Stability Conditions on a Triangulated Category

Shengxuan Liu

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This is basically part 4, which introduce the Bridgeland stability condition. In this section, we see how to construct a stability condition on a trigangulated category.

1 Definition of Stability Condition

Definition 1.1. A slicing P of a triangulated category $\mathcal{D}$ is a collection of full additive subcategories $\mathcal{P}(\phi)$ for each $\phi \in \mathbb{R}$ satisfying:

(1) $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$

(2) For any $\phi_1 > \phi_2$ we have $Hom(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$

(3) For each $0 \neq e \in \mathcal{D}$ there is a sequence $\phi_1 > \phi_2 > ... > \phi_n$ of real numbers and a sequence of exact triangles

$$0 = E^0 \to E^1 \to E^2 ... E^{n-1} \to E^n = E$$

with exact triangles $E^i \to E^{i+1} \to A_{i+1}$, and $A_i \in \mathcal{P}(\phi_i)$, and we call this Harder-Narasimhan filtration of E.

Remark 1.1. (0) Compare to the definition of heart of a bounded $t-$structure, we find out that this is kind of continuous analogue.

(1) We call the objects in $\mathcal{P}(\phi)$ semistable of phase ϕ.

(2) Given the slicing $\mathcal{P}$, the sequence of ϕ_i and the HN filtration are automatically unique. We set $\phi^+_{\mathcal{P}}(E) = \phi_1$ and $\phi^-_{\mathcal{P}}(E) = \phi_n$. Well, this is not that automatical to me, but I can use a similar method of HN-filtration to prove it. I omit my proof here.

(3) If $\mathcal{P}(\phi) \neq 0$ only for $\phi \in \mathbb{Z}$, then the slicing is equivalent to the datum of a bounded $t-$structure, with heart $\mathcal{P}(0)$.

(4) If $\phi^-(A) > \phi^+(B)$, then $Hom(A, B) = 0$.

(5)More generally, given a slicing $\mathcal{P}$, let $\mathcal{A} = \mathcal{P}((0, 1])$ be the full extension-closed subcategory generated by all $\mathcal{P}(\phi)$ for $\phi \in (0, 1]$. Then $\mathcal{A}$ is the heart of a bounded $t-$structure. In other words, a slicing is always a refinement of a bounded $t-$structure.

While this gives a notation of semistable objects and HN-filtration, we need to tell what are semistable objects at the first place. We try to go around with it with stability functions.

Definition 1.2. A stability condition on a triangulated category $\mathcal{D}$ is a pair $(Z, \mathcal{P})$ where $Z : K(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism (called central charge) and $\mathcal{P}$ is a slicing, so that for every $0 \neq E \in \mathcal{P}(\phi)$ we have

$$Z(E) = m(E) \cdot e^{i\pi\phi}$$

for some $m(E) \in \mathbb{R}_{>0}$.

It seems that we still need to give data about semistable object. But we have the following theorem by Bridgeland.

Proposition 1.1. To give a stability condition $(Z, \mathcal{P})$ on $\mathcal{D}$ is equivalent to giving a heart $\mathcal{A}$ of a bounded $t-$structure with a stability function $Z_{\mathcal{A}} : K(\mathcal{A}) \to \mathbb{C}$ such that $(\mathcal{A}, Z_{\mathcal{A}})$ have the Harder-Narasimhan property, i.e., any object in $\mathcal{A}$ has a HN-filtration by $Z_{\mathcal{A}}-$stable objects.

We will show how to get a stability condition from $(\mathcal{A}, Z_{\mathcal{A}})$.

Proof. If $\mathcal{A}$ is the heart of a bounded $t-$structure on $\mathcal{D}$, then we have $K(\mathcal{D}) = K(\mathcal{A})$, as Grotendieck group only defined up to extension.

Given $(\mathcal{A}, Z_{\mathcal{A}})$, we define $\mathcal{P}(\phi)$ for $\phi \in (0, 1]$ to be the $Z_{\mathcal{A}}-$semistable objects in $\mathcal{A}$ of phase $\phi(E) = \phi$. Then we extend this to whole real numbers by $\mathcal{P}(\phi + n) = \mathcal{P}(\phi)[n] \subset \mathcal{A}[n]$ for $\phi \in (0, 1]$ and $0 \neq n \in \mathbb{Z}$. The compatiibiliity (3) of HN filtration is satisfied by construction, so we just need to show that $\mathcal{P}$ satisfies the remaining properties. (2) is following from definition of the bounded $t-$structure and semistability. Given a $E \in \mathcal{D}$, we get a filtration given by bounded $t-$structure, and the HN-filtration $0 \to A_{i1} \to A_{i2} \to ... \to A_{im_i} = A_i$ given by the HN-property inside $\mathcal{A}$ can be combined into a HN-filtration of E: it begins with as

$$0 \to F_1 = A_{11}[k_1] \to F_2 = A_{12}[k_1] \to ... \to F_{m_1} = A_1[k_1] = E_1,$$

i.e., with the HN-filtration of A_1. then the following filtration steps F_{m_1+i} are an extensions of $A_{2i}[k_2]$ by E_1 can be constructed as the cone of the composition $A_{2i}[k_2] \to A_2[k_2] \to E_1$ the octahedral axiom shows that these have the same filtration quotients as $0 \to A_{21}[k_2] \to A_{22}[k_2]...$ continuing this we obtain a filtration of E as desired.

Conversely, given the stability condition, we set $\mathcal{A} = \mathcal{P}((0, 1])$ as before. Then we would like to see it has HN filtration property and has a stability function. The HN filtration propery is direct by taking the sequence of the definiton of slicing. Also by the definition of the stability pair, we see the central charge takes values on the upper half plane. We remain to check the compatibility of semistable objects. However, if an object is semistable with respect to $\mathcal{A}$, then it is obvious that the semistable is compatible. More precisely, taking $0 \to A \to B$, then we take the HN filtration w.r.t. triangulated category of A and B/A. Then by extension we form a HN filtration of B. □

2 Examples

If X is a smooth projective curve, and $\mathcal{D} = D^b(X)$. Let $\mathcal{A}$ be the heart of the standard $t-$structure, and $Z(E) = -deg(E) + irk(E)$. Then as Z is a stability function with HN filtration property, this gives a stability condition on $\mathcal{D}$. ITs semistable objects are shifts of slope semistable vector bundles, and shifts of trosion sheaves. Similarly, we get a stability condition on Quiver representations.

Remark 2.1. The remark said that the heart of a stability function is hard to construct, and no one has succeed to construct a heart with a stability function on Calabi-Yau threefold. I think this is not true anymore. If my understanding to Chunyi Li's paper correctly, he constructed such a stability condition and heart. Wow!