Derived Categories and T-Structures

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In this short note, we will summerize some results of derived categoriies and introduce T-structures. This will follow the note written by Arend Bayer.

1 Basics of derived categories

We recall some facts of homological algebra and derived categories.

Let $\mathcal{A}$ be an abelian category, like $Coh(X)$ or $Rep(Q)$ for some projectiive vareity X or quiver Q. We assume $\mathcal{A}$ has enough injectives. One should note that this is not alwyas ture. Like $Coh(X)$ does not have enough injectives always. One can see Huybrechets for the treatment of $D_{Coh}(X)$.

Then we construct the bounded deriived category $D^b(\mathcal{A})$ as follows:
(1) We let $C^b(\mathcal{A})$ be the cateogry of bounded complexes, with objects are bounded chain complexes, and cohomologically finite. Morphisms are morphism of chain complexes.
(2) The homotopy category $K^b(\mathcal{A})$ is denoted to be the quotient category of above by chain homotopy. (I am not quite sure whether I use the correct term, but hope one can understand.)
(3) $D^b(\mathcal{A})$ is obtained by let morphisms of kind of roof, i.e., a morphism $A^\cdot \to B^\cdot$ is a roof $C^\cdot \to A^\cdot$ and $C^\cdot \to B^\cdot$, with the first is a quasi-isomorphism, i.e., the induced map on cohomology is identity.

We have the following facts to shorten our proof:

Proposition 1.1. (1) If $A^\cdot$ is any complex in $D^b(\mathcal{A})$ and $I^\cdot$ is a complex of injectives, then

$$Hom_{D^b(\mathcal{A})}(A^\cdot, I^\cdot) = Hom_{K^b(\mathcal{A})}(A^\cdot, I^\cdot).$$

(2) If $P^\cdot$ is a complex of projectives, then

$$Hom_{D^b(\mathcal{A})}(P^\cdot, A^\cdot) = Hom_{K^b(\mathcal{A})}(P^\cdot, A^\cdot).$$

One basic result in homological algebra is that when the category has enough injectives, then any complex is quasi-isomorphic to a complex of injectives. Then we have the following beautiful fact:

Proposition 1.2. Let $F \in \mathcal{A}$, let $F[i]$ be the complex with $-i-$th position F and other positions are zero. If the category has engouth injectives, we have

$$Hom_{D^b(\mathcal{A})}(F[p], G[q]) = Ext^{q-p}(F, G)$$

if $p \leq q$ and 0 otherwise.

Proof. As it has enough injectives, we replace G with an injective complex that is quasi-isomorphic to G. If $p > q$, then the only map between these two complexes is the zero map as the first nonzero position of the injective resolution is at position $-q$. Otherwise, this is just the definitioin of the Ext functor. (Actually the last step need some more argument, but are fundamental.) $\square$

We give an example to see how derived category help the language of homological algebra.

Example 1.1. Let $0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0$ is a short exact sequence of complexes, then there is a map $C^{\cdot} \to A^{\cdot}[1]$ in $D^b(\mathcal{A})$.

When the complex has only one degree, say $0 \to A \to B \to C \to 0$, then C is quasi-isomorphic to the comple $A \to B$, which has a natural map to $A[1]$.

A basic fact is that for a short exact sequence, we can replace $C^{\cdot}$ with the so called mapping of the complex $A^{\cdot} \to B^{\cdot}$. We omit the detailed construction, but only note that it is an extension of $Ext^1(A^{\cdot}[1], B^{\cdot})$ as we are in a triangulated category, which means that the mapping cone is determined by $f \in Hom(A^{\cdot}, B^{\cdot}) = Ext^1(A^{\cdot}[1], B^{\cdot})$. In particular, the mapping cone splits if and only if f is zero.

Thus by above, we have so called exact triangle, which is central to the definition of triangulated category: $A \to B \to C \to A[1]$. This is used to replace the concept of exact sequence in derived category, as for general case, a derived category need not be abelian.

2 Octahedral axiom

The octahedral axiom is one of the axioms show up in derived category. It is quite complicated to fomulate (one can see a octahedral in Wikipedia, and that is how the name comes from). However, the idea is very simple. We slightly mention it here.

The idea is that given a morphism of chain complex $A \to B \to C$, is the cone of the maps has some relations? The answer is that the cone forms an exact triangle. As I do not know how to draw a graph, I omit the graph.

Take a special case, $D^b(\mathcal{A})$, and A, B, C are three objects, regarded as complex in degree 0, and we have a map $A \to B \to C$, then the octahedral axiom is just $(C/A)/(B/A) = C/B$.

3 Filtration by cohomology

The fact that two chain complexes with same cohomology need not be quasi-isomorphic tells us that there are more information in the chain complex than the cohomology groups. However we have the following proposition:

Proposition 3.1. Given a complex $E \in D^b(\mathcal{A})$, there exists a sequence of maps

$$0 \to E_k \to E_{k-1} \to ... \to E_{j+1} \to E_j = E$$

and exact triangles $E_i \to E_{i-1} \to H^{-i+1}(E)[k-1] \to E_i[1]$.

We call this the filtration of E by its cohomology objects.

Arend said that the cohomology filtration is extremely useful, especially when the proof involves elementary spectral sequence argument. Let's see as we proceed.

4 Bounded t-structure

The motiviation of this is: Suppose we have an equivalence of derived categories $D^b(\mathcal{A}) \sim D^b(\mathcal{B})$, how should we understand the image of $\mathcal{A}$ in $D^b(\mathcal{B})$?

Definition 4.1 (Heart of $t-$structure)**.** The heart of a bounded $t-$sturcture in a triangulated category $\mathcal{D}$ is a full additive subcategory $\mathcal{A}^{\#}$ such taht
(1) For $k_1 > k_2$, we have $Hom(\mathcal{A}^{\#}[k_1], \mathcal{A}^{\#}[k_2]) = 0$.
(2) For every object E in $\mathcal{D}$ there are integers $k_1 > k_2 > ... > k_n$ and a sequence of exact triangles

$$0 = E^0 \to E^1 \to E^2 \to ... \to E^{n-1} \to E^n = E$$

with distinguished triangles $E^0 \to E^1 \to A_1$, with $A_i \in \mathcal{A}^{\#}[k_i]$.

Remark 4.1. (1) A bounded $t-$structure is determined by its heart. So we only introduce the heart of such structure.

(2) By proposition 1.2 and proposition 3.1, the subcategory $\mathcal{A}[0] \subset D^b(\mathcal{A})$ is the heart of a $t-$structure.

(3) The herat $\mathcal{A}^{\#}$ is abelian: A morphism $A \to B$ of two objects is defiend to be an inclusion if its cone is also in $\mathcal{A}^{\#}$, and it is defined to be surjective if the cone is in $\mathcal{A}^{\#}[1]$. (I did not see this point so far ....)

(4) The objects A_i are called the cohomology objects $H^i_{\#}(E)$ of E with respect to $\mathcal{A}^{\#}$. They are functorial and indueces a long exact cohomology sequence for any exact triangle.

One can construct many nontrivial $t-$structures are given by tilting at a torsion pair.

Definition 4.2. A torsion pair in an abelian category $\mathcal{A}$ is a pair (T, F) of full additive subcategories with

(1) $Hom(\mathcal{T}, \mathcal{F}) = 0$

(2) For all $E \in \mathcal{A}$ there exists a short exact sequence

$$0 \to T \to E \to F \to 0$$

with $T \in \mathcal{T}, F \in \mathcal{F}$.

Property (1) implies that the short exact sequence in (2) is unique and functorial.

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5 Examples

(1) The canonical example of a torsion pair is $\mathcal{A} = Coh(X)$, where we defien $\mathcal{T}$ to be torsion sheaves and $\mathcal{F}$ to be torsion-free sheaves.

(2) Consider a finite quiver $Q = (Q_0, Q_1)$ with relations R, and assume that the vertex n is a sink, i.e., it has no outgoing arrows. Then let $\mathcal{T}$ be the subcategory of representations $\underline{V}$ concentrated at vertex n, i.e., with $V_i = \mathbb{C}^0$ for $i \neq n$, and $\mathcal{F}$ be the subcategory of representations $\underline{V}$ with $V_n = \mathbb{C}^0$. As n is assumed to be a sink, any representation $\underline{V}$ has a subrepresentation $(\mathbb{C}^0, ..., \mathbb{C}^0, V_n)$, inducing the short exact sequence (2).

(3) There are two ways to generalize the previous construction to the case where n is allowed to have outgoing arrows; for simplicity we will assume that there are no loops going from the vertex n to itself:

(a) Let $\mathcal{T}$ consist of representation generated by V_n, and $\mathcal{F}$ of representations with $V_n = \mathbb{C}^0$ as before.

(b) Let $\mathcal{T}$ consist of representations concentrated at the vertex n, i.e., it consist of the direct sums $S_n^{\oplus k}$ where S_n is the simple one-dimensional representation concentrated at vertex n. Let $\mathcal{F}$ be the subcategory of representations with $Hom(\underline{S}_n, \underline{V}_n = 0$ for the equivalently, we can characterize $\mathcal{F}$ as the set of representations for which the intersection of the kernels of all maps $\phi_j : V_n \to V_i$ going out of V_n is trivial.

(4) Again consider a finite quiver, with possibly relations R, and again assume that vertex n has not loops. This time we let $\mathcal{F}$ consist of $S_N^{\oplus k}$, and $\mathcal{T}$ consist of representations with $Hom(\underline{V}, S_n) = 0$; more explicitly, $\underline{V} \in \mathcal{F}$ if the images of the ingoing maps $\phi_i : V - i \to V_n$ span V_n.

(5) Let $\mathcal{A} = Coh(X)$, where X is a smooth curve, and $\mu \in \mathbb{R}$ is a real number. Let $\mathcal{A}_{\geq \mu}$ be the subcategory generated by torsion sheaves and vector bundles all of whose $HN-$filtration quotiens have slope $\geq \mu$, and similarly for $\mathcal{A}_{<\mu}$. Then $()\mathcal{A}_{\geq \mu}, \mathcal{A}_{<\mu})$ is a torsion pair: (1) follows form the lemma of the morphisms, and (2) is botained by collapsing the $HN-$filtration into two parts: let $T = E_i$ for I maximal such that $\mu_i \geq \mu$.

Now we show how to give a heart of a bounded $t-$structure form these data.

Proposition 5.1. Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, the following defines the heart of a bounded $t-$structure in $D^b(\mathcal{A})$:

$$\mathcal{A}^{\#} := \{E \in D^b(\mathcal{A}) | H^0(E) \in \mathcal{T}, H^{-1}(E) \in \mathcal{F}, H^i(E) = 0 \; for \; i \neq 0, -1\}.$$

Objects in $\mathcal{A}$ can be thought as an extension of F by T determined by an element in $Ext^1(F, T)$. Objects of $\mathcal{A}^{\#}$ are instead an extension of some T by some $\mathcal{F}[\infty]$, determined by $Ext^1(T, F[1]) = Ext^2(T, F)$. More concretely, every object in $\mathcal{A}^{\#}$ can be represented by a two-term complex $E^{-1} \xrightarrow{d} E^0$ with $ker(d) \in \mathcal{F}$ and $coker(d) \in \mathcal{T}$. There is a very beautiful picture in Arend's note, one can check. (Tilt is a very fancy English word to me. I never saw this word until I study stability.)

In the picture, there are no morphiisms form the left to ther right, and any objects can be written as a successive extension of objects cotained in one of the building blocks, starting with its right-most building block and extending it by objects further and further to the left. This picture also tells how to prove the proposition: the Hom-vanishing follows by extending known Hom-vanishings to extensions, and the filtration step $E^{\#}_K$ of E with respect to $\mathcal{A}^{\#}$ is given by an extension of the filtration step E^{k+1} with respect to $\mathcal{A}$ and the torsion part of $H^k(E)$.

There is a more general result associated to this, will study in the future.

6 Exercises

Exercise 6.1. Prove proposition 1.2.

Proof. This is an elanged proof of that part. For $p > q$ the reason is trivial. We prove the other part. First, we replace G with an injective resolution $I^{\cdot}$, and put the shift, i.e., put $I^{\cdot}[q]$, and we would like to compute $Hom(F[p], I^{\cdot}[q])$. First observe that we resolve two shift by one, making it into $Hom(F, I^{\cdot}[p-q])$ which is obvious. Then we compute this one. To give such a map $f^{\cdot}$, it is equivalent to give a compatible morphism $f : F \to I(p-q)$. It is compatible means that $d_I \circ f = 0$ by conductivity of the chain map. Also it is 0 means that it is chain homotopic to 0, this means that there is a map $g : F \to I^{p-q-1}$, and $d_I \circ g = f$. This exact means that f is an element of $Ext^{p-q}(F, G)$ by the defition that resolving G by an injective resoltuion. Then we get the result by direct applying proposition 1.1. $\square$

Exercise 6.2. Translate the lemma of 9 to a triangulated cateogry, and prove it.

Well, I do not understand which lemma. I will add it up when I understand. Please email me if you know which one.

Exercise 6.3. Pick a spectral sequence proof in a derived categories textbook and replace it by an argument using the filtration by cohomology.

Proof. This example is very interesting. Let me try this. [Corollary 2.68 of Hybrechets] Suppose $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ is an exact functor which admits a right derived functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$, and assume that $\mathcal{A}$ has enough injectives. Suppose that $\mathcal{C} \subset \mathcal{B}$ is a thiick subcategory with $R^iF(A) \in \mathcal{C}$ for all $A \in \mathcal{A}$ and there exists $n \in \mathbb{Z}$ with $R^iF(A) = 0$ for $i < n$ and all $A \in \mathcal{A}$. Then RF takes values in $D^+_{\mathcal{C}}(\mathcal{B})$, i.e., $RF : D^+(\mathcal{A}) \to D^+_{\mathcal{C}}(\mathcal{B})$.

I will change the statement into D^b as the statement we have is for a bounded version. Then take $A \in \mathcal{A}$, we apply R^iF to the filtration of A. As it is a functor, we have a sequence $R^iF(A_i)$, where A_i is the filtered object. As right derived functor of an excat functor is exact (I forget whether we have a defintion for right derived functor for a nonexact funcotr, never mind.), we keep exact triangles. By a tracking of cohomology, we get the result. $\square$

Remark 6.1. I do not know whether it is because of the cohomology filtration makes it easy or just because the original statement is also an easy statement.

Exercise 6.4. Use the filtration by cohomology to show that for a smooth projective curve X, every object in $D^b(X)$ is the direct sum of its cohomology sheaves. (The same statement holds for the cateogry $Rep(Q)$ of representations of a quiiver Q without relations.)

Proof. We do this by induction on the length of cohomological filtration. When there is only one, it is obvious. Now suppose that we have already done for cohomological length n, we prove it for an object in the derived category with cohomological length $n+1$. By induction, we know that $E_n = \oplus H^i(E)[-i]$, which $i \leq n$. Then as we have an exact triangle, we just need to show that $Ext^1(H^{n+1}(E)[-n-1], E_n) = 0$, then we get splitness. As the second one is a direct sum, we just need to prove $Ext^1(H^{n+1}(E)[-n-1], H^i(E)[-i]) = 0$ for $i \leq n$. Then this is equal to $Ext^1(H^{n+1}(E), H^i(E)[n+1-i]) = Ext^{1+n+1-i}(H^{n+1}(E), H^i(E))$. However, $1+n+1-i > 1$, and as the curve is smooth, every local ring is regular local ring, then $pd(F) < 2$, so such a term is 0. Then we get the result.

For the quiver with no relations, every representation can split into funcamental ones, thus we are done. □

Exercise 6.5. (1) Prove that if $\mathcal{A}^\# \subset \mathcal{D}$ is a heart of a bounded $t-$structure, $A \to B \to C$ is an exact triangle in $\mathcal{D}$ with $A, B \in \mathcal{A}^\#$, then the cohomology object $H^i_\#(C)$ of C are zero except for $i = 0, -1$.

(2) Show that $\mathcal{A}^\#$ is an abelian category if we define the kernel of $f : A \to B$ to be $H^{-1}_\#(conef)$ and the cokernel to be $H^0_\#(conef)$

Proof. (1) A filtration is given by $0 \to B \to C$, with $A_1 = A[1]$ and $A[0] = B$. Then we get the result. (I think the difference is due to the definition and the grading. I will correct it after I find the correct one. (Possibly from my second supervisor Chunyi Li..........................).

(2) The checking is very length. We just show that what we get is really kernal and image. First, as we take

I am confused about the whole exercise. Will update it later. □

Exercise 6.6. Consider one of the examples (3)(a), (3)(b), or (4), and prove that it is a torsion pair in the category of $Rep(Q)$ of a quiver Q. Prove that it is indeed a torsion pair, and check the explicit descriptions of $\mathcal{F}$ and $\mathcal{T}$.

Proof. They are quite similar, so I just show the first one. For any quiver representation, we take out the largest subrepresentation generated by V_n. Then it belongs to $\mathcal{T}$, and the quotient is in $\mathcal{F}$ as we take out the largest. Also, $Hom(\mathcal{F}, \mathcal{T}) = 0$ is obvious. So we are done. □

Exercise 6.7. Let $X = \mathbb{P}^1$, $\mathcal{A} = Coh(X)$, and let $\mathcal{A}^\#$ be the heart of the tilting. Let Q be Kronecker quiver, and let $\Phi_T : D^b(X) \to D^b(Rep(Q))$ be the equivalence of the derived category induced by tilting bundle $O \oplus O(1)$. Show that $\mathcal{A}^\#$ is the inverse of the heart of the standard $t-$strucure.

Let me know what is the standard $t-$strucure first..............

Exercise 6.8. Consider an elleptic curve E, and its autoequivalence induce by the Fourier-Mukai transform associated to the Poincare line bundle. Determine the image of the standard $t-$strucure.

Again, let me know what is the standard $t-$strucure first..............