Topics in Algebraic K-Theory

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Contents

1 What is Algebraic K-Theory?

We briefly summarise the main idea of (Higher) Algebraic K-Theory:

Algebraic K-Theory starts with (a certain type of) ***category****, applies some type construction to it to produce a* ***topological space*** *or* ***topological spectrum****, and then takes* ***homotopy groups****. The resulting homotopy groups are referred to as the (adjective depending on the details of the construction)* ***K-Theory*** *of the starting category.*

The exact details of the above procedure will give different types of K-Theory. Mathematicians are interested in these K groups because it turns out that they give deep information about your starting category.

The first example of this power came from Grothendieck, who was (arguably) the first to discover a glimpse of this theory. Specifically, given a noetherian scheme X, Grothendieck defined $K(X)$ to be the quotient of the free abelian group generated by all isomorphism classes of coherent sheaves on X, by the subgroup generated by all expressions

$$\mathcal{F} - \mathcal{F}' - \mathcal{F}''$$

whenever there is an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

of coherent sheaves on X.

(Grothendieck used the letter K to stand for word 'klasse', which is german for the english word 'class'. Grothendieck, having had a background in functional analysis, was hesitant to use the letter C).

In today's language, what Grothendieck had defined was K_0 of the abelian category of isomorphism classes of coherent sheaves on X, denoted $\mathrm{Coh}(X)$ (see definition 2.9). Despite only being a 'lower' K-group in today's language, this group is still very powerful: Grothendieck used this group to prove his far-reaching Grothendieck–Riemann–Roch theorem, see [4] for details.

As to **why** homotopy groups are required to present the theory in full generality is not at all obvious. Indeed, the discovery of these definitions, as well as the development of the theory in full generality, was the reason why Quillen was awarded his Fields Medal. Some attempts to explain the homotopy groups will be given in this essay, after we have reviewed classical K-theory.

For a detailed history of Algebraic K-Theory, we refer the reader to Weibel's article [23], available on his webpage.

2 Classical Algebraic K-Theory

The phrase 'Classical Algebraic K-Theory' refers to the groups K_0, K_1, and K_2 that were discovered before Quillen's general definition in terms of homotopy groups. The group K_0 may be defined in various contexts whereas the groups K_1 and K_2 were defined exclusively in the context of rings.

A lot can be said about these three groups. For example, Milnor wrote a whole book [13] about them. For the sake of brevity, we will in this section only briefly review the theory of these groups and we will explain how these groups are related, which hopefully serve as a motivation to define Higher Algebraic K-Theory.

2.1 The Grothendieck group K_0

As mentioned above, the group K_0 may be defined in various contexts. The common theme amongst these various contexts is the use of the so called **group completion** (also known as the **Grothendieck group**), which we now explain.

2.1.1 The group completion of a monoid

Group completion is an easy procedure that takes as input an (abelian) monoid and produces as an output an (abelian) group. It is perhaps best stated in terms of its universal property.

Definition 2.1. *The group completion of an (abelian) monoid M is an (abelian) group M^{gp}, together with a monoid morphism $[\cdot] : M \to M^{gp}$ that has the following universal property:*

For any group A and any monoid morphism $\alpha : M \to A$, there exists an unique group homomorphism $\tilde{\alpha} : M^{gp} \to A$ that makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{[\cdot]} & M^{gp} \\ & \searrow^{\alpha} & \downarrow{\scriptstyle \tilde{\alpha}} \\ & & A \end{array}$$

commute.

Note, by universality, if the group completion exists, it is unique up to unique isomorphism. To construct the group completion for abelian monoids (the construction for monoids being analogous), we may define

$$M^{gp} := F(M)/R(M)$$

where $F(M)$ is the free abelian group generated by symbols $[m]$ for $m \in M$ and $R(M)$ is the subgroup of $F(M)$ generated by all relations $[m+n]-[m]-[n]$ for $m, n \in M$.

2.1.2 K_0 of a ring

Let R be a ring. We will denote by $\mathbf{P}(R)$ the set of all isomorphism classes of finitely generated projective (left) modules over R (we will explain shortly why we restrict our attention to *finitely generated* projective modules). Note, $\mathbf{P}(R)$ is an abelian monoid under direct sum $\oplus$, with additive identity 0.

Definition 2.2. *In the above notation, we define*

$$K_0(R) := \boldsymbol{P}(R)^{gp}.$$

Remark 1. The reason why we restrict our attention to *finitely generated* projective modules is because of the so called *Eilenberg Swindle*, which would imply that $K_0(R) = 0$.

Indeed, let R^∞ denote the (countable) infinitely generated on R. Let P be a finitely generated projective module. By definition, there exists a R module Q such that $P \oplus Q \cong R^n$ for some $n \in \mathbb{N}$. Therefore, we deduce that

$$\begin{aligned} P \oplus R^\infty &\cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \\ &\cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \\ &\cong R^n \oplus R^n \oplus \cdots \\ &\cong R^\infty. \end{aligned}$$

Therefore, in $K_0(R)$, we would have

$$[P] + [R^\infty] = [P \oplus R^\infty] = [R^\infty]$$

i.e. $[P] = 0$. As $R^\infty \cong R^\infty \oplus R^\infty$, we deduce that $K_0(R) = 0$.

Remark 2. The motivation behind the definition of $K_0(R)$ perhaps comes from the following theorem, which says that when R is commutative, there is a correspondence between finitely generated projective modules over R and vector bundles (i.e. locally free sheaves of finite rank) over Spec(R).

Theorem 2.3. *Let R be a commutative ring and let M be a R-module. Then, the following are equivalent:*

- *M is finite projective*
- *M is finite locally free i.e. we can cover Spec(R) by standard opens $D(f_i)$, $i \in I$ such that the localizations M_{f_i} are finite free R_{f_i}-modules for all $i \in I$.*

Proof. See the Stacks project [2], tag 00NV. □

The correspondence then follows from the equivalence of categories between the category of R-modules and the category of quasi-coherent $\mathcal{O}_{\mathrm{Spec}(R)}$-modules (see Hartshorne [10], Corollary 5.5, chapter II).

It would be beneficial to see an example of a computation of K_0.

Example 2.4. Let R be a commutative PID (for example, R could be a field). Let P be a finitely generated projective R-module. If we can show that P is necessarily free, then we would have shown that

$$K_0(R) \cong \mathbb{Z}.$$

Indeed, if P is finitely generated, then there exists a surjection $\pi : R^n \twoheadrightarrow P$ for some $n > 0$. Therefore, as P is projective, the short exact sequence

$$0 \longrightarrow \ker(\pi) \longrightarrow R^n \longrightarrow P \longrightarrow 0,$$

splits, so that $P \oplus \ker(\pi) \cong R^n$. Thus, P may be identified with a submodule of R^n. But, R is a PID and R^n is free over R. Therefore, so is P.

2.1.3 K_0 of an symmetric monoidal category

Let S be a symmetric monoidal category and suppose the isomorphism classes of objects in S form a set, denoted S^{iso}. We claim that $(S^{iso}, \otimes, e)$ is an abelian monoid with product $[s] \otimes [t] := [s \otimes t]$. Indeed, this follows from the fact that if $s \cong s'$ and $t \cong t'$, then $s \otimes t \cong s' \otimes t'$ since $\otimes : S \times S \to S$ is a bifunctor.

Remark 3. We remark that we really need to take isomorphisms classes as $s \otimes e \neq s$ in general, only naturally isomorphic to it.

We define K_0 of S in terms of this abelian monoid.

Definition 2.5. *Let S be a symmetric monoidal category such that the isomorphism classes of objects in S, denoted S^{iso}, form a set. Then, we define K_0 of S as the group completion*

$$K_0(S) := (S^{iso})^{gp}$$

of the abelian monoid $(S^{iso}, \otimes, e)$.

For example, this is precisely the way we defined $K_0(R)$, with the symmetric monoidal category S being the category of finitely generated projective modules over R.

2.1.4 K_0 of an exact category

Exact categories were discovered by Quillen in [16]. Intuitively, they generalise abelian categories in the sense that one may talk about short exact sequences without insisting on the existence of kernels and cokernels.

In the literature, there are two common ways people define exact categories. One is via an embedding into an ambient abelian category and the other is via defining an exact structure. The author prefers the latter because it is more intrinsic and it comes with a theorem (see Theorem 2.8) which gives an embedding into an abelian category that preserves the exact structure. For more information, we refer the reader to Bühler's article [5].

Definition 2.6. *Let $\mathcal{A}$ be an additive category. Then, a kernel-cokernel pair (i, p) in $\mathcal{A}$ is a pair of composable morphisms*

$$A' \xrightarrow{i} A \xrightarrow{p} A''$$

such that i is the kernel of p and p is the cokernel of i.

If a class $\mathcal{E}$ of kernel-cokernel pairs is fixed, we say that i is an admissible monic if there exists a morphism p such that $(i, p) \in \mathcal{E}$. Admissible epics are defined dually. We will denote admissible monics by $\rightarrowtail$ and admissible epics as $\twoheadrightarrow$.

Definition 2.7. *Let $\mathcal{A}$ be an additive category. An exact structure on $\mathcal{A}$ is a class $\mathcal{E}$ of kernel-cokernel pairs which is closed under isomorphisms and satisfy the following axioms:*

- $E0$) *For any $A \in \mathcal{A}$, 1_A is an admissible monic.*
- $E0^{op}$) *For any $A \in \mathcal{A}$, 1_A is an admissible epic.*
- $E1$) *Admissible monics are closed under composition.*
- $E1^{op}$) *Admissible epics are closed under composition.*
- $E2$) *For any admissible monic $A \rightarrowtail B$ and for any morphism $A \to A'$, there exists a pushout diagram*

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ A' & \rightarrowtail & B' \end{array}.$$

- $E2^{op}$) *For any admissible epic $A \rightarrowtail B$ and for any morphism $B' \to B$, there exists a pullback diagram*

$$\begin{array}{ccc} A' & \twoheadrightarrow & B' \\ \downarrow & & \downarrow \\ A & \twoheadrightarrow & B \end{array}.$$

The pair $(\mathcal{A}, \mathcal{E})$ is called an exact category. If the exact structure $\mathcal{E}$ is understood, we may simply write $\mathcal{A}$. The elements of $\mathcal{E}$ are called short exact sequences.

Remark 4. Some of these axioms are actually redundant. See Remark 2.4 [5].

Remark 5. Note that for any $A, B \in \mathcal{A}$, the sequence

$$A \overset{\binom{1}{0}}{\rightarrowtail} A \oplus B \overset{(0\ 1)}{\twoheadrightarrow} B$$

is a short exact sequence because we have pushout diagram

$$\begin{array}{ccc} 0 & \rightarrowtail & B \\ \downarrow & & \downarrow \\ A & \rightarrowtail & A \oplus B \end{array}$$

and short exact sequences are closed under isomorphisms.

Here is the relevant embedding theorem:

Theorem 2.8 (Gabriel-Quillen)**.** *Let $(\mathcal{A}, \mathcal{E})$ be a small exact category. Then, there is an embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ into an abelian category $\mathcal{B}$ such that $\mathcal{A}$ is closed under extensions in $\mathcal{B}$ and $\mathcal{E}$ is the class of all sequences in $\mathcal{A}$ which are short exact in $\mathcal{B}$.*

Proof. See appendix of [5]. □

Remark 6. There is also a 'recognition theorem' for exact categories. Specifically, let $\mathcal{B}$ be an abelian category and let $\mathcal{A}$ be a full subcategory of $\mathcal{B}$ that is closed under extensions. Then it follows that $\mathcal{A}$ is an exact category with exact structure $\mathcal{E}$ given by all short sequences in $\mathcal{A}$ which are exact in $\mathcal{B}$. For more details, see appendix of [5], remark A.2.

We will refer to $\mathcal{B}$ as the *ambient abelian category* of $\mathcal{A}$.

Abelian categories are obvious examples of exact categories. Another important example of an exact category is the category $\mathbf{P}(R)$ of finitely generated projective modules of ring R. To see this, firstly note that we have a canonical embedding $\mathbf{P}(R) \hookrightarrow R - \mathbf{mod}$. Secondly, suppose

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence in $R - \mathbf{mod}$ with M' and M'' finitely generated projective. Then, as M'' is projective, the above short exact sequence splits to give (noncanonical) isomorphism $M \cong M' \oplus M''$. Therefore, we deduce that M is finitely generated projective, so that $\mathbf{P}(R)$ is closed under extensions and therefore by remark 6 is an exact category. This is the prototypical example of an exact category to keep in mind. Moreover, this argument shows that $\mathbf{P}(R)$ is actually a **split** exact category i.e. every short exact sequence splits.

Next, let us define K_0 of a small exact category.

Definition 2.9. *Let $\mathcal{A}$ be a small exact category. We define $K_0(\mathcal{A})$ as the abelian group with presentation*

$$K_0(\mathcal{A}) := \langle G | R \rangle$$

where

$$G := \{[C] : C \in Obj\mathcal{A}\}$$

and

$$R := \{[C] - [B] - [D] : 0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0 \textit{ short exact sequence in } \mathcal{A}\}.$$

In $K_0(\mathcal{A})$, we have the following identities, whose proof is left as a routine exercise.

Proposition 2.10. *We have the following identities in $K_0(\mathcal{A})$:*

- $[0] = 0_{K_0(\mathcal{A})}$
- $C \cong C' \Rightarrow [C] = [C']$
- $[C' \oplus C''] = [C'] + [C'']$.

Proof. Exercise. □

As an example, we compute that for exact category $\mathbf{P}(R)$,

$$\begin{aligned} K_0(\mathbf{P}(R)) &= \left\langle [M] \middle| [M] = [M'] + [M''] \text{ for } 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \text{ short exact sequence in } \mathbf{P}(R) \right\rangle \\ &= \left\langle [M] \middle| [M' \oplus M''] = [M'] + [M''] \right\rangle \text{ (short exact sequences split)} \\ &\cong K_0(R). \end{aligned}$$

This example also shows that if $\mathcal{A}$ is a split exact category, then

$$K_0(\mathcal{A}) \cong K_0^{\oplus}(\mathcal{A})$$

where we regard $(\mathcal{A}, \oplus, 0)$ as a symmetric monoidal category. If $\mathcal{A}$ is not a split exact category, this isomorphism in general will not hold.

2.2 K_1 of a ring

Denote by $GL_n(R)$ the $n \times n$ general linear matrices on R. Note, there is an embedding $GL_n(R) \hookrightarrow GL_{n+1}(R)$ given by $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. This gives a directed system

$$GL_1(R) \subset GL_2(R) \subset \cdots \subset GL_n(R) \subset GL_{n+1}(R) \subset \cdots. \tag{1}$$

The direct limit $GL(R) := \varinjlim_n GL_n(R)$ is called the *infinite general linear group* of R.

The group $K_1(R)$ is then defined as the **abelianization** of this infinite general linear group:

Definition 2.11. *We define*

$$K_1(R) := GL(R)/[GL(R), GL(R)].$$

Remark 7. This group, also known as the *Whitehead group* of a ring, was first explicitly defined by Milnor in the appendix of [11]. It is related to an obstruction in homotopy theory, see [23] for references.

It is beneficial to state an alternate description of $K_1(R)$, first discovered by Whitehead. To state it, let us first establish some notation.

For $i \neq j$ and $r \in R$, define the *elementary matrix* $e_{ij}(r)$ to be the matrix in $GL(R)$ which has 1 in every diagonal spot; has r in the (i, j)-spot and has zero everywhere else. Let $E_n(R)$ denote the subgroup of $GL_n(R)$ generated by all elementary matrices $e_{ij}(r)$ with $1 \leq i, j \leq n$.

Note, we have a directed system for the $E_n(R)$ analogous the directed system 1. The direct limit $E(R) := \varinjlim_n E_n(R)$ is called the *infinite group of elementary matrices* on R.

Lemma 2.12 (Whitehead's lemma)**.** *In the above notation, we have*

$$E(R) = [GL(R), GL(R)].$$

For the sake of brevity, we refer the reader to Srinivas [21] proposition 1.5 for a proof (the proof uses elementary techniques). For now, let us see an application of this result.

Proposition 2.13. *Let $\mathbb{F}$ be a field. Then,*

$$K_1(\mathbb{F}) \cong \mathbb{F}^{\times}.$$

Proof. By standard linear algebra arguments, we have that for $n \geq 1$,

$$E_n(\mathbb{F}) = SL_n(\mathbb{F}).$$

Therefore, $E(\mathbb{F}) = SL(\mathbb{F})$, where $SL(\mathbb{F}) := \varinjlim_n SL_n(\mathbb{F})$ under the usual embedding.

Then, notice that

$$\ker(GL(\mathbb{F}) \xrightarrow{\det} \mathbb{F}^{\times}) = SL(\mathbb{F}).$$

Hence,

$$GL(\mathbb{F})/SL(\mathbb{F}) \cong \mathbb{F}^{\times}.$$

Thus, we have

$$GL(\mathbb{F})/E(\mathbb{F}) = GL(\mathbb{F})/SL(\mathbb{F}) \cong \mathbb{F}^{\times}.$$

The proposition then follows from Whitehead's lemma. □

2.3 K_2 of a ring

The group K_2 of a ring was originally defined by Milnor in 1967 following a paper by R.Steinberg on universal central extensions of Chevally groups. To define $K_2(R)$, we first need to define the *Steinberg group.*

Definition 2.14. *For $n \geq 3$ [1], the Steinberg group $St_n(R)$ of a ring R is defined as the group generated by symbols $x_{ij}(r)$ where (i, j) distinct pair of integers between 1 and n, $r \in R$; subject to the following 'Steinberg relations':*

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s) \tag{2}$$

$$[x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq l \\ x_{il}(rs) & \text{if } j = k \text{ and } i \neq l \\ x_{kj}(-sr) & \text{if } j \neq k \text{ and } i = l. \end{cases} \tag{3}$$

Note, it may be checked that the elementary matrices $e_{ij}(r)$ which generate $E_n(R) \leq GL_n(R)$ also satisfy the Steinberg relations (indeed, they probably inspired the definition), so that there is a canonical group surjection

$$\phi_n : St_n(R) \longrightarrow E_n(R)$$

sending $x_{ij}(r)$ to $e_{ij}(r)$.

In addition, note that the Steinberg relations for $n+1$ clearly include the Steinberg relations for n, so that there is a canonical map $St_n(R) \to St_{n+1}(R)$. These maps form a directed system, and allow us to write $St(R) := \varinjlim_n St_n(R)$, the *infinite Steinberg group.* Note, by the universal property of colimit, from the $\phi_n : St_n(R) \to E_n(R)$, we obtain a map

$$\phi : St(R) \longrightarrow E(R).$$

Notice that $\phi : St(R) \to E(R)$ is a surjection as the ϕ_n are.

Definition 2.15. *In the above notation, we define*

$$K_2(R) := \ker(\phi : St(R) \longrightarrow E(R)).$$

Remark 8. Intuitively speaking, we may think of $K_2(R)$ as the set of all *nontrivial* relations between elementary matrices, the Steinberg relations 2 and 3 being the 'trivial' relations.

The following theorem shows that $K_2(R)$ is in fact an abelian group:

Theorem 2.16. *We have*

$$K_2(R) = Z(St(R)),$$

[1] To avoid technical complications, $St_2(R)$ is not defined.

where $Z(St(R))$ denotes the center of $St(R)$.

Proof. See Weibel's K-book [24], Theorem 5.2.1 III. □

In addition, one may actually prove that

$$0 \longrightarrow K_2(R) \longrightarrow St(R) \longrightarrow E(R) \longrightarrow 0$$

is a *universal central extension* of $E(R)$, see [24] Theorem 5.5 III.

Then, since $E(R)$ is perfect (for $n \geq 3$, $E_n(R)$ is perfect as for i, j, k distinct, $e_{ij}(r) = [e_{ik}(r), e_{kj}(1)]$), it follows (see [24], Theorem 5.4 III) that

$$K_2(R) \cong H_2(E(R); \mathbb{Z}), \tag{4}$$

the second group homology of $E(R)$. This isomorphism is used for example to show that the definition for K_2 of a ring in terms of higher K-theory is in agreement with the classical definition up to isomorphism, see Corollary 3.4.

2.3.1 K_2 of fields

Recall that if $\mathbb{F}$ is a field, then $K_0(\mathbb{F}) \cong \mathbb{Z}$ and $K_1(\mathbb{F}) \cong \mathbb{F}^\times$. For K_2, we have the following theorem, due to Matsumoto:

Theorem 2.17 (Matsumoto)**.** *Let $\mathbb{F}$ be a field. Then, $K_2(\mathbb{F})$ is the free abelian group generated by the set of symbols (often called the 'Steinberg symbols') $\{x, y\}$ with $x, y \in \mathbb{F}^\times$, subject to the following relations:*

1. *(Bilinearity) $\{xx', y\} = \{x, y\}\{x', y\}$ and $\{x, yy'\} = \{x, y\}\{x, y'\}$.*
2. *(Steinberg identity) $\{x, 1 - x\} = 1$ for all $x \neq 0, 1$.*

From these two relations, it is also possible to prove that for any $x, y \in \mathbb{F}^\times$, we have relation $\{x, y\}^{-1} = \{y, x\}$.

The proof of Matsumoto's theorem is difficult and we therefore omit it from this essay. We refer the reader to Milnor [13], section 12.

Notice that from the above presentation, we may **identify** $K_2(\mathbb{F})$ with the quotient of $\mathbb{F}^\times \otimes \mathbb{F}^\times$ by the subgroup generated by the elements $x \otimes (1 - x)$, $x \neq 0, 1$. That is to say, we have

$$K_2(\mathbb{F}) \cong \frac{\mathbb{F}^\times \otimes \mathbb{F}^\times}{\langle x \otimes (1-x) | x \neq 0, 1\rangle}.$$

Note that the module structure on $\mathbb{F}^\times$ is taken to be multiplication, and we often write elements of the right hand side as $\{x, y\}$.

This isomorphism gives us the following snappy corollary:

Corollary 2.18. *Let $\mathbb{F}_q$ be a finite field. Then,*

$$K_2(\mathbb{F}_q) = 1.$$

Proof. Recall a famous result that $\mathbb{F}_q^\times$ is **cyclic**. Let $x \in \mathbb{F}_q^\times$ generate it. Note, $x \otimes x$ generates cyclic group $\mathbb{F}_q^\times \otimes \mathbb{F}_q^\times$. Thus, it suffices to prove that $x \otimes x$ vanishes in K_2.

Suppose q is even. Note, by Lagrange's theorem applied to finite group $(\mathbb{F}_q, +)$, it follows that $\mathrm{char}(\mathbb{F}_q) = |\langle 1 \rangle|$ divides q. But, $\mathrm{char}(\mathbb{F}_q)$ is prime and q is even and hence some power of 2 (as it is the cardinality of some finite field). Therefore, we deduce that $\mathrm{char}(\mathbb{F}_q) = 2$. Thus, we have $x = -x$ and therefore, $\{x, x\} = \{x, -x\} = 1$.

Now suppose q is odd. Then, note in $\mathbb{F}_q^\times$ we have $x^{\frac{q-1}{2}} = -1$. Therefore, we have

$$\{x, x\}^{\frac{q+1}{2}} = \{x, x^{\frac{q+1}{2}}\} = \{x, -x\} = 1.$$

Also, note that $x^{q-1} = 1$. Therefore

$$\{x, x\}^{q-1} = \{x, x^{q-1}\} = \{x, 1\} = 1.$$

Thus, if we define $d := |\{x, x\}|$, we deduce that $d \big| \frac{q+1}{2}$ and $d \big| q - 1$. But, we also have $\gcd(\frac{q+1}{2}, q - 1) = 2$ as $2\frac{q+1}{2} - (q - 1) = 2$. Hence, we deduce that $d \leq 2$.

Therefore, to prove $d = 1$ (which will finish the proof), it suffices to prove d is odd.

Note, $v \mapsto 1 - v$ is an involution of $\mathbb{F}_q^\times \setminus \{1\}$. Also, note that $\mathbb{F}_q^\times \setminus \{1\} = \{x, x^2, \ldots, x^{q-2}\}$. Therefore, we deduce that $\mathbb{F}_q^\times \setminus \{1\}$ has $\frac{q-3}{2}$ even powers of x and $\frac{q-1}{2}$ odd powers of x.

Thus, by the pigeonhole principle, we conclude that there exists a $v \in \mathbb{F}_q^\times \setminus \{1\}$ such that v and $1 - v$ are odd powers of x, say $v = x^m$ and $1 - v = x^n$. Therefore, we have

$$1 = \{v, 1 - v\} = \{x^m, x^n\} = \{x, x\}^{mn}.$$

Hence, it follows $d \big| mn$. But mn is odd, so d must be odd. □

2.4 Milnor K-theory of fields

Let $\mathbb{F}$ be a field. Recall that $K_0(\mathbb{F}) \cong \mathbb{Z}$, $K_1(\mathbb{F}) \cong \mathbb{F}^\times$ and by Matsumoto

$$K_2(\mathbb{F}) \cong \frac{\mathbb{F}^\times \otimes \mathbb{F}^\times}{\langle x \otimes (1 - x) | x \neq 0, 1 \rangle}.$$

Milnor took the hypothesis that these were the only relations and in his paper [12] gave an ad-hoc [2] definition of what is now known as *Milnor K-theory*.

[2] to quote from his paper: 'for $n \geq 3$, the definition is purely ad-hoc'

Define $T^0(\mathbb{F}^\times) := \mathbb{Z}$ and for $k \geq 1$, define

$$T^k(\mathbb{F}^\times) := \mathbb{F}^\times \otimes \cdots \otimes \mathbb{F}^\times$$

tensored k times over $\mathbb{Z}$ and given multiplicative module structure. We then define the *tensor algebra* of $\mathbb{F}^\times$ to be

$$T(\mathbb{F}^\times) := \bigoplus_{k=0}^{\infty} T^k(\mathbb{F}^\times).$$

It is customary to write $l(x)$ for the element of degree one in $T(\mathbb{F}^\times)$ corresponding to $x \in \mathbb{F}^\times$.

Definition 2.19. *In the above notation, we define the Milnor K-theory of a field* $\mathbb{F}$ *to be the graded ring*

$$K_*^M(\mathbb{F}) := \frac{T(\mathbb{F}^\times)}{\langle l(x) \otimes l(1-x) | x \neq 0, 1\rangle}.$$

Remark 9. These definitions make perfect sense for commutative rings. They are stated in terms of fields to emphasis the connection with the above discussions.

We define the Milnor K-group $K_n^M(\mathbb{F})$ as the abelian subgroup generated by elements of degree n and it is customary to write $\{x_1, \ldots, x_n\}$ for the image of $l(x_1) \otimes \cdots \otimes l(x_n)$ in $K_n^M(\mathbb{F})$. Thus, $K_n^M(\mathbb{F})$ may be presented as the abelian group generated by symbols $\{x_1, \ldots, x_n\}$ subject to bilinearity of multiplication in each slot (coming from the tensor product), and equals to zero if $x_i + x_{i+1} = 1$ for some i.

Note, by construction, we have that $K_i^M(\mathbb{F}) \cong K_i(\mathbb{F})$ for all $i = 0, 1, 2$.

Despite the ad-hoc nature of the construction, Milnor K-theory has deep connections to other areas of mathematics. We refer the reader to the MathOverflow answer [1] for an interesting discussion on this.

2.5 Motivation for Higher Algebraic K-Theory

Motivation to define a Higher Algebraic K-Theory of course depends on personal taste, but we will briefly mention the author's perspective on this question.

Let $I \trianglelefteq R$ be a two sided ideal of a ring R. Then, it is possible to define $K_i(I)$ for $i = 0, 1, 2$ so that the functor K_2 is related to K_1 and K_0 for example by means of the following exact sequence:

$$\begin{array}{ccccc}
K_2 I & \longrightarrow & K_2 R & \longrightarrow & K_2(R/I) \\
K_1 I & \longrightarrow & K_1 R & \longrightarrow & K_2(R/I) \\
K_0 I & \longrightarrow & K_0 R & \longrightarrow & K_0(R/I).
\end{array}$$

For more details, we refer the reader to Milnor [13], section 6.

(This exact sequence perhaps also justifies the indexing of these classical K-groups.)

Any algebraist looking at this sequence will have the urge to extend this to a long exact sequence, which in the author's opinion provides a good motivation to define the higher K-groups. We will later see how the higher algebraic K-groups fit into a long exact sequence.

3 Higher Algebraic K-Theory

We will shall define Higher Algebraic K-Theory is three settings:

- For rings,
- For exact categories,
- For symmetric monoidal categories.

(For the sake of brevity, we have unfortunately decided not to write about K-Theory for Waldhausen categories. However, this is an important subject and the reader is encouraged to read Waldhausen's paper [22] for more details).

In each case, the idea is to define a 'K-theory space' $K\mathcal{A}$ from you starting category $\mathcal{A}$ and take homotopy groups. Of course, the K-theory spaces are defined so that we have an agreement isomorphism with classical K-theory, but this will be far from obvious when first introduced to the definitions.

The ideas here are primarily due to Daniel Quillen, with important contributions made also by Daniel Grayson, Graeme Segal and Friedhelm Waldhausen.

3.1 BGL^+ definition for rings

Quillen's first definition of Higher Algebraic K-Theory for a ring R is given in terms of a topological space $BGL(R)^+$. This definition first appeared in [15].

The space $BGL(R)$ is simply the classifying space of the infinite general linear group $GL(R)$ defined in section 2.2. The '$+'$ on the other hand is something Quillen discovered for the purposes of Higher Algebraic K-Theory. It is called the Quillen +-construction. It has properties which are uniquely characterised up to homotopy. This is described by the following theorem:

Theorem 3.1 (Quillen)**.** *Let (X, x) be a path connected CW complex, $N \trianglelefteq \pi_1(X, x)$ a perfect normal subgroup. Then, there exists a continuous map between CW complexes*

$$f : (X, x) \longrightarrow (X^+, x^+)$$

such that

1. *There is an exact sequence*

$$0 \longrightarrow N \longrightarrow \pi_1(X, x) \xrightarrow{f_*} \pi_1(X^+, x^+) \longrightarrow 0.$$

2. *For all $n \geq 0$, we have isomorphisms*

$$f_* : H_n(X, \mathbb{Z}) \longrightarrow H_n(X^+, \mathbb{Z})$$

   (this is actually true for any local coefficient system L on X^+).

3. *If $g : (X, x) \longrightarrow (Y, y)$ is a continuous map between CW complexes such that*

$$N \subset \ker\left(g_* : \pi_1(X, x) \longrightarrow \pi_1(Y, y)\right),$$

   then there exists a continuous map $h : (X^+, x^+) \longrightarrow (Y, y)$, unique up to homotopy, making the diagram

$$\begin{array}{ccc} (X, x) & \xrightarrow{\quad f \quad} & (X^+, x^+) \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & (Y, y) & \end{array}$$

   commute.

Proof. See Srinivas [21]. □

Definition 3.2 (Quillen)**.** *Let R be a ring. Then, we define the K-theory space of R, denoted $K(R)$, as*

$$K(R) := K_0(R) \times BGL(R)^+$$

where $K_0(R)$ is the classical K_0 group of R endowed with the discrete topology, and the plus construction is taken with respect to normal subgroup $E(R) = [GL(R), GL(R)] \trianglelefteq GL(R)$.

Remark 10. Note that

$$\begin{aligned}\pi_0(K_0(R) \times BGL(R)^+) &\cong \pi_0(K_0(R)) \times \pi_0(BGL(R)^+) \\ &\cong K_0(R)\end{aligned}$$

and for $n \geq 1$,

$$\begin{aligned}\pi_n(K_0(R) \times BGL(R)^+) &\cong \pi_n(K_0(R)) \times \pi_n(BGL(R)^+) \\ &\cong 1 \times \pi_n(BGL(R)^+) \\ &\cong \pi_n(BGL(R)^+).\end{aligned}$$

Thus, for $n \geq 1$, it suffices to study $\pi_n(BGL(R)^+)$. Indeed, $BGL(R)^+$ is the 'main' part of the definition.

Thus, we have that $\pi_0(K(R)) \cong K_0(R)$ and

$$\begin{aligned}\pi_1(K(R)) = \pi_1(BGL(R)^+) &\cong \frac{\pi_1(BGL(R))}{E(R)} \quad \text{(by definition of plus construction)} \\ &\cong \frac{GL(R)}{E(R)} \\ &= K_1(R).\end{aligned}$$

On the other hand, the fact that $\pi_2(K(R)) \cong K_2(R)$ is far from obvious. We give a proof in which we unashamedly cite big hammers.

Proposition 3.3. *Let P be a perfect normal subgroup of a group G, with corresponding $+$-construction $f : BG \to BG^+$. If $F(f)$ is the homotopy fiver of f, then $\pi_1 F(f)$ is the universal central extension of P, and*

$$\pi_2(BG^+) \cong H_2(P, \mathbb{Z}).$$

Proof. Consider the exact sequence

$$\begin{array}{l}\pi_2 F(f) \longrightarrow \pi_2 BG = 0 \longrightarrow \pi_2 BG^+ \\ \hookrightarrow \pi_1 F(f) \longrightarrow G \longrightarrow \frac{G}{P} \longrightarrow 1.\end{array}$$

By identifying the long exact sequence of homotopy groups of pair (BG^+, BG) and the long exact sequence of fibration associated to homotopy fibre $F(f)$, by Corollary 3.5 IV of Whitehead's Elements of Homotopy Theory [25], it follows

$$\mathrm{Im}(\pi_2 BG^+ \to \pi_1 F(f)) \subseteq Z(\pi_1 F(f)).$$

Therefore, we deduce that

$$0 \to \pi_2 BG^+ \to \pi_1 F(f) \to P \to 1 \tag{5}$$

is a central extension of P. But be Lemma 1.6 IV of Weibel [24], $F(f)$ is acyclic (i.e. has the homology of a point). Therefore, by Lemma 1.3.1 IV in *op.cit.*, $\pi_1 F(f)$ is perfect and $H_2(\pi_1 F(f), \mathbb{Z}) = 0$.

Moreover, as $\pi_1 F(f)$ is perfect, we also have

$$\begin{aligned} H_1(\pi_1 F(f), \mathbb{Z}) &\cong (\pi_1 F(f))^{ab} \\ &\cong 0. \end{aligned}$$

Therefore, by the Recognition Theorem 5.4 III in *op.cit.*, it follows that exact sequence 5 is a universal central extension of P. Thus, as P is perfect, it follows

$$\pi_2 BG^+ \cong H_2(P; \mathbb{Z}).$$

□

Corollary 3.4. *We have*

$$\pi_2(BGL(R)^+) \cong K_2(R).$$

Proof. Note, $E(R) = [GL(R), GL(R)] \trianglelefteq GL(R)$ is a perfect normal subgroup. Then, by the preceding proposition,

$$\pi_2(BGL(R)^+) \cong H_2(E(R), \mathbb{Z}) \cong K_2(R),$$

where the last isomorphism is the isomorphism 4. □

3.2 Quillen's $\mathcal{Q}$-construction for exact categories

The reader is referred to section 2.1.4 to refresh her memory on exact categories. We will in this section define Quillen's $\mathcal{Q}$-construction for exact categories. This definition first appeared in Quillen's paper [16].

Definition 3.5 (Quillen)**.** *Let $\mathcal{A}$ be a small exact category. Then, the category $\mathcal{QA}$ is given by the following datum:*

- *The objects of $\mathcal{QA}$ are the objects of $\mathcal{A}$.*
- *For any objects $A, B \in \mathcal{QA}$,*

$$Hom_{\mathcal{QA}}(A, B) = \frac{\left\{ A \overset{j}{\twoheadleftarrow} E \overset{i}{\rightarrowtail} B : j \text{ admissible epic, } i \text{ admissible monic} \right\}}{\sim}$$

where $(A \overset{j}{\twoheadleftarrow} E \overset{i}{\rightarrowtail} B) \sim (A \overset{j'}{\twoheadleftarrow} E' \overset{i'}{\rightarrowtail} B)$ *if there exists commutative diagram*

$$\begin{array}{ccccc} A & \twoheadleftarrow & E & \rightarrowtail & B \\ \| & & \downarrow{\scriptstyle\sim} & & \| \\ A & \twoheadleftarrow & E' & \rightarrowtail & B \end{array},$$

the middle map being an isomorphism.

Composition of $(A \overset{j}{\twoheadleftarrow} E \overset{i}{\rightarrowtail} B)$ *and* $(B \overset{j}{\twoheadleftarrow} E' \overset{i}{\rightarrowtail} C)$ *is given by pullback diagram*

$$\begin{array}{ccccccc} & & E'' & \rightarrowtail & E' & \rightarrowtail & C \\ & & \big\downarrow & \lrcorner & \big\downarrow & & \\ A & \twoheadleftarrow & E & \rightarrowtail & B & & \end{array}.$$

in the ambient abelian category of $\mathcal{A}$.

One may check that these definitions do not depend on choice.

Remark 11. Note, by construction, we have that

$$\ker(E'' \to E) \cong \ker(E' \twoheadrightarrow B) \in \mathcal{A}.$$

Therefore, as $\mathcal{A}$ is closed under extensions in the ambient abelian category of $\mathcal{A}$, it follows $E'' \in \mathcal{A}$ and $E'' \twoheadrightarrow E$ is an admissible epic. Similarly, $E'' \rightarrowtail E'$ is an admissible monic.

Remark 12. Note that there are two distinguished types of morphisms that play a special role in $\mathcal{QA}$. They are the admissible monics

$$(i_! : A \to B) := (A = A \overset{i}{\rightarrowtail} B)$$

and the (oppositely oriented) admissible epics

$$(q^! : A \to B) := (A \overset{q}{\twoheadleftarrow} B = B)$$

These are closed under composition and every morphism in $\mathcal{QA}$ factors as a composition of these (uniquely up to isomorphism).

We can now define the K-theory of a small exact category. Again, the definition is due to Quillen [16].

Definition 3.6. *Let* $\mathcal{A}$ *be a small exact category. Define*

$$K\mathcal{A} := \Omega B \mathcal{QA},$$

the loop space of the classifying space of $\mathcal{QA}$. *We define the K-Theory of* $\mathcal{A}$ *as*

$$K_n(\mathcal{A}) := \pi_n K\mathcal{A}.$$

Remark 13. For $n = 0$, this definition coincides with definition 2.9 but this is not easy to prove. We refer the reader to Srinivas [21], Example 4.10 for a 'bare hands' proof.

3.2.1 The long exact sequence

For the sake of brevity, we unfortunately do not include all of the 'fundamental' theorems for Higher Algebraic K-Theory for exact categories. These theorems were first proved by Quillen in his paper [16]. The reader is referred to Quillen's paper or Srinivas [21] for statements and proofs. However, to make a connection with the exact sequence in section 2.5, we state one of these theorems about how the higher K-groups fit into a long exact sequence.

The theorem will be stated in terms of quotients of abelian categories. The reader is referred to the appendix of Srinivas [21] for more details.

Theorem 3.7. *Let $\mathcal{B}$ be a Serre subcategory of an abelian category $\mathcal{A}$, with quotient abelian category $\mathcal{A}/\mathcal{B}$. Let $e : \mathcal{B} \to \mathcal{A}$ and $s : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ be the canonical functors. Then, there is a long exact sequence*

$$\cdots \to K_{i+1}(\mathcal{A}/\mathcal{B}) \to K_i(\mathcal{B}) \stackrel{e_*}{\to} K_i(\mathcal{A}) \stackrel{s_*}{\to} K_i(\mathcal{A}/\mathcal{B}) \to K_{i-1}(\mathcal{B}) \cdots$$
$$\stackrel{e_*}{\to} K_0(\mathcal{A}) \stackrel{s_*}{\to} K_0(\mathcal{A}/\mathcal{B}) \to 0.$$

Further, the sequence is functorial for exact functors $(\mathcal{A}, \mathcal{B}) \longrightarrow (\mathcal{A}', \mathcal{B}')$.

Proof. See Srinivas [21], Theorem 4.9. □

Remark 14. We should remark that this long exact sequence comes from the long exact sequence of homotopy groups associated to a homotopy fibration. This perhaps gives some very vague intuition as to why homotopy groups are used, but we emphasis that there are other more significant reasons for the use of homotopy groups in K-theory.

3.3 The $S^{-1}S$-Construction

The Higher K-theory of symmetric monoidal categories is defined in terms of the $S^{-1}S$-Construction. It first appeared in [9]. The motivation for this construction comes from the group completion of an abelian monoid (see section 2.1.1). This will be an analogous group completion, but this time in the context of topological spaces.

Definition 3.8. *Let S be a symmetric monoidal category. Define the category $S^{-1}S$ by the following datum:*

- *Objects in $S^{-1}S$ are pairs (m, n) where $m, n \in obj(S)$.*

- *A morphism in $S^{-1}S$ is an equivalence class of composites*

$$(m_1, m_2) \xrightarrow{s\otimes -} (s \otimes m_1, s \otimes m_2) \xrightarrow{f,g} (n_1, n_2).$$

This composite is equivalent to the composite

$$(m_1, m_2) \xrightarrow{t\otimes -} (t \otimes m_1, t \otimes m_2) \xrightarrow{f',g'} (n_1, n_2)$$

precisely when there exists an isomorphism $\alpha : s \xrightarrow{\simeq} t$ such that $f = f' \circ (\alpha \otimes m_1)$ and $g = g' \circ (\alpha \otimes m_2)$.

Remark 15. Note, there are two distinguished types of morphisms in $S^{-1}S$. First we have maps of the form $(f_1, f_2) : (m_1, m_2) \to (n_1, n_2)$; and second we have maps of the form $s \otimes - : (m, n) \to (s \otimes m, s \otimes n)$ obtained by taking $(f, g) = (id, id)$.

Remark 16. Note that a strict symmetric monodial functor $S \longrightarrow T$ induces a functor $S^{-1}S \longrightarrow T^{-1}T$ in the canonical way.

Remark 17. Note that $S^{-1}S$ is itself a symmetric monoidal category with product

$$(m, n) \otimes (m', n') := (m \otimes m', n \otimes n').$$

The functor in the preceding remark then becomes a strict symmetric monoidal functor between symmetric monoidal categories.

The Higher K-Theory of S is defined in terms of the following construction.

Definition 3.9. *Let S be a symmetric monoidal category. Define*

$$KS := B(S^{-1}S)$$

the classifying space of $S^{-1}S$. We define the K-Theory of S as

$$K_n(S) := \pi_n(KS).$$

Remark 18. For $n = 0$, this definition agrees with definition 2.5 when all arrows in S are isomorphisms. See proposition 3.18.

Remark 19. Recall that $S^{-1}S$ is symmetric monoidal and note that the functor $S \longrightarrow S^{-1}S$, $m \mapsto (m, e)$ is monoidal. Thus, we have that $B(S^{-1}S)$ is a homotopy commutative, homotopy associative H-space and the induced map $BS \longrightarrow B(S^{-1}S)$ is a H-space map.

Indeed, if S is any monoidal category with product $\otimes : S \times S \longrightarrow S$, then the H-space multiplication is given by

$$\mu : BS \times BS \cong B(S \times S) \xrightarrow{B\otimes} BS.$$

The natural isos $e \otimes s \cong s \otimes e \cong s$ imply that e is the homotopy identity of μ. Also, any monoidal functor $S \longrightarrow T$ between monodial categories induces a map of H-spaces

$BS \longrightarrow BT$. If in addition S is a symmetric monoidal category, then the symmetric monoidal axioms imply that BS is a homotopy commutative, homotopy associative H-space. We refer the reader to Whitehead [25] Chapter III for more details, including precise definitions of the homotopy theory language used here.

Remark 20. Let Y be a (based) H-space, with multiplication $\mu : Y \times Y \longrightarrow Y$. We claim that for any (based) space X, we can define a product in $[X, Y]$, the homotopy classes of (based) maps.

Indeed, given $f_1, f_2 : X \longrightarrow Y$, define product $f_1 \cdot f_2$ to be the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f_1 \times f_2} Y \times Y \xrightarrow{\mu} Y$$

where Δ is the diagonal map $x \mapsto (x, x)$. This product is compatible with homotopy and therefore fines a product in $[X, Y]$. This product is associative **precisely** when Y is homotopy associative and commutative **precisely** when Y is homotopy commutative. See Whitehead [25] Chapter III for more details.

Remark 21. By the above remark, if Y is a homotopy commutative and homotopy associative H-space, then $\pi_0(Y)$ becomes an abelian monoid and $H_0(Y; \mathbb{Z})$ is the monoid ring $\mathbb{Z}[\pi_0(Y)]$. Moreover, with the same assumption,

$$H_*(Y; \mathbb{Z}) = \bigoplus_{n=0}^{\infty} H_n(Y; \mathbb{Z})$$

becomes an associative graded commutative ring with unit, multiplication being defined as

$$H_i(Y; \mathbb{Z}) \times H_j(Y; \mathbb{Z}) \xrightarrow{\times} H_{i+j}(Y \times Y; \mathbb{Z}) \xrightarrow{\mu_*} H_{i+j}(Y; \mathbb{Z}),$$

the first map being the homology cross product. Note that, in this context, $\pi_0(Y)$ is a multiplicatively closed subset of ring $H_*(Y; \mathbb{Z})$ as in the sense of commutative algebra.

We are now in a position to define group completion in the context of topology. The above remarks will make the following definition make sense.

Definition 3.10. *Let X be a homotopy commutative, homotopy associative H-space. A group completion of X is a H-space Y, together with a H-space map $X \longrightarrow Y$ such that $\pi_0(Y)$ is the group completion of the abelian monoid $\pi_0(X)$ and the canonical map*

$$\pi_0(X)^{-1} H_*(X; \mathbb{Z}) \longrightarrow H_*(Y; \mathbb{Z})$$

from the localization $\pi_0(X)^{-1} H_*(X; \mathbb{Z})$ into $H_*(Y; \mathbb{Z})$ induced by $X \longrightarrow Y$ is an isomorphism of graded rings.

We would like to show that for S symmetric monoidal category, $B(S^{-1}S)$ is a group completion of BS when S satisfies some suitable conditions (see Theorem 3.17 and the remark after it). However, this will involve fitting $S^{-1}S$ into a more general framework.

Definition 3.11. *A monoidal category* $(S, \otimes, 1)$ *is said to act upon a category* $\mathcal{X}$ *by a functor* $\oslash : S \times \mathcal{X} \longrightarrow \mathcal{X}$ *if for every* $s, t, u \in S$ *and* $x \in \mathcal{X}$*, there exists natural isomorphisms*

$$s \oslash (t \oslash x) \xrightarrow{\simeq} (s \otimes t) \oslash x$$
$$1 \oslash x \xrightarrow{\simeq} x$$

such that diagrams

$$\begin{array}{ccc} s \otimes (1 \oslash x) & \longrightarrow & (s \otimes 1) \oslash x \\ & \searrow \quad \swarrow & \\ & s \oslash x & \end{array}$$

$$\begin{array}{ccc} & (s \otimes t) \oslash (u \oslash x) & \\ \swarrow & & \searrow \\ ((s \otimes t) \otimes u) \oslash x & & s \oslash (t \oslash (u \oslash x)) \\ \downarrow & & \downarrow \\ (s \otimes (t \otimes u)) \oslash x & \longrightarrow & s \oslash ((t \otimes u) \oslash x) \end{array}$$

commute.

For example, S acts on itself by $\otimes$.

From an action, we can obtain a category.

Definition 3.12. *Let* S *be a monoidal category acting on* $\mathcal{X}$*. Then, the category* $\langle S, \mathcal{X} \rangle$ *is defined by the following datum:*

- *The objects of* $\langle S, \mathcal{X} \rangle$ *are the same as the objects of* $\mathcal{X}$*.*
- *A morphism between* x *and* y *in* $\langle S, \mathcal{X} \rangle$ *is an equivalence class of pairs*

  $$(s, s \oslash x \xrightarrow{\rho} y).$$

  This is equivalent to composite

  $$(s', s' \oslash x \xrightarrow{\rho'} y)$$

  precisely when there exists an isomorphism $\psi : s' \xrightarrow{\simeq} s$ *such that* ρ' *is equal to the composite*

  $$s' \oslash x \xrightarrow{(-\oslash id)(\psi)} s \oslash x \xrightarrow{\rho} y.$$

If S is a monoidal category acting on $\mathcal{X}$, we write

$$S^{-1}\mathcal{X} := \langle S, S \times \mathcal{X} \rangle,$$

where $\oslash : S \times (S \times \mathcal{X}) \longrightarrow S \times \mathcal{X}$ is the canonical action that acts on both factors. Setting $\mathcal{X} = S$, we recover our original definition:

Proposition 3.13. *We have an isomorphism of categories*

$$\langle S, S \times S \rangle \cong S^{-1}S.$$

Proof. The isomorphism is given by

$$F : \langle S, S \times S \rangle \longrightarrow S^{-1}S$$
$$F(s, s \oslash (m_1, m_2) \xrightarrow{(f,g)} (n_1, n_2)) := (m_1, m_2) \xrightarrow{s \otimes -} (s \otimes m_1, s \otimes m_2) \xrightarrow{(f,g)} (n_1, n_2).$$

□

The following definition will be important:

Definition 3.14. *An action $\oslash : S \times \mathcal{X} \longrightarrow \mathcal{X}$ is said to be invertible if for any $s \in S$, the translation map*

$$s \oslash - : \mathcal{X} \longrightarrow \mathcal{X}$$
$$x \mapsto s \oslash x$$

is a homotopy equivalence i.e. induces a homotopy equivalence on classifying spaces.

We will need to prove the following proposition:

Proposition 3.15. *If S is a symmetric monoidal category acting on $\mathcal{X}$ by $\oslash$, then the action*

$$\hat{\oslash} : S \times (S^{-1}\mathcal{X}) \longrightarrow (S^{-1}\mathcal{X})$$
$$s \hat{\oslash} (t, x) := (t, s \oslash x)$$

is invertible.

Proof. Given $s \in S$, we claim that the functor $F : (t, x) \mapsto (t, s \oslash x)$ has homotopy inverse $G : (t, x) \mapsto (s \otimes t, x)$. Indeed, both $F \circ G$ and $G \circ F$ are the functors

$$(t, x) \mapsto (s \otimes t, s \oslash x)$$

and there is a natural transformation in $S^{-1}\mathcal{X}$

$$\eta_{(t,x)} : (t, x) \longrightarrow (s \otimes t, s \oslash x)$$
$$\eta_{(t,x)} := (s, s \oslash (t, x) = (s \otimes t, s \oslash x)).$$

□

Another definition that will be important to us will be that of a faithful action.

Definition 3.16. *Let S be a monoidal category. The action $\otimes : S \times S \longrightarrow S$ is said to be (left) faithful if for every $s \in S$, the functor*

$$s \otimes - : S \longrightarrow S$$
$$t \mapsto s \otimes t$$

is a faithful functor.

Now, let S be a symmetric monoidal category. Recall, BS is a homotopy commutative, homotopy associative H-space. Therefore, as explained above, $\pi_0(S) = \pi_0(BS)$ is an abelian monoid. Furthermore, $\pi_0(S)$ is a multiplicatively closed subset of commutative ring $H_0(S) = \mathbb{Z}[\pi_0(S)]$. Therefore, if S acts on $\mathcal{X}$, then $\pi_0(S)$ acts on $H_p(\mathcal{X})(= H_p(B\mathcal{X}))$ via the homology cross product

$$H_0(S) \times H_p(\mathcal{X}) \longrightarrow H_{0+p}(S \times \mathcal{X}) \longrightarrow H_p(\mathcal{X}).$$

In addition, in proposition 3.13 we showed that S acts invertibly on $S^{-1}\mathcal{X}$ via $s\hat{\oslash}(t, x) = (t, s \oslash x)$. Therefore, immediately from the definition of invertible action, we deduce that for every $\gamma \in \pi_0(BS)$, the map induced by the action of γ

$$\gamma\cdot : H_p(S^{-1}\mathcal{X}) \longrightarrow H_p(S^{-1}\mathcal{X})$$

is an **isomorphism**.

Thus, given the map

$$\varphi : H_*(\mathcal{X}; \mathbb{Z}) \longrightarrow H_*(S^{-1}\mathcal{X}; \mathbb{Z})$$

induced by the functor $x \mapsto (1, x)$, we localize $H_*(\mathcal{X}; \mathbb{Z})$ at multiplicatively closed subset $\pi_0(S)$ to obtain $\pi_0(S)^{-1}H_*(\mathcal{X}; \mathbb{Z})$ and we have the induced map

$$\overline{\varphi} : \pi_0(S)^{-1}H_*(\mathcal{X}; \mathbb{Z}) \longrightarrow H_*(S^{-1}\mathcal{X}; \mathbb{Z})$$
$$\frac{\sigma}{\gamma} \mapsto (\gamma\cdot)^{-1}\varphi(\sigma)$$

where $(\gamma\cdot)^{-1}$ is the inverse of $\gamma\cdot : H_p(S^{-1}\mathcal{X}) \longrightarrow H_p(S^{-1}\mathcal{X})$.

Theorem 3.17. *Let S be a symmetric monoidal category such that all arrow in S are* *isomorphisms and the action $\otimes : S \times S \longrightarrow S$ is faithful (see definition 3.16). Suppose S acts on $\mathcal{X}$.*

Then, for every $p \geq 0$, the maps described above

$$\pi_0(S)^{-1}H_p(\mathcal{X};\mathbb{Z}) \longrightarrow H_p(S^{-1}\mathcal{X};\mathbb{Z})$$

are isomorphisms.

Remark 22. Setting $\mathcal{X} = S$, this theorem together with proposition 3.18 says that $B(S^{-1}S)$ is a group completion of BS in the sense of definition 3.10 when S satisfies the conditions of the theorem.

Proof. See Srinivas [21], Theorem 7.2. □

Proposition 3.18. *Let S be a symmetric monoidal category. Recall, $\pi_0(S)$ is an abelian monoid. We have*

$$\pi_0(S)^{gp} \cong \pi_0(S^{-1}S).$$

Proof. Let $A := \pi_0(S)^{gp}$. Consider

$$\begin{aligned} \alpha : S^{-1}S &\longrightarrow A \\ \alpha(m,n) &:= [m] - [n]. \end{aligned}$$

We want to show α induces a map

$$\overline{\alpha} : \pi_0(S^{-1}S) \longrightarrow A.$$

To do this, we need to show α maps objects which are connected by a morphism to the same element in A.

For $s \in S$ and $m, n \in S$, we have

$$\begin{aligned} \alpha(s \otimes m, s \otimes n) &= [s \otimes m] - [s \otimes n] \\ &= [s] + [m] - ([s] + [n]) \\ &= [m] - [n] \\ &= \alpha(m,n). \end{aligned}$$

Moreover, for $f_i : m_i \to n_i$ morphisms in S, have

$$\alpha(m_1, m_2) = [m_1] - [m_2] = [n_1] - [n_2] = \alpha(n_1, n_2).$$

Therefore, deduce α induces a map

$$\overline{\alpha} : \pi_0(S^{-1}S) \longrightarrow A.$$

Now, by universal property of group completion, we have a diagram

$$\begin{array}{ccc} \pi_0(S) & \xrightarrow{[\cdot]} & A \\ & \searrow_{\varphi} & \Big\downarrow{\scriptstyle \overline{\varphi}} \\ & & \pi_0(S^{-1}S) \end{array}$$

where φ induced map $m \mapsto (m, e)$. The claim is that $\overline{\alpha}^{-1} = \overline{\varphi}$. The proof is a simple computation. □

3.4 The $+ = \mathcal{Q}$ theorem

Let R be a ring and consider $\mathbf{P}(R)$ the category of finitely generated projective R-modules.

Note, we may consider $\mathbf{P}(R)$ as an exact category and take K-Theory via the $\mathcal{Q}$-construction

$$K_n(\mathbf{P}(R)) = \pi_n(\Omega B\mathcal{Q}\mathbf{P}(R)).$$

But in addition, note that we may also consider the symmetric monoidal category $S :=$ iso$(\mathbf{P}(R))$ where iso$(\mathbf{P}(R))$ is the subcategory of $\mathbf{P}(R)$ with the same objects as $\mathbf{P}(R)$, whose arrows are all isomorphisms. We can take the K-Theory of S via the $S^{-1}S$ construction to obtain

$$K_n^{\oplus}(\mathbf{P}(R)) = \pi_n B(S^{-1}S).$$

Finally, we may just take the K-Theory of R via the the BGL^+ construction

$$K_n(R) = \pi_n(K_0(R) \times BGL(R)^+).$$

We commented earlier that as $\mathbf{P}(R)$ is a split exact category, all of these constructions coincide for $n = 0$. Rather miraculously, this is true for all $n \geq 0$:

Theorem 3.19. *In the above notation,, we have that $B(S^{-1}S)$ is homotopy equivalent to $K_0(R) \times BGL(R)^+$ and there is a natural homotopy equivalence*

$$\Omega B\mathcal{Q}\boldsymbol{P}(R) \xrightarrow{\simeq} B(S^{-1}S).$$

Proof. See Srinivas [21], Theorem 7.4 and Theorem 7.7. □

In some sense the $S^{-1}S$ construction provides a 'bridge' between the BGL^+ construction and the $\mathcal{Q}$-construction.

4 Conclusion

In summary, we hope that the reader has been convinced that the author is worthy enough to continue his PhD.

Bibliography

References

[1] https://mathoverflow.net/questions/4246/why-is-milnor-k-theory-not-ad-hoc.

[2] https://stacks.math.columbia.edu/tag/00nv.

[3] Armand Borel and Jean-Pierre Serre. Le théoreme de riemann-roch. *Bulletin de la Société mathématique de France*, 1958.

[4] Theo Bühler. Exact categories. *Expositiones Mathematicae*, 2010.

[5] Daniel Grayson. Higher algebraic k-theory: Ii. In *Algebraic K-theor*. Springer, 1976.

[6] Robin Hartshorne. *Algberaic Geometry*. Springer, 1977.

[7] John Milnor. Two complexes which are homeomorphic but combinatorially distinct. *Annals of Mathematics*, 1961.

[8] John Milnor. Algebraic k-theory and quadratic forms. *Inventiones mathematicae*, 1970.

[9] John Milnor. *Introduction to Algebraic K-Theory*. Princeton University Press, 1971.

[10] Daniel Quillen. Cohomology of groups. Actes du Congres International des Mathématiciens (Nice, 1970), 1970.

[11] Daniel Quillen. Higher algebraic k-theory: I. In *Higher K-theories*, pages 85–147. Springer, 1973.

[12] V Srinivas. *Algebraic K-Theory*. Birkhauser, 1995.

[13] Friedhelm Waldhausen. Algebraic k-theory of spaces. In *Algebraic and geometric topology*. Springer, 1985.

[14] Charles A Weibel. The development of algebraic k-theory before 1980. *Algebra, K-theory, groups, and education (New York, 1997)*, 1999.

[15] Charles A Weibel. *The K-book: An Introduction to Algebraic K-theory*. American Mathematical Society, 2013.

[16] George W. Whitehead. *Elements of Homotopy Theory*. Springer, 1978.