The Commutativity Theorem

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1. Review of Intersection product

In this section, we review the definition of Intersection product as well as state the main theorem of this report: The Commutativity Theorem.

To begin, we recall the definitions and important properties of the three types of divisors we discussed in the last talk: Weil divisors, Cartier diviors and psuedo-diviors. We only discuss material that is used in this report. A more complete treatment can be found in [Fu, Section 2.1].

1.1. **Definition.** Let X be an n-dimensional variety. A Weil divisor α on X is an element of $Z_{n-1}(X)$ i.e. α is an $(n-1)$- cycle on X.

1.2. **Example.** If X is a curve, a Weil divisor is simply an integral linear combination of points of X with only finitely many coefficients non-zero.

1.3. **Definition.** A Cartier divisor on X is given by the data (U_α, f_α) where U_α form an open covering of X and each $f_\alpha \in R(X)^\times$ is a rational function subject to the condition that $f_\alpha/f_\beta \in \mathcal{O}_X(U_\alpha \cap U_\beta)^\times$. The rational functions f_α are called local equations for D; they are determined up to multiplication by units on U_α.

1.4. **Example.** Let $X = \mathbb{A}^1 - \{1\}$. The data $U_\alpha = X$ and $f_\alpha = 1/t$ determines a Cartier divisor on X. The data $U_\alpha = X$ and $f_\alpha = (t-1)/t$ determines the same Cartier divisor on X.

1.5. **Remark.** The Cartier divisors form an abelian group $\mathrm{Div}(X)$: if D and E are given by the data (U_α, f_α) and (V_β, g_β), then $D + E$ is given by $(U_\alpha \cap V_\beta, f_\alpha g_\beta)$.

1.6. **Remark.** A Cartier divisor $D = (U_\alpha, f_\alpha)$ on a scheme X determines a line bundle $\mathcal{O}_X(D)$ on X such that $\mathcal{O}_X(D)(U_\alpha) = (f_\alpha^{-1})$. Moreover, $\mathcal{O}_X(D)$ comes with a canonical section s_D such that $s_D|_{U\alpha} = f_\alpha$.

1.7. **Definition.** The support of a Cartier divisor $|D|$ is the union of all subvarieties Z of X such that a local equation for D in the local ring $\mathcal{O}_{Z,X}$ is not a unit. The support $|D|$ is a closed algebraic set of codimension 1.

1.8. **Definition.** A Cartier divisor is called effective if the local equation f_α lies in $\mathcal{O}_X(U_\alpha)$ for each α.

1.9. **Remark.** We will also often use the following alternative definition of effective Cartier divisors: a closed subscheme $D \subseteq X$ is called an effective Cartier divisor if for every $x \in D$, there exists an affine neighbourhood $U = \text{Spec } A$ of x such that $U \cap D = \text{Spec } (A/(f))$ with $f \in A$ not a zero divisor.

Having discussed both Cartier and Weil divisors, it is natural to ask whether there is any relation between them. The following definition tells us that a Cartier divisor gives rise to a Weil divisor:

1.10. **Definition.** If D is a Cartier divisor, define its associated Weil divisor to be

$$[D] := \sum \text{ord}_V D \cdot [V],$$

where the sum is taken over all codimension 1 subvarieties V of X and where $\text{ord}_V D := \text{ord}_V(f_\alpha)$ for any local equation f_α such that $U_\alpha \cap V \neq \emptyset$.

1.11. **Definition.** A psuedo divisor on a scheme X is a triple (L, Z, s), where L is a line bundle on X, Z is a closed subset of X and s is a nowhere vanishing section of L on $X - Z$.

1.12. **Remarks.** The most important property of psedo-divisors is that they always pullback. Moreover, given a Cartier divisor D, we get an associated psuedo divisor $(\mathcal{O}_X(D), |D|, s_D)$. Conversely, if (L, Z, s) is a psuedo divisor, we say that a Cartier divisor represents (L, Z, s) if $|D| \subseteq Z$ and if there is an isomorphism from $\mathcal{O}_X(D)$ to L which takes s_D to s outside Z. It is a fact (see [Fu, Lemma 2.2]) that if $Z \neq X$ then any psuedo divisor is uniquely represented by a Cartier divisor D and if $Z = X$, then any psuedo divisor is represented by a Cartier divisor that is determined up to linear equivalence.

We have seen that Cartier divisors give rise to Weil divisors; the following definition shows us that psuedo divisors give rise to Weil divisor classes.

1.13. **Definition.** If D is a psuedo-divisor on an n-dimensional variety X and $|D|$ is its support, we define the Weil divisor class $[D] \in A_{n-1}(|D|)$ as follows: take a Cartier divisor which represents D and let $[D]$ be the class in $A_{n-1}(|D|)$ of the Weil divisor associated to the Cartier divisor.

1.14. **Definition.** Let D be a psuedo-divisor on a scheme X and let V be a k-dimensional subvariety of X. We define the intersection product $D \cdot [V] := [j^*D] \in A_{k-1}(V \cap |D|)$, where $j : V \hookrightarrow X$ is the inclusion and $[j^*D]$ is the Weil divisor class associated to the psuedo divisor j^*D according to Definition 1.13. For a k-cycle, we extend this definition linearly: namely, if $\alpha = \sum n_V[V]$ is a k-cycle, we define $D \cdot \alpha := \sum n_V D \cdot [V] \in A_{k-1}(|D| \cap |\alpha|)$, where $|\alpha|$ is the union of subvarieties V appearing with non-zero coefficient.

The intersection product satisfies a number of nice properties. We recall only the following two, since they are used in the rest of this report.

1.15. **Theorem.** (a) (Linearity) If D is a psuedo-divisor on X and α, α' are k-cycles on X, then

$$D \cdot (\alpha + \alpha') = D \cdot \alpha + D \cdot \alpha'$$

(b) (Projection formula) Let D be a psuedo-divisor on X, $f : X' \to X$ a proper morphism, α a k-cycle on X' and g the morphism from $f^{-1}(|D|) \cap |\alpha|$ to $|D| \cap f(|\alpha|)$ induced by f. Then

$$g_*(f^*D \cdot \alpha) = D \cdot f_*(\alpha)$$

.

Proof. See [Fu, Proposition 2.3] □.

1.16. **Remark.** The projection formula is analogous to the following situation: if $f : X \to Y$ is any set map with $V \subseteq X$ and $W \subseteq Y$, then

$$f(f^{-1}(W) \cap V) = W \cap f(V).$$

Indeed if we compare this formula with the projection formula, the set theoretic intersection corresponds to the intersection product, the set theoretic inverse image corresponds with pullback and the set theoretic direct image corresponds to the pushforward.

Having recalled the necessary material, we can now state the main theorem of this report.

1.17. **Theorem (Commutativity Theorem).** Let D and D' be Cartier divisors on an n-dimensional variety X. Then

$$D \cdot [D'] = [D'] \cdot [D]$$

in $A_{n-2}(|D| \cap |D'|)$.

2. Blow up

In this section, we introduce an important geometric construction: blow-up. This construction is one of the key tools needed to prove the Commutativity theorem.

2.1. **Definition.** Let X be a scheme, let $\mathcal{I}$ be a coherent sheaf of ideals on X and let $f : Y \to X$ be a morphism of schemes. The inverse image ideal sheaf of $\mathcal{I}$, which we denote by $f^{-1}I\mathcal{O}_Y$, is defined as follows: for $U \subseteq Y$ open, $f^{-1}\mathcal{I}\mathcal{O}_Y(U)$ is the ideal generated by $f^{-1}(U)$ under the ring map $f^{-1}\mathcal{O}_X(U) \to \mathcal{O}_Y(U)$.

2.2. **Definition.** Let X be a scheme and let $\mathcal{I}$ be a coherent sheaf of ideals on X. Then the blow up of X along $\mathcal{I}$ is a morphism $\pi : \tilde{X} \to X$ satisfying the following universal property: $\pi^{-1}\mathcal{I}O_{\tilde{X}}$ is an invertible sheaf of ideals and if $f : Z \to X$ is morphism such that $f^{-1}I\mathcal{O}_Z$ is an invertible sheaf of ideals, then there is a unique morphism $g : Z \to \tilde{X}$ making the following diagram commute:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\pi"] \\ Z \arrow[ur, "g"] \arrow[r, "f"] & X \end{tikzcd}$$

2.3. **Remark.** It is a fact that blow-ups exist and we also have an explicit construction for it: $\tilde{X} = \mathbf{Proj} \oplus_{n\in\mathbb{N}} \mathcal{I}^n$.

2.4. **Remark.** We give an alternative definition in terms of closed subschemes. If X is a scheme and $D \subseteq X$ is a closed subscheme, then the blow up of X along D is a morphism $\pi : \tilde{X} \to X$ such that $\pi^{-1}D$ is an effective Cartier divisor satisfying the same universal property as above. We remark that $\pi^{-1}D$ is often called an effective Cartier divisor.

2.5. **Example.** Let $X = \mathbb{A}^2$ and $D = \{(0,0)\}$. View $\mathbb{P}^1$ as the set of all lines in $\mathbb{A}^2$ passing through the origin. Let $\tilde{X} = \{(p,l)|p \in l\} \subseteq \mathbb{A}^2 \times \mathbb{P}^1$ and let $\pi : \tilde{X} \to X$ be the projection i.e. $\pi((p,l)) = p$. Then it is a fact that $\pi : \tilde{X} \to X$ is the blow-up of the origin in $\mathbb{A}^2$. Note that the preimage of each point except the origin has cardinality 1; on the other hand, the preimage of the origin is 'blown up': it is an entire copy of $\mathbb{P}^1$.

2.6. **Example.** If X is any scheme and D is an effective Cartier divisor on X, then $\mathrm{id}_X : X \to X$ is the blow up of X along D.

3. Proof of the Commutativity Theorem

3.1. **Lemma.** If D and D' are Cartier divisors on X, $\pi : \tilde{X} \to X$ is a proper birational map of varieties, $\pi^*D = B \pm C$, $\pi^*D' = B' \pm C'$ for Cartier Divisors B, C, B', C' on $\tilde{X}$ with $|B| \cup |C| \subseteq \pi^{-1}(|D|)$, $|B'| \cup |C'| \subseteq \pi^{-1}(|D'|)$ and the theorem holds for for each of the pairs $(B, B'), (B, C'), (C, B')$ and (C, C') on $\tilde{X}$, then the theorem holds for (D, D') on X.

Proof. Firstly, we note that:

$$\begin{aligned}
\pi_*([B \pm C]) &= \pi_*(\pi^*D) \\
&= \pi_*((\pi^*D) \cdot [\tilde{X}]) \\
&= D \cdot \pi_*([\tilde{X}]) && \text{(By the projection formula)} \\
&= D \cdot ([(R(\tilde{X}) : R(X)][X]) && \text{(By definition of pushforward)} \\
&= D \cdot [X] && \text{(Since } \pi \text{ is birational)} \\
&= [D].
\end{aligned}$$

Similarly, we get that $\pi_*([B' \pm C']) = [D']$. Let g be the induced morphism $\pi^{-1}(|D| \cap |D'|) \to |D| \cap |D'|$. Then:

$$\begin{aligned}
D \cdot [D'] &= D \cdot \pi_*([B' \pm C']) && \text{(From above)} \\
&= g_*(\pi^*D \cdot [B' \pm C']) && \text{(By the projection formula)} \\
&= g_*((B \pm C) \cdot [B' \pm C']) && \text{(By hypothesis)} \\
&= g_*(B \cdot [B'] \pm B \cdot [C'] \pm C \cdot [B'] \pm C \cdot [C']) && \text{(By linearity)} \\
&= g_*(B' \cdot [B] \pm C' \cdot [B] \pm B' \cdot [C] \pm C' \cdot [C]) && \text{(By hypothesis)} \\
&= g_*((B' \pm C') \cdot [B \pm C]) && \text{(By linearity)} \\
&= g_*(\pi^*D' \cdot [B \pm C])) && \text{(By hypothesis)} \\
&= D' \cdot \pi_*([B \pm C]) && \text{(By the projection formula)} \\
&= D' \cdot [D] && \text{(From above)}
\end{aligned}$$

□

We now state the Commutaivity theorem again and give a proof of it:

3.2. **Theorem (Commutativity Theorem).** Let D and D' be Cartier divisors on an n-dimensional variety X. Then

$$D \cdot [D'] = [D'] \cdot [D]$$

in $A_{n-2}(|D| \cap |D'|)$.

Proof. The proof is split into four cases. The first case is purely algebraic. The remaining three cases are geometric and involve the blow-up construction.

Case 1: D and D' are both effective and intersect properly i.e. no codimension 1 subvariety is contained inside $D \cap D'$. Let W be any codimension 2 subvariety of X, let $A = \mathcal{O}_{W,X}$ and let a, a' be local equations for D, D' in X. The subvarieties V of X which contain W correspond to height 1 primes $\mathfrak{p}$ of A. By definition, one checks that the coefficient of $[V]$ in $[D']$ is $l_{A_\mathfrak{p}}(A_\mathfrak{p}/a'A_\mathfrak{p})$. The coefficient of W in $D \cdot [V]$ is $l_{A/\mathfrak{p}}(A/(\mathfrak{p}+aA))$. The coefficient of $[W]$ in $D \cdot [D']$ is therefore

$$\sum_{\mathfrak{p} \subseteq A, \mathrm{ht}(\mathfrak{p})=1} l_{A_\mathfrak{p}}(A_\mathfrak{p}/a'A_\mathfrak{p}) \cdot l_{A/\mathfrak{p}}(A/(\mathfrak{p}+aA)) := e_A(a, A/a'A).$$

By symmetry, the coefficient of $[W]$ in $D' \cdot [D]$ is $e(a', A/aA)$. Using [Fu, Lemma A.2.7] and [Fu, Lemma A.2.8], we conclude that $e_A(a, A/a'A) = e(a', A/aA)$.

Before proving the rest of the theorem, we need some preparation. If D and D' are effective Cartier divisors, define the excess of intersection by the formula:

$$\epsilon(D, D') := \max\{\mathrm{ord}_V(D) \cdot \mathrm{ord}_V(D') | \mathrm{codim}(V, X) = 1\},$$

where the maximum runs over all codimension one subvarieties of X. Note that $\epsilon(D, D') = 0$ if and only if D and D' intersect properly since:

$$\epsilon(D, D') > 0 \iff \mathrm{ord}_V(D), \mathrm{ord}_V(D') > 0 \iff V \subseteq D \text{ and } V \subseteq D'.$$

for some codimension 1 subvariety V of X.

Let $D \cap D'$ denote the intersection scheme of D and D'. This is the subscheme on X which on an open affine U is defined by (a, a'), where a and a' are local equations for D and D' in U. Let $\pi : \tilde{X} \to X$ be the blow-up of X along $D \cap D'$ and let $E = \pi^{-1}(|D \cap D'|)$ be the exceptional divisor. Then since $\pi^{-1}(|D \cap D'|) \subseteq \pi^{-1}(|D|)$ and since $\pi^{-1}(|D \cap D'|) \subseteq \pi^{-1}(|D'|)$, the local equations for π^*D and π^*D' are divisible by the local equations for E and so:

$$\pi^*D = E + C, \quad \pi^*D' = E + C'$$

for effective Cartier divisors C, C' on $\tilde{X}$.

We now claim the following:

Claim: With the above notation:
(a) C and C' are disjoint.
(b) If $\epsilon(D, D') > 0$, then $\epsilon(C, E)$ and $\epsilon(C', E)$ are strictly smaller than $\epsilon(D, D')$.

<u>Proof of claim:</u> Since the assertions are local (disjointness is a local property and part (b) involves calculations in local rings), we will assume that $X = \text{Spec A}$ and that $D = \text{div}(a)$ and $D' = \text{div}(a')$ (this means that all local equations for D are given by a and all local equations for D' are given by a').

(a) If $I = (a, a')$ then from the blow-up construction, we get that $\tilde{X} = \text{Proj} \oplus_{n \in \mathbb{N}} I^n$. From the graded algebra surjection $A[S,T] \to \oplus I^n$ given by sending S to a and T to a', we get a closed immersion $f : \tilde{X} \hookrightarrow X \times \mathbb{P}^1$ making the following diagram commute:

$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[d, "\pi"'] & X \times \mathbb{P}^1 \arrow[dl, "\text{pr}_1"] \\ X & \end{tikzcd}$$

By definition of the surjection, $\tilde{X}$ is contained in the subscheme $X \times \mathbb{P}^1$ where $a'S - aT$ vanishes. Let $\mathcal{O}(1)$ be the pullback of the standard line bundle on $\mathbb{P}^1$ to X and let s, t be sections of $\mathcal{O}(1)$ induced by S, T. We show that C is the zero scheme $Z(s)$ of s and similarly that $C' = Z(t)$. On the affine patch $s \neq 0$, $a' = \frac{t}{s} \cdot a$ and so $(a, a') = (a, \frac{t}{s} \cdot a) = (a)$ as ideals and so $\pi^* D = E$ on the affine patch $s \neq 0$. On the affine patch $t \neq 0$, $a = \frac{s}{t} \cdot a'$ and so $(a) = (\frac{s}{t} a') = (\frac{s}{t})(a') = (\frac{s}{t})(a, a')$ which implies that $\pi^* D = Z(s) + E$ on $t \neq 0$. Thus, overall, $\pi^* D = Z(s) + E$ and similarly, $\pi^* D = Z(t) + E$ and so $C = Z(s)$ and $C' = Z(t)$ as claimed. Since $Z(s) \cap Z(t) = \emptyset$, $C \cap C' = \emptyset$ as well.

(b) Suppose for contradiction that $\epsilon(C, E) \geq \epsilon(D, D') > 0$. Choose a codimension subvariety $\tilde{V}$ of $\tilde{X}$ contained in $C \cap E$ such that $\text{ord}_{\tilde{V}} C \cdot \text{ord}_{\tilde{V}} E = \epsilon(C, E)$. Note that $\text{ord}_{\tilde{V}} E$ must then be greater than zero. From part (a), we see that $C \subseteq X \times \{0\}$ and $C' \subseteq X \times \{\infty\}$. Then $V := \pi(\tilde{V})$ is a codimension one subvariety of X. Since $E = \pi^{-1}(D \cap D')$ and $\tilde{V} \subseteq E$, $\pi(\tilde{V}) \subseteq D \cap D'$. We know that $\pi^* D = E + C$ and using the projection formula in the same way as in Lemma 3.1, $[D] = \pi_*([E] + [C])$. Thus, $\text{ord}_V D = \text{ord}_V E + \text{ord}_V C$. Since there could be other codimension 1 subvarieties mapping onto V from inside C and E which would increase the coefficient of $[V]$ inside the pushforward, we have

$$\text{ord}_V D \geq \text{ord}_{\tilde{V}} E + \text{ord}_{\tilde{V}} C.$$

By a similar argument,

$$\text{ord}_V D' \geq \text{ord}_{\tilde{V}} E + \text{ord}_{\tilde{V}} C'.$$

Thus:

$$\begin{aligned}\epsilon(D,D') &\geq \mathrm{ord}_V D \cdot \mathrm{ord}_V D' \\ &\geq (\mathrm{ord}_{\tilde{V}} E + \mathrm{ord}_{\tilde{V}} C)(\mathrm{ord}_{\tilde{V}} E + \mathrm{ord}_{\tilde{V}} C') \\ &= (\mathrm{ord}_{\tilde{V}} E)^2 + \mathrm{ord}_{\tilde{V}} E \mathrm{ord}_{\tilde{V}} C + \mathrm{ord}_{\tilde{V}} E \mathrm{ord}_{\tilde{V}} C' + \mathrm{ord}_{\tilde{V}} C \mathrm{ord}_{\tilde{V}} C' \\ &= (\mathrm{ord}_{\tilde{V}} E)^2 + \epsilon(C,E) + \mathrm{ord}_{\tilde{V}} E \mathrm{ord}_{\tilde{V}} C' + \mathrm{ord}_{\tilde{V}} C \mathrm{ord}_{\tilde{V}} C' \\ &\geq (\mathrm{ord}_{\tilde{V}} E)^2 + \epsilon(C,E) \qquad \text{(Since } E, C, C' \text{ are effective)}.\end{aligned}$$

As noted before, $\mathrm{ord}_{\tilde{V}} E > 0$ which implies that $\epsilon(D,D') > \epsilon(C,E)$ contradicting the first line in part (b). Thus, in fact $\epsilon(C,E) < \epsilon(D,D')$ as desired. By a similar argument, $\epsilon(C',E) < \epsilon(D,D')$. This finishes the proof of our claim. □

<u>Case 2:</u> D and D' are effective. We proceed by induction on $\epsilon(D,D')$. When $\epsilon(D,D') = 0$, we are back in Case 1. Suppose that $\epsilon(D,D') = n$ and the theorem holds whenever $\epsilon(D,D') < n$ (Inductive hypothesis). Blow up X along $D \cap D'$. By part (b) of the previous claim, $\epsilon(E,C'), \epsilon(C,E) < \epsilon(D,D') = n$. By the induction hypothesis, the theorem holds for (E,C') and (C,E). Now the theorem is true for (E,E) since $E \cdot [E] = E \cdot [E]$. The theorem is also true for (C,C') since $C \cap C' = \emptyset$ by part (a) of the previous claim and so $C \cdot [C'] = C' \cdot [C] = 0$. By Lemma 3.1, the proof of this case is complete.

<u>Case 3:</u> Either D or D' is effective. Without loss of generality, assume D' is effective. Let $\mathcal{J}$ be the ideal sheaf of denominators for D. In particular, this means that if $U = \mathrm{Spec}\ A$ is open affine where D has local equation d, then:

$$\mathcal{J}(U) = \{a \in A | ad \in A\}.$$

Let $\pi : \tilde{X} \to X$ be the blow-up of X along $\mathcal{J}$ and let E denote the exceptional divisor. Then $\pi^* D + E$ is an effective divisor by definition of $\mathcal{J}$ so that

$$\pi^* D = F - E$$

for an effective Cartier divisor F on $\tilde{X}$. Note that $|F| \cup |E| \subseteq \pi^{-1}(|D|)$ and that the theorem also holds for $(F,0)$ and $(E,0)$. Moreover, since F, E and $\pi^* D'$ are all effective, we can use Case 2 to conclude that the theorem holds for $(F, \pi^* D')$ and $(E, \pi^* D')$. Thus, in the setting of Lemma 3.1, if we let $B = F$, $C = E$, $B' = \pi^* D'$ and $C' = 0$, we complete the proof for this case.

<u>Case 4:</u> D and D' are arbitrary Cartier divisors. As in Case 3, let $\mathcal{J}$ be the ideal sheaf of denominators for D, $\pi : \tilde{X} \to X$ be the blow-up

of X along $\mathcal{J}$, E the exceptional divisor and so:

$$\pi^* D = F - E$$

for an effective Cartier divisor F on $\tilde{X}$. Since both F and E are effective, the theorem holds for $(F, \pi^* D')$ and $(E, \pi^* D')$ by Case 3. The theorem also holds for $(F, 0)$ and $(E, 0)$. Thus, in the setting of Lemma 3.1, if we let $B = F$, $C = E$, $B' = \pi^* D'$ and $C' = 0$, we complete the proof for this case.

This completes the proof of the Commutativity Theorem in its entirety. □

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