Space of Stability Conditions

Shengxuan Liu

June 16, 2020

This is basically part 5, which we apply the Bridgeland stability to get some results.

1 Deformation Result

We mention here that how the extension of stability condition to triangulated category gives us more flexibility of the deformation.

We will restrict our attention to stability condition $\sigma = (Z, \mathcal{P})$ satisfying two additional assumptions:

(1) We fix a finite-dimensional lattice Λ, a map $\lambda : K(\mathcal{D}) \to \Lambda$, and we assume that our stability conditions factor through this map. If $K(\mathcal{D})$ is finite dimensional, then this assumes nothing. If it it infinite dimensional, a typical choice is the numerical Grothendieck group.

(2) Let $|| \cdot ||$ be arbitrary fixed norm on $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$. (I think any norm is fine, as we only use it to define a topology, and on finite dimensional vector space, any two norms are equivalent.) We assume that σ satisfies the support property:

$$inf\{\frac{|Z(E)|}{||[E]||} | E\ is\ \sigma - semistable\} > 0.$$

Here we use the notation $[E]$ actually means $\lambda([E])$. Then we use $Stab(\mathcal{D})$ to denote the set of all the stability condition satisfy the above two conditions (we omit (Λ and similar restrictions in the notation.) The result main result in this section is that this is a complex manifold and structure of it.

We define a general metric on between slicings by

$$d_S(\mathcal{P}, \mathcal{Q}) = sup_{E \neq 0 \in \mathcal{D}}\{|\phi^-_{\sigma^2}(E) - \phi^-_{\sigma^2}(E)|, |\phi^+_{\sigma^2}(E)| - \phi^+_{\sigma^2}(E)\} \in [0, \infty],$$

where σ^i are stability conditions asscociated to the two different slicings. Then we define a metric on $Stab(\mathcal{D})$ by combining this metric with the metric of dual lattice, i.e.,

$$d(\sigma, \tau) = sup_S\{d_S(\mathcal{P}, \mathcal{Q}), ||Z - W||\}.$$

Then we define a topology on $Stab(\mathcal{D})$. Then we have the following main theorem, inspired by Douglas' $\pi-$stability:

Theorem 1.1. The space $Stab(\mathcal{D})$ is a complex manifold of finite dimensional, with the natural map

$$Z : Stab(\mathcal{D}) \to Hom(\Lambda, \mathbb{C})$$

with the projection to the stability function is a local homeomorphism.

Then this theorem tells us that we deform the stability condition on a triangulated category is enough to deform the stability function. This result also tells us that the complex structure on $Stab(\mathcal{D})$ is the unique stability condition determined by the condition that the function of evaluation of E is holomorphic. The proof will be at the end of this note.

We already see that we can deform the stability condition on the representation of a quiver by choosing z_i freely on the upper half plane. Then the theorem tells us that we can also deform z_i to lower half plane.

2 Example 1: Wall-Crossing within an Abelian Catogory

Here we show one interesting phenomenon can happen in a deformation, called wall crossing. Suppose we have a deformation form Z to W, such that we do not change the heart $\mathcal{A}$ of stability condition. Suppose we have a short exact sequence $0 \to A \to B \to C \to 0$, such that there are no nontrivial subobjects of $[A]$ and $[C]$. We say there is a wall between Z and W if the orientation of the stability parallelogram change the orientation, i.e., the image of $0, A, B, C$ under stability function. There are two cases for B, stable, which gives it in one side of the wall, and B is unstable, and $0 \to A \to B$ gives the HN filtration of B, as $C = B/A$ is semistable by trivial reason. This is the most trivial case of a wall0crossing, but this shows that we can hope to get a new stability after we deform the stability function.

3 Example 2: Stability and Tilting for Quivers

Consider $\mathcal{A} = Rep(Q)$, where Q is a quiver, and the stability on $D^b(Rep(Q))$ is constructed as an earlier example. Now we want to see what will happen if we deform one z_n into negative half plane. Suppose now that all other z_i keep in the upper half plane. Let S_n be the simple one-dimensional representation supported at the vertex n.

(1) If z_n crosses the negative real line, say form $-x + i\epsilon$ to $-x - i\epsilon$, then $\mathcal{A} = \mathcal{P}((0, 1])$ is replaced by $\mathcal{Q}((0, 1]) = \mathcal{A}^{\#}[-1]$, where $\mathcal{A}^{\#}$ is the tilting with respect to the torsion pair $\mathcal{A} = \{S_n^{\oplus k}\}$, $\mathcal{F} = \{\underline{V} | Hom(S_n, \underline{V}) = 0\}$. Then it is obvious that when ϵ is small enough, the new central charge W satisfies the positivity property for $\mathcal{Q}((0, 1])$. On the other hand, it is also obvious that this is the only tilt choice: the object S_n is $Z-$stable, and its phase in W is larger than 1, then we should consider $S_n[-1]$ to make the phase falls in $(0, 1]$. But

other stable objects are still stable, so these objects should remain unchanged. The extension closure of these objects are given by $\mathcal{F}$. Thus we form this tilting.

(2) If z_n crosses the positive real line, the we are in the other condition. The torsion pair is dual to the above, i.e., we take $\mathcal{F} = \{S_n^{\oplus k}\}$. This time, we replace S_n by $S_n[1]$.

Then the note said that the tilted heart is again the category of a representations of a quiver. Then study space of stability condition can gives lots of information of tilting of quivers.

4 Example 3: Stability conditions for $\mathbb{P}^1$

Here we consider the category of coherent sheaves $Coh(X)$ for $X = \mathbb{P}^1$. Set the stability function

$$Z(F) = -deg(F) + z \cdot rank(F).$$

If $z = i$, we get the classical slope stable function. We would like to study what happens when z passes real line. One notes that we do not change the stable objects if we do not pass through wall. (The one in the notes is obviously wrong, as one can see from below.)

A picture of z close to real line, but still above the real line, is in the note. By Grothendieck theorem, all the vector bundle of rank higher than 1 splits. So only stable objects are line bundles and skyscraper sheaves. Now suppose we let z change form $x+i\epsilon$ to $x-i\epsilon$ vertically. Then first notice that the central charge of stable object will never be zero. Thus we need to pick $Re(z) \in \mathbb{R} \backslash \mathbb{Z}$, as otherwise $Z(O(n)) = 0$ for some n. So we pick $Re(z) \in (0, 1)$. Then we would like to see what will happen to $\mathcal{P}((0, 1])$. However, it will be easier to understand what happens to $\mathcal{P}((\frac{[}{1}]2, \frac{3}{2}])$. This consists of extension closure of $\{O(2), O(3), ...\}$ and $\{O(1)[1], O[1], O(-1)[1], ...\}$. One thing happen in this process is that the stable objects when we change z. For example, $O(n) \to O(n+1) \to O_x$ for $n \geq 2$ destabilizes $O(3), O(4), ...$ Similarly, $O_x \to O(n)[1] \to O(n+1)[1]$ destabilizes $O[1], O(-1)[1], ..,$ and $O(2) \to O_x \to O(1)[1]$ destabilizes O_x. Thus the only remaining stable objects are $O(2), O(1)[1]$. In fact, the heart $\mathcal{A}^{\#}$ is isomorphic to $Rep(P_2)$, representation of Kronecker quiver, with two arrows, and $O(2), O(1)[1]$ corresponds to the simple objects. One may say that we move from a geometric chamber to an chamber of algebraic stability conditions.

Now we describe new heart $\mathcal{A}^{\#\#} = \mathcal{Q}((0, 1])$: The only stable objects are $O(1)[1]$ and $O(2)[-1]$. Then by definition, all the objects in the heart is extension of these two objects. However, the only extension of these two objects are trivial extensions. Thus the category $\mathcal{A}^{\#\#}$ is equivalent to category of pairs of vector spaces V_0, V_1.

5 Proof of the Deformation Theorem

Here we present the proof of the main theorem. First we show the injectivity.

Lemma 5.1. If two stability condition σ, τ satisfies $d_S(\mathcal{P}, \mathcal{Q}) < \frac{1}{4}$, and $Z = W$, then $\sigma = \tau$.

Proof. Let E be a σ-semistable of phase ϕ. Then by definition of d_S, we get

$$\mathcal{P}(\phi) \subset \mathcal{Q}((\phi - \frac{1}{4}, \phi + \frac{1}{4})) \subset \mathcal{P}((\phi - \frac{1}{2}, \phi + \frac{1}{2}]) =: \mathcal{A}.$$

Any HN-factor of E with respect to τ will have central charge inside the range of the angle of $\frac{\pi}{2}$ with respect to the ray of E. Thus if E is not $t-$semistable, it has a nontrival HN-filtration, with all factors inside this range. We pick the first HN-object $A \to E$ with respect to τ, then
(1) the morphism $A \to E$ is an inclusion inside $\mathcal{A}$
(2) the phase of $Z(A)$ is bigger than the phase of E.
Then this contradicts to the fact that E is semistable. □

Then we prove the local surjectivity. The point here is that we make the ball in $\Lambda^\wedge$. The size will dependent on the following quantity:

$$S(\sigma) = inf\{\frac{|Z(E)|}{||[E]||} | E \ is \ \sigma - semistable\}.$$

It is larger than 0 is by the positivity assumption. This quantity measures how fast the central charge $Z(E)$ can vary relative to the original central charge:

Remark 5.1. If E is $\sigma-$semistable and W is another central charge close to σ by $||W - Z|| < S(\sigma) \cdot \epsilon$, then $W(E)$ contained in the ball of radius $\epsilon|Z(E)|$ around $Z(E)$.

Another measure of how unstable of an object be:

Definition 5.1 (Mass)**.** Given a stability condition $\sigma = (Z, \mathcal{P})$, and an object E, the mass is defined to be

$$m_\sigma(E) = \sum_i |Z(A_i)|,$$

where A_i are HN-factors of E.

In the proof of HN-filtration of coherent sheaf of curves, the mass is given by the length of thw left bound.

Lemma 5.2. The function $S(\sigma) : Stab(\mathcal{D}) \to \mathbb{R}_{>0}$ is continuous.

Proof. Take another stability $\tau = (\mathcal{Q}, W)$, and $d_S(\mathcal{P}, \mathcal{Q}) < \epsilon$, and $||Z - W|| < \epsilon$. Assume E is $\tau-$seimistable of phase ψ. By the definition of norm of central charge and triangular inequality, we get $\frac{|Z(E)|}{||E||} \geq \frac{|Z(E)|}{||E||} - \epsilon$. Then by definition of d_S, we get:

$$\frac{|Z(E)|}{||E||} > \frac{\sum_i |Z(A_i)| cos(2\pi\epsilon)}{||E||} > cos(2\pi\epsilon) \frac{\sum_i |Z(A_i)|}{||A_i||} > cos(2\pi\epsilon) S(\sigma).$$

Then we are done by $\delta - \epsilon$ criterion. □

Then the last step is next lemma:

Lemma 5.3. There exists $\epsilon > 0$ such that for any stability condition $\sigma = (Z, \mathcal{P}) \in Stab(\mathcal{D})$ and any group homomorphism $W : \Lambda \to \mathbb{C}$ with $||W - Z|| < \epsilon S(\sigma)$ and either

(1) $Im(W) = Im(Z)$ or

(2) $Re(W) = Re(Z)$,

then there exists a stability condition $\tau = (W, \mathcal{Q})$ with $d_S(\mathcal{P}, \mathcal{Q}) < \epsilon$.

To reduce the theorem to this lemma, we see that in a small neighbourhood, we can connect two points by horizontal then vertical lines. The thing we need to ensure is that when we move horizontally, the change of the neighbourhood we care about does not change rapidly. This is exactly what lemma 5.2 and remark 5.1 tells us.

Proof. The proof is similar to the proof of the existence of HN-filtraion on the curve. We prove case (1) here and case (2) will follow a similar argument. If $Im(Z) = Im(W)$, then W is also a stability function for $\mathcal{A}^{\#} = \mathcal{P}((0, 1])$. Then what we need to show is that W has HN-filtration property.

Let $H_W(E)$ be the convex hull of all the subobjects with respect to W. Then follow the idea of curve, we will prove that it is left bounded, and it has finite line segment of the left bound from 0 to $W(E)$.

For $0 \to A \to E$, consider $H_Z(A)$, $H_Z(W)$, and the line segments $0 \to v_1 \to v_2, .., v_n = Z(A)$ which gives the HN filtration of A with respect to Z.

The the mass $m_\sigma(A)$ is obviously bounded by the left boundary of $H_Z(E)$ up to the height of $Z(A)$. Then there is a constant $C(E)$ such that

$$Re(Z(A)) \geq C(E) + m_\sigma(A).$$

Then by remark 5.1, the real part of W is just in a small neghbourhood of Z, to be precise, $\epsilon \cdot |Z(A)|$. Then

$$Re(W(A)) \geq Re(Z(A)) - \epsilon \cdot m_\sigma(A) \geq C(E) + (1 - \epsilon) m_\sigma(A) > C(E).$$

Thus we get the left bounded of $H_W(A)$. Compare with the proof of HN-filtration of curves, the only thing left is to prove that we only have finitely many extreme points. This is by observing that for any extreme subobject A of E, $Re(A)$ is bounded by $\frac{max(0, Re(W(E)) - C(E)}{1 - \epsilon}$. As we assume $K(\mathcal{D})$ factor through a finite dimensional lattice, we get the finiteness.

The support property follows from lemma 5.2, as the proof does not need to know it is a stability condition with support property at first. ☐

6 Exercises

Exercise 6.1. Prove that $\mathcal{A} = \mathcal{P}((0, 1])$ is the heart of a bounded $t-$structure.

Proof. This is quite direct. Property (1) is satisfied for property (2) of slicing. Property (2) is satisfied by property (3). ☐

Exercise 6.2. Giving a slicing $\mathcal{P}$ with heart $\mathcal{A} = \mathcal{P}((0,1])$, and $\phi \in (0,1)$, prove that $\mathcal{T} = \mathcal{P}((\phi,1])$, $\mathcal{F} = \mathcal{P}((0.\phi])$ define a torsion pair in $\mathcal{A}$. Prove that the heart $\mathcal{A}^{\#}$ obtained by tilting at this torsion pair is equal to $\mathcal{P}((\phi,\phi+1])$.

Proof. The proof is also direct. Property (1) is due to property (2) of slicing. Property (2) is by definition of slicing. The second is also direct by difinition of tilting. □

Exercise 6.3. Find the Harder-Narasimhan filtrations of all $O_{\mathcal{P}^1}(n)$ in the algebraic stability condition considered in section 4.

Proof. For the first two cases $O(1)$, $O(2)$, this is obvious.

For $O(n)$ with $n > 2$, we want to find the HN-filtration of it. First $Hom(O(2), O(n)) = Hom(O, O(n-2)) = H^0(O(n-2))$. So $E^1 = O(2)^{n-1}$. Next, we calculate $Hom(O(n), O(1)[1]) = Ext^1(O(n), O(1)) = Ext^1(O, O(1-n)) = H^1(O(1-n))$, so $A_2 = O(1)^{n-2}$. Similar calculation we can get the one for $O(-n)$. □