Basepoint-free Theorem

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In this short note, we will present the proof of basepoint-free theorem, adopated from Kollar-Mori, which adopted from Reid. We will work over character 0.

Theorem 0.1 (Basepoint-free)**.** Let (X, Δ) be a proper klt pair with Δ effective. Let D be a nef Cartier divisor such that $aD - K_X - \Delta$ is nef and big for some $a > 0$. Then $|bD|$ has no basepoints for all $b \gg 0$.

This theorem is one of the three theorems needed to prove general cone theorem. For the proof, we will use Non-vanishing Theorem, but we will prove this first following Kollar-Mori.

Proof. The proof will have several steps.

Step 1. As $aD - K_X - \Delta$ is big and nef, by theorem 2.61 of Kollar-Mori, there exists a log resolution $f : X \to Y$ such that
(1) $K_Y \equiv f^*(K_X + \Delta) + \sum a_j F_j$ with all $a_j > -1$ as (X, Δ) is klt pair and Δ is effective.
(2) $f^*(aD - (K_X + \Delta)) - \sum p_j F_j$ is ample for some $a > 0$ and for suitable $0 < p_j \ll 1$.

Note, the F_j need not be f-exceptional. Then on Y, we can write

$$f^*(aD - (K_X + \Delta)) - \sum p_j F_j$$

$$= af^*D + \sum (a_j - p_j)F_j - (f^*(K_X + \Delta) + \sum a_j F_j)$$

$$\equiv af^*D + G - K_Y,$$

where $G = \sum (a_j - p_j)F_j$. By assumption, $\lceil G \rceil$ is an effective $f-$exceptional divisor ($a_j > 0$ only when F_j is $f-$exceptional as Δ is effective), $af^*D + G - K_Y$ is ample, and $H^0(Y, mf^*D + \lceil G \rceil) = H^0(X, mD)$. Then by nonvanishing theorem, $|mD| \neq 0$ for $m \gg 0$.

Step 2. For $s \in \mathbb{Z}_{>0}$, let $B(s)$ be the (reduced) base locus of $|sD|$. Then obviously $B(s^u) \subset B(s^v)$ for any positive integers $u > v$. Then Noetherian induction implies that the sequence $B(s^u)$ stablizes, and we call the limit B_s. So either B_s is non-empty for some s or B_s and $B_{s'}$ are empty for two relatively prime integers s and s' (by number theoretic reason). For the latter case, take u and v such that $B(s^u)$ and $B(s'^v)$ are empty, and use some basic number theory we can conclude that $|mD|$ is basepoint-free for $m \gg 0$. Then we get the conclusion in this case.

For the other case, let $m = s^u$ such that $B(m)$ stablizes at this point and by assumption, it is nonempty. Then we take the linear system obtained form the Nonvanishing theorem, and we make a further blow up to obtain a new $f : Y \to X$ for which the conditions in step 1 is still satisfied, also for some $m > 0$,

$$f^*|mD| = |L|(moving\ part) + \sum r_j F_j(fixed\ part)$$

with $|L|$ basepoint free. Therefore the base locus of $|mD|$ is given by $\cap\{f(F_j)|r_j > 0\}$. Then we will find a contricition by finding some F_j such that $r_j > 0$ such that, for all $b \gg 0$, F_j is not contained in the base locus of $|bf^*D|$.

Step 3. For an integer $b > 0$ and rational number $c > 0$, such that $b > cm+a$, we define divisors

$$\begin{aligned} N(b,c) := bf^*D - K_Y + \sum 9 - cr_j + a_j - p_j)F_j \\ \equiv (b - cm - a)f * D \\ +c(mf^*D - \sum r_j F_j) \\ +f^*(aD - K_X - \Delta) - \sum p_j F_j, \end{aligned}$$

where the first line is nef by $b > cm + a$, the second line is basepoint free by Step 2, and the third line is ample by step 1. Thus $N(b,c)$ as a classical Hartshorne exercise and nef+ample is ample. Then by Kodaira vanishing theorem, $H^1(Y, \lceil N(b,c)\rceil + K_Y) = 0$, and

$$\lceil N(b,c)\rceil = bf^*D + \sum \lceil -cr_j + a_j - p_j\rceil F_j - K_Y.$$

Step 4. c and p_j can be chosen so that $\sum(-cr_j + a_j - p_j)F_j = A - F$ for some $F = F'_j$, where $\lceil A\rceil$ is effective and A does not have F as a component. In fact, we choose $c > 0$ so that $min_j(-cr_j + a_j - p_j) = -1$. If the minimum is acheived nonuniquely, we choose p_j slightly different to get uniqueness as we have freedom of p_j. This j satisfies $r_j > 0$ and $\lceil N(b,c)\rceil + K_Y = bf^(D + \lceil A\rceil - F$. Then step 3 tells us that

$$H^0(Y, bf^*D + \lceil A\rceil) \to H^)(F, (bf^*D + \lceil A\rceil)|F)$$

is surjective for $b \geq cm + a$.

Step 5. Notice that

$$N(b,c)|F = (bf^*D + A_F - K_Y)|F = (bf^*D + A)|F - K_F.$$

So we can apply the nonvanishing theorem on F to get $H^)(F, (bf^*D + \lceil A\rceil)|F) \neq 0$. So $H^)(Y, bf^*D + \lceil A\rceil)$ has a section not vanishing on F. Since $\lceil A\rceil$ is $f-$exceptional and effective,

$$H^)(Y, bf^*D + \lceil A\rceil) = H^0(X, bD).$$

So $f(F)$ is not contained in the base locus of $|bD|$ for all $b \gg 0$. This completes the proof. □

One of the application of the Basepoint-free Theorem is provided in the book

Theorem 0.2. Let (X, Δ) be a proper klt pair, Δ effective. Assume that $K_X + \Delta$ is nef and big. Then the canonical ring, defined as

$$\oplus_{m=0}^{\infty} H^0(X, O_X(mK_X + \lfloor m\Delta \rfloor)),$$

is finitely generated over $\mathbb{C}$.

Remark 0.1. The canonical ring given as above is idded a ring, as $\lfloor m_1\Delta \rfloor + \lfloor m_2\Delta \rfloor \leq \lfloor m_1 + m_2\Delta \rfloor$. Another remark is that the general case is proved by BCHM and Siu.

Proof. By basepoint-free theorem, there is an $r > 0$ such that $r\Delta$ is an integral divisor and $O_X(rK_X + r\Delta)$ is generated by global sections by taking $D = K_X + \Delta$. These sections define a morphism $f : X \to Z$ and there is an ample invertible sheaf L on Z such that $f^*L = O_X(rK_X + r\Delta)$. Let $G_m = f_*O_X(mK_X + \lfloor m\Delta \rfloor)$. Then

$$\oplus_{m=0}^{\infty} H^0(X, O_X(mK_X + \lfloor m\Delta \rfloor)) = \oplus H^0(Z, G_m).$$

The G_m are coherent sheaves and $G_{m+r} \cong G_m \otimes L$ by projection formula and $r\Delta$ is integral. Since L is ample, $R = \oplus H^0(Z, L^m)$ is a finitely generated ring over $\mathbb{C}$ and $\oplus H^0(Z, G_{j+rm})$ is a finitely generated $R-$module for every $0 \leq j < r$. Thus

$$\oplus H^0(Z, G_m) = \oplus_{j=0}^{r-1}(\oplus H^0(Z, G_{j+rm}))$$

is a finitely generated ring over $\mathbb{C}$. $\square$

The above theorem is a special case of the following conjecture:

Conjecture 0.1 (Abundance Conjecture)**.** Let (X, Δ) be a proper log canonical pair, Δ effective. Then
(1) $\oplus_{m=0}^{\infty} H^0(X, O_X(mK_X + \lfloor m\Delta \rfloor))$ is a finitely generated ring.
(2) If $K_X + \Delta$ is nef then $|m(K_X + \Delta)|$ is basepoint free for some $m > 0$.

Remark 0.2. If $K_X + \Delta$ is nef then (2) implies (1). In general, the minimal model program can reduce (1) to (2). So many people call (2) abundance conjecture. The conjecture is known to be true for surfaces and 3-fold. For $\Delta = 0$ follows Miyaoka and Kawamata. The general case (1) for $\Delta = 0$ is done by BCHM and Siu. Siu claimed a proof of the abundance conjecture.