$. Let C_Q be the complex plane curve with equation $Q = 0$, and let $\hat{C}_Q \subset \mathbb{P}^1(\mathbb{C})^2$ be the Zariski closure of $\iota \circ g^{-1}(C_Q)$. Let C_P be the complex curve with equation $P = 0$ and let $q = \Gamma^2 \circ f$. The commutativity of the diagram (2.1) implies that $A := C_P \cap q^{-1}(\hat{C}_Q)$ contains $\widetilde{\mathcal{V}}$.

2.3. Analytic continuation from the complexification of a real algebraic curve. Viewing C_P as a complex analytic space with respect to the Euclidean topology, we thus see that A is an analytic subset of C_P, consisting of the points of C_P which lie on the vanishing locus of the holomorphic function $Q \circ q$. Using the fact that A contains $\widetilde{\mathcal{V}}$, and hence segments of the real algebraic curve $\mathcal{V}$, we show below that this condition is rigid enough to force the entire complex curve C_P to vanish on $Q \circ q$, hence yielding that $f(C_p)$ is bialgebraic for Γ^2. We shall employ the formalism of thin sets and a version of the identity principle from [GR84].

We recall (cf. [GR84, page 132]) that a closed subset A of a complex analytic space X is called *thin* if every $p \in A$ has an open neighbourhood U such that $A \cap U$ is contained in a nowhere dense analytic subset of X. In particular, we will use in Proposition 2.1 below that if a subset of a complex curve contains an open set (in the complex analytic topology), then it is not thin. We will also use [GR84, Theorem, page 168] which implies that if X is connected, then every proper analytic set of X is thin in X.

Proposition 2.1. *The curve $f(C_P)$ is bialgebraic for Γ^2.*

Proof. Since $\mathcal{V}$ is infinite, $\widetilde{\mathcal{V}}$ contains a subset I which is homeomorphic to an open interval. Since any complex curve has only finitely many singular points, we may choose a smooth point x of C_P lying in I. By the implicit function theorem, there exists an open subset $U \subseteq C_P$ (in the complex analytic topology) containing x, an open disc $D \subseteq \mathbb{C}$, and a biholomorphic map $\phi : D \to U$. Thus, $(Q \circ q) \circ \phi$ vanishes on $\phi^{-1}(I \cap U)$; by the identity theorem, this function must vanish on D as well. So $Q \circ q$ vanishes on U and hence $U \subseteq A$. This implies that the analytic subset A of C_P is not thin. Since C_P is irreducible, a standard result implies that it is connected in the Euclidean topology and so we must have $A = C_P$. Therefore $q(C_P) \subseteq \hat{C}_Q$ and so $\Gamma^2(f(C_P)) \subseteq \hat{C}_Q$, proving that $f(C_P)$ is bialgebraic for Γ^2. ∎

2.4. Application of Theorem 1.1. By Theorem 1.1 and Proposition 2.1, we conclude that $f(C_P)$ must equal a set of the form

(1) $\{(x, y) \in \mathbb{C}^2 : x = c\}$ for some $c \in \mathbb{C}$.
(2) $\{(x, y) \in \mathbb{C}^2 : y = c\}$ for some $c \in \mathbb{C}$.
(3) $\{(x, y) \in \mathbb{C}^2 : x = y\}$.

Since $f(\mathcal{V}) \subseteq f(C_P)$, we deduce that $f(\mathcal{V})$ is contained in a set of the form described above.

2.5. Endgame. We note that $f(\mathcal{V})$ cannot be contained in a horizontal or vertical line; indeed, in that case there would be an $\alpha = a+ib \in \mathbb{C}$ such that every $(x, y) \in \mathcal{V}$ satisfies $x \pm iy = a+ib$, contradicting the assumption that $\mathcal{V}$ is not a point. Thus, we must have that $f(\mathcal{V}) \subseteq \{(x, y) \in \mathbb{C}^2 : x = y\}$, which implies that $x + iy = x - iy$ for all $(x, y) \in \mathcal{V}$. This forces $y = 0$ and so $\mathcal{V}$ must be contained in, and hence equal to, the x-axis. Conversely, using Equation (1.1), we see that the image of the x-axis is contained in the x-axis, so the x-axis is indeed weakly bialgebraic. This completes the proof of Theorem 1.4.

3. The image of the x-axis under G

Proposition 3.1. *Let $\mathcal{V}$ denote the x-axis in $\mathbb{R}^2$. We have that*

$$G(\mathcal{V} \cap (\mathbb{R}^2 \setminus S)) = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}, x \neq 0\}.$$

Proof. It suffices to show that $\Gamma(\mathbb{R} \setminus \mathbb{Z}_{\leq 0}) = \mathbb{R} \setminus \{0\}$. Since Γ has a global minimum on the interval $(0, \infty)$ where it achieves the value ≈ 0.88, we instead study its behaviour on the negative real axis to prove Proposition 3.1. We recall the reflection formula $\Gamma(x) = \frac{\pi}{\sin(\pi x)\Gamma(1-x)}$, which is valid for all $x \notin \mathbb{Z}$. It follows that for each $n \geq 0$, $\Gamma(x)$ tends to $-\infty$ (resp. $+\infty$) as x approaches either of the end-points of the interval $(-2n-1, -2n)$ (resp. $(-2n-2, -2n-1)$). Let m_n denote the global maximum (resp. global minimum) of $\Gamma(x)$ on $(-2n-1, -2n)$ (resp. $(-2n-2, -2n-1)$); from the reflection formula and the fact that $\Gamma(x) \to \infty$ as $x \to +\infty$, it follows that $m_n \to 0$ as $n \to \infty$. Since Γ is continuous on each of the above intervals, we conclude that the image of $\mathbb{R}_{\leq 0} \setminus \mathbb{Z}_{\leq 0}$ under Γ equals $\mathbb{R} \setminus \{0\}$. ∎

In particular, Proposition 3.1 implies that the image of the x-axis is a semialgebraic set. This is in harmony with the real geometric transcendence results of [Tam23, ST25] where images of weakly bialgebraic sets were often also semialgebraic.

Remark 3.2. *The algebraic rigidity of the x-axis under G is in sharp contrast with the behavior of the purely imaginary y-axis. Setting $z = it$ in the reflection formula for $t \in \mathbb{R}$, using Equation (1.1) and the equation $\Gamma(z+1) = z\Gamma(z)$, it follows that*

$$|\Gamma(it)|^2 = \frac{\pi}{-it\sin(\pi it)} = \frac{\pi}{t\sinh(\pi t)}.$$

Thus, $|\Gamma(it)|$ decays exponentially to zero as $t \to \infty$. Stirling's approximation implies that, for a continuous branch of the logarithm along the imaginary axis, $\mathrm{Im}(\log \Gamma(it))$ grows asymptotically as $t \log t$. Since this imaginary part is unbounded, the image of the y-axis under G must intersect the real axis infinitely many times as it converges to the origin. By Bézout's theorem, the image cannot be contained in any real algebraic curve in $\mathbb{R}^2$, providing a concrete geometric illustration of a highly transcendental image under G.

4. Manin–Mumford analogues for Γ

4.1. Recollection of general set-up. Let f be a transcendental meromorphic function on $\mathbb{C}$ and let $S \subset \mathbb{C}$ be its set of poles. A point $a \in \mathbb{C}$ is called *f-special* if $a \in \overline{\mathbb{Q}}$ and $a = f(b)$ for some $b \in \overline{\mathbb{Q}}$. A point $(a_1, a_2) \in \mathbb{C}^2$ is called *f-special* if both a_1 and a_2 are f-special. If $\Omega_f : (\mathbb{C} \smallsetminus S)^2 \to \mathbb{C}^2$ is given by $(z_1, z_2) \mapsto (f(z_1), f(z_2))$, then an irreducible algebraic curve $W \subseteq \mathbb{C}^2$ is called f-special if $\Omega_f(V \cap (\mathbb{C} \smallsetminus S)^2) \subseteq W$ for some algebraic curve $V \subseteq \mathbb{C}^2$. A *Manin–Mumford type conjecture* in this setting predicts that if an algebraic curve $W \subseteq \mathbb{C}^2$ contains infinitely many f-special points, then W must be f-special.

Example 4.1. If $f = \exp(z)$, then by the Gelfond–Schneider theorem, the f-special points are exactly the roots of unity in $\mathbb{C}$. It follows from the discussion in §1 that the f-special curves in $\mathbb{C}^2$ are precisely curves defined by a polynomial of the shape $X^m Y^n = \alpha$, where $m, n \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$. It was shown by Lang [Lan65] (with independent proofs by Serre, Ihara and Tate) that if a curve in $\mathbb{C}^2$ contains infinitely many f-special points, then it must be f-special with the corresponding α being a root of unity.

We refer to the survey article [KUY18] for other examples, and generalisations, of Manin–Mumford type conjectures.

4.2. Complex analogue for Γ. In view of the discussion in Section 4.1, a point a in $\mathbb{C}$ is called Γ-special if $a \in \overline{\mathbb{Q}}$ and if $a = \Gamma(b)$ for some $b \in \overline{\mathbb{Q}}$. Since $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{Z}_{\geq 1}$, it follows that the factorials are Γ-special. A special case of the Lang–Rohrlich conjecture implies that the converse is also true; see, for instance, the discussion in [Riv12, page 239]. We record this special case here as follows.

Conjecture 4.2. *Suppose $z \in \overline{\mathbb{Q}}$. Then $\Gamma(z) \in \overline{\mathbb{Q}}$ if and only if $z \in \mathbb{Z}_{\geq 1}$.*

For example, $\Gamma(1/2) = \sqrt{\pi}$, and $\Gamma(1/3)$ and $\Gamma(1/4)$ are known to be transcendental by the work of Chudnovsky [Chu76], but the transcendence of $\Gamma(1/5)$ is still open.

We now proceed to study analogues of Manin–Mumford type conjectures for the Gamma function. We first recall the background about the asymptotic behaviour of algebraic curves from [Wal92, Section 3] that we shall use in our study.

Remark 4.3 (Asymptotic behaviour of algebraic curves)**.** *Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be an integer polynomial with positive degree in both X and Y, which is irreducible in $\mathbb{Q}[X, Y]$. Write*

$$F(X, Y) = A_n(X)Y^n + A_{n-1}(X)Y^{n-1} + \cdots + A_0(X).$$

Puiseux's theorem asserts the existence of n distinct formal series $Y_i(X) = \sum_{k=-f_i}^{\infty} c_{k,i} x^{-k/e_i}$, where each e_i is a positive integer, each f_i is an integer chosen such that $c_{-f_i,i} \neq 0$ and the $c_{k,i} \in \mathbb{C}$, such that $F(X, Y) = A_n(X) \prod_{i=1}^{n} (Y - Y_i(X))$ as formal power series. As explained in [Wal92, page 162], there exists $R \in \mathbb{R}_{>0}$ such that each $Y_i(X)$ converges when $|X| > R$. It follows that there exists $R' \in \mathbb{R}_{>0}$ such that if $(x, y) \in \mathbb{C}^2$ satisfies $F(x, y) = 0$ with $|x| > R'$, *then there exists $i \in \{1, \ldots, n\}$ such that $y = Y_i(x)$. This result describes the asymptotic behaviour of points (x, y) lying on the curve $F = 0$ as $x \to \infty$; for sufficiently large x, the points on the curve lie on graphs of finitely many fractional power series in X.*

Notation 4.4. Let $(a_n)_n$ and $(b_n)_n$ be two sequences of positive real numbers. We say that $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, $a_n = O(b_n)$ if there exists a constant $C > 0$ such that $a_n \leqslant Cb_n$ for all n and $a_n = o(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Proposition 4.5. *Suppose $P \in \mathbb{C}[X, Y]$ is an irreducible polynomial such that it has infinitely many roots of the form $(n!, m!)$ where $n, m \geqslant 1$. Then the curve $C_P \subseteq \mathbb{C}^2$ defined by $P = 0$ must be a vertical line, a horizontal line or the diagonal in $\mathbb{C}^2$.*

Proof. Let the infinite sequence of roots be denoted as $(n_k!, m_k!)$ for $k = 1, 2, \ldots$; since P is irreducible and has infinitely many integral solutions, we may assume that $P \in \mathbb{Q}[X, Y]$. By clearing denominators, we may further assume that $P \in \mathbb{Z}[X, Y]$.

We first assume that the sequence $(n_k)_k$ is bounded. There exists n_0 such that $n_k = n_0$ for infinitely many k. By applying the division algorithm in $\mathbb{C}[X, Y] = \mathbb{C}[Y][X]$, we obtain that $P = (X - n_0!)Q + R$ for some $Q \in \mathbb{C}[X, Y]$ and $R \in \mathbb{C}[Y]$. Upon substituting $X = n_0!$ and $Y = m_k!$, it follows that the polynomial R has infinitely many roots and so must be zero. Since P is irreducible, it follows that C_P is the zero locus of $X - n_0!$ and so is a vertical line. A similar argument shows that if $(m_k)_k$ is bounded, then C_P must be a horizontal line.

We now assume that both $(n_k)_k$ and $(m_k)_k$ are unbounded. We will show that $n_k = m_k$ for infinitely many k. This will imply, by Bézout's theorem, that C_P must be the diagonal. The discussion in Remark 4.3 implies that, by passing to an infinite subsequence if necessary and relabeling indices, there exists $r \in \mathbb{Q}$ and $c \in \mathbb{R}_{>0}$ such that $n_k! \sim c(m_k!)^r$ as $k \to \infty$. Taking logarithms yields

$$\log(n_k!) = r \log(m_k!) + \log c + o(1). \tag{4.1}$$

Applying Stirling's approximation $\log(x!) = x \log x - x + O(\log x)$ to Equation (4.1) implies $n_k \log n_k \sim rm_k \log m_k$. Applying logarithms now yields $\log n_k \sim \log m_k$ and so $n_k \sim rm_k$.

We now employ a p-adic argument to estimate the difference $n_k - rm_k$ along a suitable infinite subsequence. Let $P(X, Y) = \sum_{i,j} c_{i,j} X^i Y^j$ and for a fixed prime p, consider the p-adic valuation of each evaluated monomial $v_p(c_{i,j}(n_k!)^i(m_k!)^j) = v_p(c_{i,j}) + iv_p(n_k!) + jv_p(m_k!)$. Since $P(n_k!, m_k!) = 0$ for each $k \geqslant 1$, the minimal p-adic valuation of at least two evaluated monomials must be equal. Thus, there is a specific pair of monomials, say with exponents (i_1, j_1) and (i_2, j_2), that have equal minimum evaluated valuation for an infinite subsequence. By relabeling our indices, we may again assume without loss of generality that this equality holds for all k in our sequence. By Legendre's formula, $v_p(x!) = \frac{x - s_p(x)}{p-1}$, where $s_p(x)$ is the sum of the digits in the base-p expansion of x; note that $s_p(x) = O(\log x)$. Inserting this into $v_p(c_{i_1,j_1}(n_k!)^{i_1}(m_k!)^{j_1}) = v_p(c_{i_2,j_2}(n_k!)^{i_2}(m_k!)^{j_2})$, we obtain $i_1 n_k + j_1 m_k = i_2 n_k + j_2 m_k + O(\log n_k) + O(\log m_k)$. We now note that $\log n_k = O(\log m_k)$ and also that $i_1 \neq i_2$, since otherwise $m_k = O(\log m_k)$ contradicting the fact that $(m_k)_k$ is unbounded. Thus, $n_k - \frac{j_2 - j_1}{i_1 - i_2} m_k = O(\log m_k)$. This forces $r = \frac{j_2 - j_1}{i_1 - i_2}$ using again the fact that $n_k \sim rm_k$. In summary, if we let $E_k = n_k - rm_k$, then we have $E_k = O(\log m_k)$.

Applying Stirling's formula in Equation (4.1), substituting $n_k = rm_k + E_k$ and using $\log n_k = O(\log m_k)$ we obtain

$$(rm_k + E_k) \log(rm_k + E_k) - E_k - rm_k \log m_k = O(\log m_k). \tag{4.2}$$

Writing $\log(rm_k + E_k) = \log(rm_k) + \log(1 + E_k/rm_k) = \log(rm_k) + O\left(\frac{E_k}{rm_k}\right)$ and using $E_k = O(\log m_k)$, Equation (4.2) simplifies to

$$r \log r = O\left(\frac{(\log m_k)^2}{m_k}\right).$$

Since the right hand side tends to zero as $k \to \infty$, it follows that $r = 1$ and so $n_k! \sim cm_k!$ as $k \to \infty$, *i.e.*, $\lim_{k\to\infty} \frac{n_k!}{m_k!} = c$. Note that if $n_k > m_k$ for infinitely many k, then $\frac{n_k!}{m_k!} \geqslant m_k + 1 \to \infty$, a contradiction. Similarly, if $n_k < m_k$ for infinitely many k, then $\frac{n_k!}{m_k!} \leqslant \frac{1}{n_k+1} \to 0$, a contradiction. Thus, we must have that $n_k = m_k$ for all k sufficiently large as desired. ■

Thus, conditional on Conjecture 4.2, Theorem 1.1 and Proposition 4.5 imply that the analogue of the Manin–Mumford conjecture holds for (products of) the Gamma function.

4.3. Real analogue for Γ. We call a point (x, y) in $\mathbb{R}^2$ Γ- *special* if $(x, y) \in \overline{\mathbb{Q}}^2$ and $(x, y) = G((a, b))$ for some $(a, b) \in \mathbb{R}^2 \cap \overline{\mathbb{Q}}^2$; note that these correspond precisely to the Γ-special points of $\mathbb{C}$ under the natural identification $\mathbb{R}^2 \simeq \mathbb{C}$. We call an irreducible real algebraic curve W in $\mathbb{R}^2$ Γ-special if $W \supseteq G(\mathcal{V} \cap (\mathbb{R}^2 \setminus S))$ for a weakly bialgebraic set $\mathcal{V} \subseteq \mathbb{R}^2$. Conjecture 4.2 implies that the only Γ- special points in $\mathbb{R}^2$ are of the form $(n!, 0)$ for $n \geq 1$. On the other hand, Theorem 1.4 and Proposition 3.1 imply that the x-axis is the only Γ-special real algebraic curve in $\mathbb{R}^2$. Conditional on Conjecture 4.2, we thus obtain the following real analogue of the Manin–Mumford conjecture for the Gamma function: *suppose W is an irreducible real algebraic curve in $\mathbb{R}^2$ containing infinitely many Γ-special points. Then W must be Γ-special.*

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