Stability in Abelian Categories

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In this short note, we will present some classical result in stability in classical setting and some relationship with quivers. This will follow the note written by Arend Bayer.

1 Stable vector bundles on algebraic curves

The classical stabilities comes from GIT theory and stability of vector bundles and coherent sheaves. We will quickly present the second.

Let X be a smooth projective complex curve (i.e. compact Riemann surface). If E is a vector bundle, we use $rk(E)$ to denote its rank, and $deg(E)$ to denote the degree (define as an additive function over short exact sequence).

Definition 1.1. The slope of E is defined as $\mu(E) = \frac{deg(E)}{rk(E)}$.

Remark 1.1. If we define $\mu(F)$ be infinity for torsion sheaf, then it can be extended to any coherent sheaves.

Lemma 1.1 (See-Saw property)**.** Let $0 \to A \to E \to B \to 0$ be a short exact sequence of coherent sheaves on X. Then

$$\mu(A) < \mu(E) \iff \mu(E) < \mu(B)$$

$$\mu(A) > \mu(E) \iff \mu(E) > \mu(B)$$

Proof. Actually the proof is quite easy by algebra. But we present a diagrammatical proof.

Let $Z(E) = i\, rk(E) - deg(E)$. Then as it forms a short exact sequence, we have $Z(E) = Z(A) + Z(B)$ by the additivity of degree and rank. Then it is a simple graphical fact to produce the result. □

Remark 1.2. (1) Actually, Z is a function from Grothendieck group to $\mathbb{C}$ (actually lattice points).

(2) The image of Z is always contain in the upper half plane dut to nonnegativity of the rank, thus compare slope does not make any problem.

Definition 1.2 (Slope-(semi-)stable)**.** A vector bundle E is slope-(semi-)stable if for all subbundle $0 \to A \to E$, $\mu(A) < (\leq)\mu(E)$. Here we require $A \neq E$ obviously otherwise there is a problem of the definition of stable.

By see-saw property, we can ask similar things for quotients. (The note is wrong here with reverse inequality).

Example 1.1. (1) Any line bundle is stable.

(2) let L be a line bundle of degree 1. An extension $0 \to O_X \to E \to L \to 0$ is stable if and only if the extension does not split. Such extensions always exist when $g > 1$.

Proof. (1) This is trivial as only subbundle (note the difference between subbunlde and subsheaf) of a line bundle is 0 bundle (i.e. base space) and the line bundle itself. The rank of the first one is ∞ (as a torsion sheaf) and the second is not in our concern.

(2) If the sequence splits, then $E = O_X \oplus L$. Then $\mu(E) = \frac{1}{2}$. Also, there is an obvious injective map $0 \to L \to E$. Thus E is unstable. (Actually, similar argument can prove that if E is decomposible, then it will never be stable. Ref Hartshorne Exercise V.2.8(a))

Suppose it is nonsplit, but E is not stable. Then we have $0 \to L' \to E$ for some L' has degree no larger than 0. Then This means $H^0(E \otimes L'^{-1})$ is nonzero. Consider the sequence in the example and tensor with $L'{-1}$, and use basic fact of curve, we can imply that $L = L'$, and the map provide the split map of the exact sequence, which is a contradiction. One can also prove this use the following lemma or use some basic fact related to normalized rank 2 bundle and ruled surface. We omit them here.

For $g > 1$, we want to ask whether $Ext^1(L, O_X) \neq 0$. Then by Hartshorne, this is equivalent to ask $H^1(L^{-1}) \neq 0$. This is easily done by curve theory. □

Lemma 1.2. If E, E' are semistable and $\mu(E) > \mu(E')$, then $Hom(E, E') = 0$.

Proof. Give a nonzero morphism $\phi : E \to E'$, consider the following exact sequence: $E \to im\phi \to E'$, where the first arrow is surjective and the second is injective. By semistability of two vector bundles, we have $\mu(E) \leq \mu(im\phi) \leq \mu(E') < \mu(E)$, which is a contradiction. □

Then the following is a paragraph related to the classical theory of stability, related to the construction of moduli space of stable vector bundles. As it is not in detail. We skip this paragraph. Hope in the future I will read Huybreche's book.

However, what we care about is the existence of Harder-Narasimhan filtration. This can be generailized to many other stability conditions, and the existence of such thing is central to most of the stability conditions.

Theorem 1.1 (Harder-Narasimhan Filtration)**.** For any coherent sheaf F there is a unique increasing filtration

$$) = F_0 \subset F_1 \subset F_2 \subset ... \subset F_n = F$$

such that the filtration quotients $F_i/Fi - 1$ are semistable of slope μ_i, with μ_i strictly decreasing.

Proof. The following proof is also kind of graphical, and is due to Dan Grayson according to Arend Bayer.

We define $Z(F)$ same as the first lemma. Consider the subset $\{Z(A)|A \subset F\}$ of the complex plane, and let $H_Z(F)$ be its convex hull. By the following lemma, it is bounded from the left. As image of Z only contains lattice points, which is a discrete set, it follows that there are only finitely many extremal points $0 = v_0, v_1, ..., v_n = Z(F)$ of the set $H_Z(F)$ lies on or to the left of the line passing through origin and $Z(F)$. Let F_i be an arbitary subsheaf with $Z(F_i) = v_i$. Then we claim
(1) $F_i \subset F_{i+1}$
(2) The slopes $\mu(F_{i+1}/F_i)$ are decreasing
(3) F_{i+1}/F_i is semistable.

(1) Consider $F_i \cap F_{i+1} \subset F$ and $F_i + F_{i+1} \subset F$. Since $Z(F_i)$ and $Z(F_{i+1})$ are adjacent extremal points, both the above two sheaves are right to or on the line. On the other hand, as $F_i \cap F_{i+1} \subset F_i$ we know that it is below or horizontal to v_i. Similar argument apply to $F_i + F_{i+1}$. Then we consider the short exact sequence $0 \to F_i \cap F_{i+1} \to F_i \oplus F_{i+1} \to F_i + F_{i+1} \to 0$ with obvious map, we get $Z(F_i \cap F_{i+1}) + Z(F_i + F_{i+1}) = v_i + v_{i+1}$. Then we are done as we require exremal points.

(2) is easily done graphically.

(3) Let $\bar{A} \subset F_{i+1}/F_i$ be a destablizing object. Consider its preimage $A \subset F_{i+1}$. Then we have $Z(A) = Z(\bar{A}) + v_i$.As $Z(\bar{A})$ has a larger slope than $v_{i+1} - v_i$, we see that $Z(A)$ fall outside of the region, which is a contradiction.

By applyint the above lemma, we can prove the uniqueness easily. □

Lemma 1.3. *Let F be a coherent sheaf on X. Then there exists an integer d such that for any subsheaf $F' \subset F$, $deg(F') < d$.*

Proof. (I believe there is a fundamental proof. Feel free to contact me about this.) If one use Hirzbruch-Riemann-Roch, one gets this result easily. Proof is just inequality. □

Remark 1.3. This remark is very personal. Chunyi told me that every problem in mathematics is inequality and linear algebra.

2 Stability for quiver representations

We observe that the properties we used for $Coh(X)$ are
(1) We have invariants rank and degree, that are additive on short exact sequences.
(2) Rank is always a positive function, and $rk(F) = 0$ implies that $deg(F) > 0$.
(3) The boundedness of rank and degree.

Thus we generalize this idea to give the following definition:

Definition 2.1 (Stability)**.** Given an abelian category $\mathcal{A}$, we say Z is a stability function for $\mathcal{A}$ if $Z : K(\mathcal{A}) \to \mathbb{C}$ is a group homomorphism form the $K-$group of $\mathcal{A}$ to complex number such that for any $0 \neq E \in \mathcal{A}$, we have

$$Z(E) \in \mathbb{H} = \{z = me^{i\pi\phi} | m > 0, \phi \in (0, 1]\}.$$

One sees easily that the property (2) is preserved by $\mathbb{H}$. Then we define the phase $\phi(E)$ of a nonzero object by $\frac{1}{\pi} arg(Z(E)) \in\subset (0, 1]$. And we say that E is $Z-$semistable if$\phi(A) \leq \phi(E)$ for all subobjects $A \subset E$.

Now we relate this with linear algebra (as our last remark in last section). Consider a finite quiver (Q_0, Q_1), i.e., a directed graph with vertices Q_0 and arrows Q_1. We may also allow relations R. A relation iis a linear equation between directed paths starting and ending at the same vertex. We supress R from the notatioin, and let $Rep\ Q$ be the abelian category of its representations.

Example 2.1 (Kronecker Quiver)**.** The kronecker quiver P_2 is given by two vertices $\{0, 1\}$ and two arrows $x, y : 0 \to 1$. A representation $\underline{V}$ of P_2 is a pair of vector spaces (V_0, V_1) together with morphisms $\phi_x, \phi_y : V_0 \to V_1$.

Example 2.2 (A_1-Quiver)**.** The A_1-quiver has two vertices $\{0, 1\}$ with two morphisms $x_0, y_0 : 0 \to 1$ and two morphisms $x_1, y_1 : 1 \to 0$, also with relations $x_1 y_0 = y_1 x_0$ and $x_0 y_1 = y_0 x_1$. A representation of this quiver is a pair of vector spaces V_0, V_1 together with two morphisms $\phi_0, \psi_0 : V_0 \to V_1$ and another two morphisms $\phi_1, \psi_1 : V - 1 \to V_0$ satisfying the relations $\phi_1 \circ \psi_0 = \psi_1 \circ \phi_0$ and $\psi_0 \circ \phi_1 = \phi_0 \circ \psi_1$.

Pick complex numbers $z_i \in \mathbb{H}$. Then we define a stability function Z as

$$Z(\underline{V}) = \sum_{i=0}^{n} dim\ V_i \cdot z_i.$$

The positivity is obvious. The boundedness is because any subreprensentation $\underline{W}$ of $\underline{V}$ can only have a finite number of possible dimension vectors $(dim\ W_i)$.

3 Exercises

Exercise 3.1 (Schur's Lemma)**.** Let $\mathcal{A}$ be an abelian category and Z be a stability function. Assume that E is $Z-$stable. Show that any non-zero endomrophism $\phi \in End(E)$ is an automorphism.

When $\mathcal{A}$ is a linear category over an algebraically closed field k, it follows that $End(E)$ only consists of scalars $k \cdot Id$.

Proof. The first part is just because suppose ϕ is not zero and automorphism. Then we have $E \to im(\phi) \to E$, where the first map is monomorphism and the second map is epimorphism. As ϕ is not zero or automorphism, $im(\phi)$ is not 0 or E. One can check that see-saw property still holds for stability functions. Thus use similar method in lemma 1.2, we are done.

If it is a $k-$linear category for an algebraically closed field, then just by linear algebra, we get the conclusion. □

Exercise 3.2. (1) Consider the Kronecker quiver P_2, and stability conditioins given by z_0, z_1.

(a) Show that if the phase of z_1 is bigger than the phase of z_0, then the only stable objects are the two simple representatioins, i.e., the representations with dimension vectors $(1, 0)$ and $(0, 1)$.

(b) Now suppose the pahse of z_0 is bigger than the phase of z_1. Then there is a one to one correspondence between isomorphism classes of stable objects of dimension vector $(1, 1)$ and points on $\mathbb{P}^1$.

(2) Consider the same problem with P_{n+1}, the Kronecker quiver with $n + 1$ arrows and two vertices, and $\mathbb{P}^n$.

(3) For the second part of the above problem, consider the same siituation for the A_1-quiver, and the blow-up X of the affine plane $A^2_{\mathbb{C}}$.

Proof. (1)(a) One sees that for this quiver, there is no relations. So any kind of representatioins is allowed. Then suppose we have a nontrivial representation with dimension vectors (a, b).

Case 1: $a > 0, b = 0$. Then we take the subrepresentation given by $(1, 0)$. Then the Z value of this representaition is z_0. The Z value of the original representation is az_0. Then obviously this provide a nonstable object.

Case 2: $a = 0, b > 0$. This is similar to the above.

Case 3: $ab \neq 0$. Then we take the subrepresentation given by $(0, 1)$. Then it is obviously not stable.

(b) Let us first see what $(1, 1)$ can be stable. Then for such a representation, we will give two linear mophisms from $\mathbb{C}$ to $\mathbb{C}$. This is equivalent to give two complex numbers (x, y). A subrepresentaion can happen to be $(1, 0)$ and $(0, 1)$. For the $(0, 1)$ case, the reprensetation is given by the trivial map. For the $(1, 0)$ case, we need to show such a subrepresentation cannot happen, otherwise this will provide a nonstable subobject. To avoid this happen, we need to require $a \neq 0$ or $b \neq 0$. For the other case, it dose not affect the stability condition. Thus we do not consider this one. Then in conclusion, the stable $(1, 1)$ pair is given by $(a, b) \neq (0, 0)$. Also two quiver representations are isomorphiic if and only if $(a, b) = (a', b')$. Thus it s classified as points in $\mathbb{P}^1$.

(2) This is basically the same as (1). (a) part has same counterexample, and (b) part we require $(a_0, ..., a_n) \neq (0, ..., 0)$, which is just $\mathbb{P}^n$.

(3) The problem is quite similar, except we have a relation. Thus we have four complex numbers (a, b, x, y), and we require that $ax = by$. (We only have this one condition and it is enough for the two relations.) Also we have one more requirement that $(1, 0)$ cannot show up, whiich means that $(a, b) \neq (0, 0)$. Then this is basically $A^2_{\mathbb{C}}$ blow up at origin. □

Exercise 3.3. Let L be a line bundle of degree 1 on a smooth projective curve X. Show that an extension $0 \to O_X \to E \to L \to 0$ is stable if and only if the extension is nonsplit. Note that such extensions exist for any L when the genus is at least 2.

Proof. The detailed proof is already provided in the text above. □

Exercise 3.4. Show that the tangent bundle on $\mathbb{P}^2$ is slope-stable. (Note that to define slope-stability for a torsion-free sheaf on a vriety of dimension two or higher, one only allows "saturated" subsheaves to test stability, i.e. subsheaves such that the quotient is also torsion-free.)

Proof. Suppose we have a quotient $T \to Q \to 0$. Then as Q is a line bundle, it is stable. Also, by composition with Euler sequence, we have $O(1)^3 \to L$. Thus $L = O(a)$ for $a \geq 1$. Also by Euler sequence, we have the slope of the tangent bundle is $\frac{3}{2}$. So the only exceptional case is $L = O(1)$. However, if such case happen, then by compose with the split map of $O(1) \to O(1)^3$, we see that the tangent bundle splits. This is obviously wrong, otherwise $T = O(1) \oplus O(2)$, contradicts with the dimension of the global section. So we are done. □

Example 3.1. Show that in the construction of the proof of existence of Harder-Narasimhan filtration, the choice of F_i with $Z(F_i) = v_i$ is actually unique.

Proof. Suppose we have another F_i' such that $Z(F_i') = v_i$, then we consider $F_i \cap F_i'$ and $F_i + F_i'$. They are well defined as subsheaves of F. Then we have similar exact sequence

$$0 \to F_i \cap F_i' \to F_i \oplus F_i' \to F_i + F_i' \to 0.$$

Then we have $Z(F_i \cap F_i') + Z(F_i + F_i') = 2v_i$. Then from similar graph, we get the result. □