A Mathematical Model of Metal Ring Rolling

Dr Edward Brambley, Sunny Sood*

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Contents

1 Introduction

In this report, we will develop a mathematical model of metal ring rolling. Essentially, the idea of metal ring rolling is to use rollers to expand the diameter of a metal ring. To see this process in action, we refer the reader to this video : https://youtu.be/jvfHNNXRiYk by QSC Forge.

*s.sood.1@warwick.ac.uk

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Prerequisites and Literature recommendations

The reader is assumed to be familiar with solid mechanics. An excellent book to learn solid mechanics is "Plasticity Theory" by Jacob Lubliner [2]. We also assume that the reader is familiar with the basics of metal ring rolling. To learn more about metal rolling, we recommend the reader to read Minton's PhD thesis [3]. The relevant chapter for this report is chapter 6.

Furthermore, we advise the reader to have the research study group (RSG) report on metal ring rolling [1] with them. Indeed, this work is a continuation of their work, so it is natural that we refer to their report many times here. There is no need to study there work, but it is a good idea to have a copy of their report whilst reading this.

Finally, we recommend the reader to have Timoshenko's Applied Elasticity [5] with them as well.

Outcomes of the report

The main achievement of this report was to develop a mathematical model for metal ring rolling. The equations we derive generalise the equations derived by the RSG in [1, p12] and apply to the entire system. We also successfully generalise the friction model developed by the RSG in [1, p13] and we propose a set of constitutive laws (stress-strain relations) for different sections of the ring rolling system.

However, in this report, we fail to test our model against specific examples and see whether the results obtained agree with the results obtained using existing theory e.g. the theory given in Timoshenko's Applied Elasticity [5]. We also failed to computerise our model using software such as MATLAB or Mathematica.

Furthermore, we do not know how stress is distributed within the roll gap. In reality, the stresses within the roll gap will probably have very complicated distributions. Nevertheless, there could exist relatively simple distributions that sufficiently approximate the true distributions of the stresses and which may be used within a mathematical model. We leave the future researcher to explore these avenues!

2 The set up

Consider an arbitrary ring rolling configuration as shown in figure 1. We choose to parametrise the neutral axis of the workpiece by the arc length parameterisation $\mathbf{c}(s)$. We chose the neutral axis (as supposed to the centreline of the workpiece) because, by definition, there is no compression or tension along this axis. Hence, we know the horizontal and vertical stresses are zero along this axis. This may be useful in later calculations.

Next, we define

$$\mathbf{h}_t(s) := \mathbf{c}(s) + h_t(s)\mathbf{e}_r(s)$$

and

$$\mathbf{h}_b(s) := \mathbf{c}(s) + h_b(s)\mathbf{e}_r(s)$$

where $h_t(s)$ and $h_b(s)$ are functions that describe the perpendicular displacement between the neutral axis and the edges of the workpiece at point s. These parametrisation may be recognised as shown

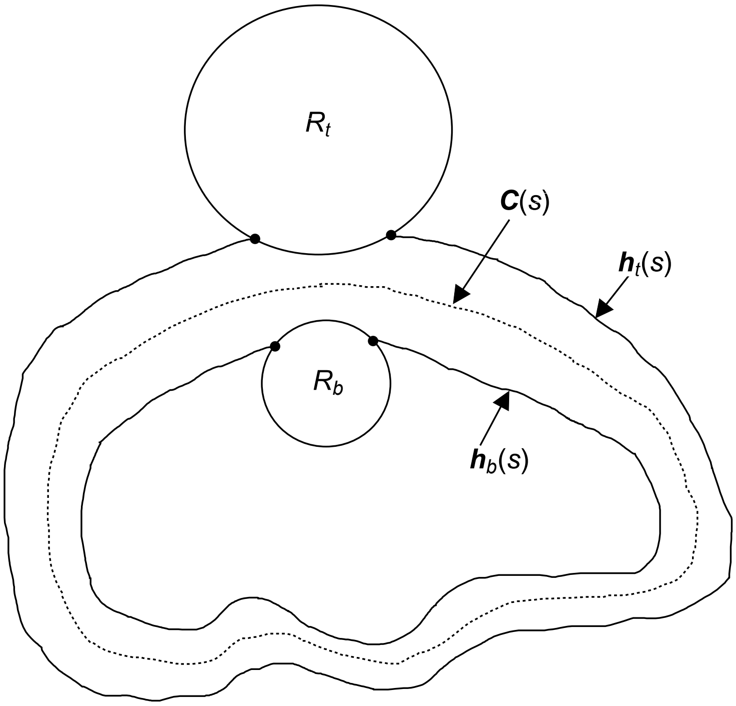

Figure 1: Showing an arbitrary ring rolling configuration

in figure 1. We define

$$\mathbf{e}_s(s) := \frac{d\mathbf{c}}{ds}(s)$$

and

$$\mathbf{e}_r(s) := \left(\frac{d\mathbf{c}}{ds}(s)\right)^{\perp}$$

where the latter is understood to be facing outwards. See figure 2 for an example.

Notice how the direction of these vectors depend on the arc length parameter s but they always remain mutually orthogonal. When clear, we will omit the arguments on the vectors. These vectors define an orthogonal curvilinear coordinate system which we will use to develop a model for the ring rolling process.

3 Developing a model

The idea is to subdivide the entire model into slices corresponding to a δs change in the arc length parameter. These are called slabs and an example of a slab is shown in figure 3.

Then, we use force and torque balance on a slab to derive three differential equations. Using equilibrium may seem inappropriate since the system is clearly dynamical, but we model the system as a *quasi-static* system. By quasi-static , we mean that the expansion of the ring is occurring slowly enough to reasonably approximate the system as a static system (to convince yourself, watch the youtube video https://youtu.be/jvfHNNXRiYk again).

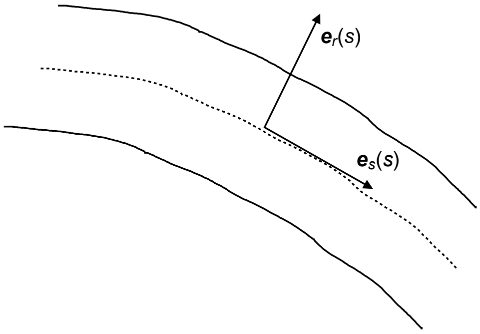

Figure 2: Showing the basis vectors

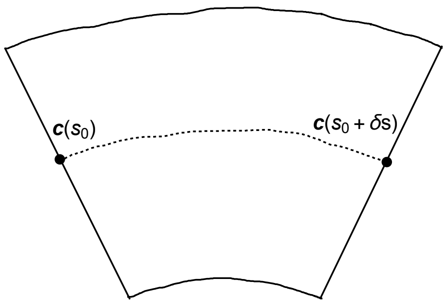

Figure 3: Showing a possible slab

Now, consider the following slab as shown in the figure 3. We seek to calculate the unit normal vectors to each side of the slab.

Firstly, if we let $\kappa(s)$ denote the curvature of the neutral axis at s, then using the Frenet - Serret formulae, we have

$$\frac{d\mathbf{e}_s}{ds} = \kappa \mathbf{e}_r$$

and

$$\frac{d\mathbf{e}_r}{ds} = -\kappa \mathbf{e}_s.$$

Next, recall

$$\mathbf{h}_t(s) = \mathbf{c}(s) + h_t(s)\mathbf{e}_r(s)$$

and

$$\mathbf{h}_b(s) = \mathbf{c}(s) + h_b(s)\mathbf{e}_r(s).$$

A simple computation shows that

$$\frac{d\mathbf{h}_t}{ds} = (1 - h_t\kappa)\,\mathbf{e}_s + h_t'\mathbf{e}_r$$

and

$$\frac{d\mathbf{h}_b}{ds} = (1 - h_b\kappa)\,\mathbf{e}_s + h_b'\mathbf{e}_r.$$

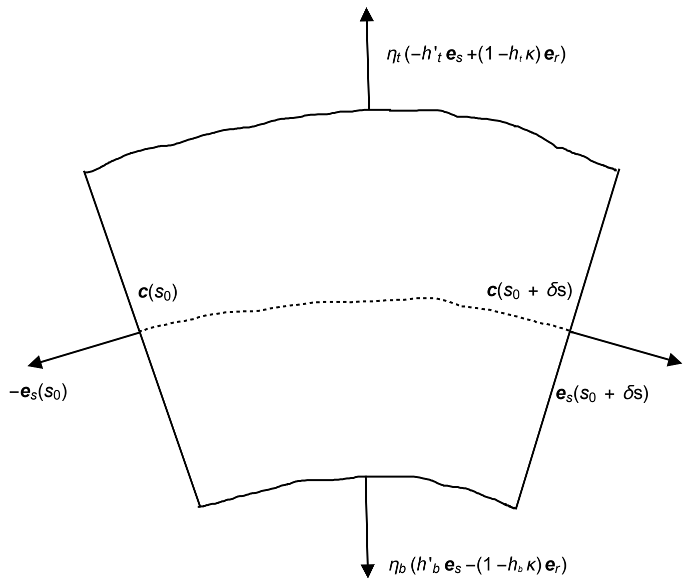

Figure 4: Showing the unit normal vectors on the slab

Then, we need two rotate the first vector 90 degrees anticlockwise, the second vector 90 degrees clockwise and normalise both. This gives us the two unit normals at the top and bottom of the slab. Using column notation ($\mathbf{e}_s$ occupying the first slot, $\mathbf{e}_r$ the second slot), these vectors are

$$\mathbf{n}_t = \eta_t \begin{pmatrix} -h_t' \\ 1 - h_t \kappa \end{pmatrix}$$

and

$$\mathbf{n}_b = \eta_b \begin{pmatrix} h_b' \\ -1 + h_b \kappa \end{pmatrix}$$

respectively where

$$\eta_t := \frac{1}{\sqrt{h_t'^2 + (1 - h_t \kappa)^2}}$$

and

$$\eta_b := \frac{1}{\sqrt{h_b'^2 + (1 - h_b \kappa)^2}}.$$

The unit normal vectors on the left and right of the slab as shown in figure 3 are $-\mathbf{e}_s(s_0)$ and $\mathbf{e}_s(s_0 + \delta s)$ respectively. These vectors are on display in figure 4. Now, suppose that the Cauchy stress tensor at $\mathbf{c}(s) + r\mathbf{e}_r$ in our curvilinear coordinate system is

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_s & \tau \\ \tau & \sigma_r \end{pmatrix}.$$

Recall that the traction vector (force per unit length) in direction $\mathbf{n}$ is given by

$$\mathbf{T}^{\mathbf{n}} = \mathbf{n} \cdot \boldsymbol{\sigma}.$$

We will now calculate the traction vectors on the edges of the slab. Note how the basis vectors will be different depending on which portion of the slab we are considering. This is why we make explicit reference to the basis vectors in the following calculations.

Firstly,

$$\begin{aligned}
\mathbf{T}^{\mathbf{n}_t} &= \mathbf{n}_t \cdot \boldsymbol{\sigma} \\
&= \eta_t(s)\Big(- h_t'(s)\mathbf{e}_s(s) + (1 - h_t(s)\kappa(s))\mathbf{e}_r(s)\Big) \cdot \begin{pmatrix} \sigma_s(s, h_t(s)) & \tau(s, h_t(s)) \\ \tau(s, h_t(s)) & \sigma_r(s, h_t(s)) \end{pmatrix} \\
&= \eta_t(s) \left(-h_t'(s)\sigma_s(s, h_t(s)) + \tau(s, h_t(s))(1 - h_t(s)\kappa(s))\right) \mathbf{e}_s(s) \\
&+ \eta_t(s) \left(-h_t'(s)\tau(s, h_t(s)) + \sigma_r(s, h_t(s))(1 - h_t(s)\kappa(s))\right) \mathbf{e}_r(s).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{T}^{\mathbf{n}_b} &= \mathbf{n}_b \cdot \boldsymbol{\sigma} \\
&= \eta_b(s)\Big(h_b'(s)\mathbf{e}_s(s) - (1 - h_b(s)\kappa(s))\mathbf{e}_r(s)\Big) \cdot \begin{pmatrix} \sigma_s(s, h_b(s)) & \tau(s, h_b(s)) \\ \tau(s, h_b(s)) & \sigma_r(s, h_b(s)) \end{pmatrix} \\
&= \eta_b(s) \left(h_b'(s)\sigma_s(s, h_b(s)) - \tau(s, h_b(s))(1 - h_b(s)\kappa(s))\right) \mathbf{e}_s(s) \\
&+ \eta_b(s) \left(h_b'(s)\tau(s, h_b(s)) - \sigma_r(s, h_b(s))(1 - h_b(s)\kappa(s))\right) \mathbf{e}_r(s).
\end{aligned}$$

Next,

$$\begin{aligned}
\mathbf{T}^{-\mathbf{e}_s(s_0)} &= -\mathbf{e}_s(s_0) \cdot \begin{pmatrix} \sigma_s(s_0, r) & \tau(s_0, r) \\ \tau(s_0, r) & \sigma_r(s_0, r) \end{pmatrix} \\
&= -\sigma_s(s_0, r)\mathbf{e}_s(s_0) - \tau(s_0, r)\mathbf{e}_r(s_0).
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathbf{T}^{\mathbf{e}_s(s_0+\delta s)} &= \mathbf{e}_s(s_0 + \delta s) \cdot \begin{pmatrix} \sigma_s(s_0 + \delta s, r) & \tau(s_0 + \delta s, r) \\ \tau(s_0 + \delta s, r) & \sigma_r(s_0 + \delta s, r) \end{pmatrix} \\
&= \sigma_s(s_0 + \delta s, r)\mathbf{e}_s(s_0 + \delta s) + \tau(s_0 + \delta s, r)\mathbf{e}_r(s_0 + \delta s) \\
&= \big(\sigma_s(s_0, r) + \delta s \partial_s \sigma_s(s_0, r) + O(\delta s^2)\big) \left(\mathbf{e}_s(s_0) + \delta s \frac{d\mathbf{e}_s}{ds}(s_0) + O(\delta s^2)\right) \\
&+ \big(\tau(s_0, r) + \delta s \partial_s \tau(s_0, r) + O(\delta s^2)\big) \left(\mathbf{e}_r(s_0) + \delta s \frac{d\mathbf{e}_r}{ds}(s_0) + O(\delta s^2)\right) \\
&= \big(\sigma_s(s_0, r) + \delta s \partial_s \sigma_s(s_0, r) + O(\delta s^2)\big) \big(\mathbf{e}_s(s_0) + \delta s \kappa(s_0)\mathbf{e}_r(s_0) + O(\delta s^2)\big) \\
&+ \big(\tau(s_0, r) + \delta s \partial_s \tau(s_0, r) + O(\delta s^2)\big) \big(\mathbf{e}_r(s_0) - \delta s \kappa(s_0)\mathbf{e}_s(s_0) + O(\delta s^2)\big) \\
&= \left(\sigma_s(s_0, r) + \delta s(\partial_s \sigma_s(s_0, r) - \kappa(s_0)\tau(s_0, r))\right) \mathbf{e}_s(s_0) \\
&+ \left(\tau(s_0, r) + \delta s(\partial_s \tau(s_0, r) + \kappa(s_0)\sigma_s(s_0, r))\right) \mathbf{e}_r(s_0) \quad \text{(To first order in } \delta s\text{)}.
\end{aligned}$$

These traction vectors are displayed in figure 5 in column notation with the arguments removed. We emphasis that the basis vectors are different corresponding to the portion of the slab under consideration.

Now, we resolve forces in the $\mathbf{e}_s(s_0)$ and $\mathbf{e}_r(s_0)$ and consider moments about the point $\mathbf{c}(s_0)$ to derive 3 differential equations. We do this as we know that the slab is in equilibrium due to our quasi-static assumption.

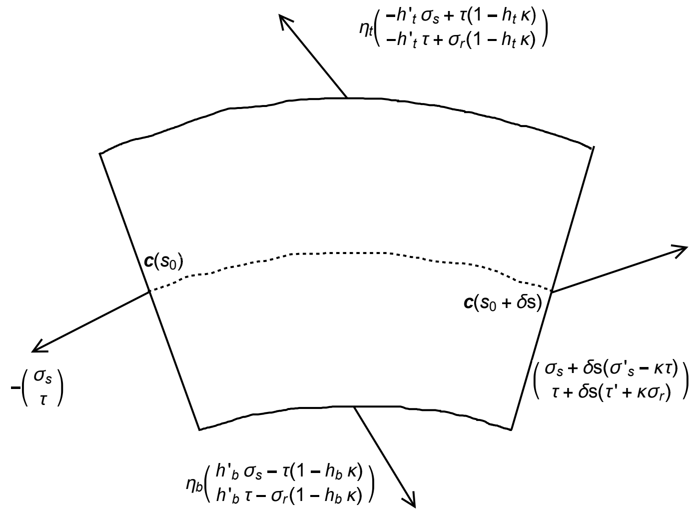

Figure 5: Showing the traction vectors on the slab

3.1 Force balance in the $\mathbf{e}_s(s_0)$ direction.

Computing the forces in the $\mathbf{e}_s(s_0)$ direction and using dl_t and dl_b to denote the elemental length on the top and bottom part of the slab respectively, we have

$$\begin{aligned} 0 = &\int_{h_b(s_0)}^{h_t(s_0)} \mathbf{T}^{-\mathbf{e}_s(s_0)} \cdot \mathbf{e}_s(s_0) \quad dr + \int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)} \mathbf{T}^{\mathbf{e}_s(s_0+\delta s)} \cdot \mathbf{e}_s(s_0) \quad dr \\ &+ \int_{s=s_0}^{s=s_0+\delta s} \mathbf{T}^{\mathbf{n}_t} \cdot \mathbf{e}_s(s_0) \quad dl_t + \int_{s=s_0}^{s=s_0+\delta s} \mathbf{T}^{\mathbf{n}_b} \cdot \mathbf{e}_s(s_0) \quad dl_b. \end{aligned}$$

Let us consider that latter two integrals first. We have

$$\begin{aligned} &\int_{s=s_0}^{s=s_0+\delta s} \mathbf{T}^{\mathbf{n}_t} \cdot \mathbf{e}_s(s_0) \quad dl_t + \int_{s=s_0}^{s=s_0+\delta s} \mathbf{T}^{\mathbf{n}_b} \cdot \mathbf{e}_s(s_0) \quad dl_b \\ &= \int_{s_0}^{s_0+\delta s} \mathbf{T}^{\mathbf{n}_t} \cdot \mathbf{e}_s(s_0) \quad \frac{dl_t}{ds} ds + \int_{s_0}^{s_0+\delta s} \mathbf{T}^{\mathbf{n}_b} \cdot \mathbf{e}_s(s_0) \quad \frac{dl_b}{ds} ds. \end{aligned}$$

Now, observe that

$$\begin{aligned} \frac{dl_t}{ds} &= \left\| \frac{d\mathbf{h}_t}{ds} \right\| = \frac{1}{\eta_t} \\ \frac{dl_b}{ds} &= \left\| \frac{d\mathbf{h}_b}{ds} \right\| = \frac{1}{\eta_b}. \end{aligned}$$

Furthermore, note that

$$\begin{aligned} \mathbf{e}_s(s) \cdot \mathbf{e}_s(s_0) &= \mathbf{e}_s(s_0 + s - s_0) \cdot \mathbf{e}_s(s_0) \\ &= \big(\mathbf{e}_s(s_0) + (s - s_0)\kappa \mathbf{e}_r(s_0) + O((s - s_0)^2)\big) \cdot \mathbf{e}_s(s_0) \\ &= 1 \quad \text{(To first order in } \delta s) \end{aligned}$$

and

$$\begin{aligned}\mathbf{e}_r(s)\cdot\mathbf{e}_s(s_0) &= \mathbf{e}_r(s_0+s-s_0)\cdot\mathbf{e}_s(s_0)\\ &= \left(\mathbf{e}_r(s_0)-(s-s_0)\kappa\mathbf{e}_s(s_0)+O((s-s_0)^2)\right)\cdot\mathbf{e}_s(s_0)\\ &= -(s-s_0)\kappa \quad \text{(To first order in } \delta s\text{)}.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{s_0}^{s_0+\delta s}\mathbf{T}^{\mathbf{n}_t}\cdot\mathbf{e}_s(s_0)\quad\frac{dl_t}{ds}ds+\int_{s_0}^{s_0+\delta s}\mathbf{T}^{\mathbf{n}_b}\cdot\mathbf{e}_s(s_0)\quad\frac{dl_b}{ds}ds\\ &=\int_{s_0}^{s_0+\delta s}-h_t'(s)\sigma_s(s,h_t(s))+\tau(s,h_t(s))(1-h_t(s)\kappa(s))\\ &+(s-s_0)\left(-h_t'(s)\tau(s,h_t(s))+\sigma_r(s,h_t(s))(1-h_t(s)\kappa(s))\right)\quad ds\\ &+\int_{s_0}^{s_0+\delta s}h_b'(s)\sigma_s(s,h_b(s))-\tau(s,h_b(s))(1-h_b(s)\kappa(s))\\ &+(s-s_0)\left(h_b'(s)\tau(s,h_b(s))-\sigma_r(s,h_b(s))(1-h_b(s)\kappa(s))\right)\quad ds.\end{aligned}$$

As δs is small, we may reasonably approximate these integrals as δs multiplied by the integrand evaluated at s_0. Hence, defining $\nu_t := 1-h_t(s_0)\kappa(s_0)$, $\nu_b := 1-h_b(s_0)\kappa(s_0)$, $\tau(s_0,h_t(s_0)) := \tau_t$, $\tau(s_0,h_b(s_0)) := \tau_b$ and if we drop the arguments involving s_0, we have

$$\begin{aligned}&\int_{s_0}^{s_0+\delta s}\mathbf{T}^{\mathbf{n}_t}\cdot\mathbf{e}_s(s_0)\quad\frac{dl_t}{ds}ds+\int_{s_0}^{s_0+\delta s}\mathbf{T}^{\mathbf{n}_b}\cdot\mathbf{e}_s(s_0)\quad\frac{dl_b}{ds}ds\\ &=\delta s\left(-h_t'\sigma_s(h_t)+\tau_t\nu_t+h_b'\sigma_s(h_b)-\tau_b\nu_b\right).\end{aligned}$$

Now, returning to our first two integrals, we have

$$\begin{aligned}&\int_{h_b(s_0)}^{h_t(s_0)}\mathbf{T}^{-\mathbf{e}_s(s_0)}\cdot\mathbf{e}_s(s_0)\quad dr+\int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)}\mathbf{T}^{\mathbf{e}_s(s_0+\delta s)}\cdot\mathbf{e}_s(s_0)\quad dr\\ &=\int_{h_b(s_0)}^{h_t(s_0)}-\sigma_s(s_0,r)\quad dr+\int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)}\sigma_s(s_0,r)+\delta s(\partial_s\sigma_s(s_0,r)-\kappa(s_0)\tau(s_0,r))\quad dr.\end{aligned}$$

The integral

$$\int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)}\delta s\partial_s\sigma_s(s_0,r)\quad dr$$

to first order in δs is equivalent to

$$\int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)}\sigma_s(s_0+\delta s,r)-\sigma_s(s_0,r)\quad dr$$

and the integral

$$\int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)}-\delta s\kappa(s_0)\tau(s_0,r)\quad dr$$

to first order in δs is equivalent to

$$-\delta s\int_{h_b(s_0)}^{h_t(s_0)}\kappa(s_0)\tau(s_0,r)\quad dr.$$

Thus, our integrals become

$$\int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)} \sigma_s(s_0+\delta s, r) \quad dr - \int_{h_b(s_0)}^{h_t(s_0)} \sigma_s(s_0, r) \quad dr - \delta s \int_{h_b(s_0)}^{h_t(s_0)} \kappa(s_0)\tau(s_0, r) \quad dr.$$

Now, if we define

$$Q(s) := \int_{h_b(s)}^{h_t(s)} \sigma_s(s, r) \quad dr$$

and compute $Q(s_0+\delta s) - Q(s_0)$ using a Taylor expansion up to first order in δs, we find that

$$\int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)} \sigma_s(s_0+\delta s, r) \quad dr - \int_{h_b(s_0)}^{h_t(s_0)} \sigma_s(s_0, r) \quad dr = \delta s \frac{d}{ds} \int_{h_b(s_0)}^{h_t(s_0)} \sigma_s(s_0, r) \quad dr.$$

Thus, our balance equation becomes

$$\begin{aligned} 0 = {} & \delta s \frac{d}{ds} \int_{h_b(s_0)}^{h_t(s_0)} \sigma_s(s_0, r) \quad dr - \delta s \int_{h_b(s_0)}^{h_t(s_0)} \kappa(s_0)\tau(s_0, r) \quad dr \\ & - \delta s \left(h_t' \sigma_s(h_t) + \tau_t \nu_t + h_b' \sigma_s(h_b) - \tau_b \nu_b \right). \end{aligned}$$

Rearranging, cancelling and dropping arguments we obtain

$$\frac{d}{ds} \int_{h_b}^{h_t} \sigma_s \quad dr = \kappa \int_{h_b}^{h_t} \tau \quad dr + \left(h_t' \sigma_s(h_t) - \tau_t \nu_t \right) - \left(h_b' \sigma_s(h_b) - \tau_b \nu_b \right).$$

3.2 Force balance in the $\mathbf{e}_r(s_0)$ direction.

Computing the forces in the $\mathbf{e}_r(s_0)$ direction and using equilibrium, we have

$$\begin{aligned} 0 = {} & \int_{h_b(s_0)}^{h_t(s_0)} \mathbf{T}^{-\mathbf{e}_s(s_0)} \cdot \mathbf{e}_r(s_0) \quad dr + \int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)} \mathbf{T}^{\mathbf{e}_s(s_0+\delta s)} \cdot \mathbf{e}_r(s_0) \quad dr \\ & + \int_{s=s_0}^{s=s_0+\delta s} \mathbf{T}^{\mathbf{n}_t} \cdot \mathbf{e}_r(s_0) \quad dl_t + \int_{s=s_0}^{s=s_0+\delta s} \mathbf{T}^{\mathbf{n}_b} \cdot \mathbf{e}_r(s_0) \quad dl_b. \end{aligned}$$

Continuing in the exact same way as above, we find that

$$\begin{aligned} & \int_{s=s_0}^{s=s_0+\delta s} \mathbf{T}^{\mathbf{n}_t} \cdot \mathbf{e}_r(s_0) \quad dl_t + \int_{s=s_0}^{s=s_0+\delta s} \mathbf{T}^{\mathbf{n}_b} \cdot \mathbf{e}_r(s_0) \quad dl_b \\ & = \delta s \left(-h_t' \tau_t + \sigma_r(h_t) \nu_t + h_b' \tau_b - \sigma_r(h_b) \nu_b \right). \end{aligned}$$

Similarly, we also find that

$$\begin{aligned} & \int_{h_b(s_0)}^{h_t(s_0)} \mathbf{T}^{-\mathbf{e}_s(s_0)} \cdot \mathbf{e}_r(s_0) \quad dr + \int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)} \mathbf{T}^{\mathbf{e}_s(s_0+\delta s)} \cdot \mathbf{e}_r(s_0) \quad dr \\ & = \delta s \frac{d}{ds} \int_{h_b(s_0)}^{h_t(s_0)} \tau(s_0, r) \quad dr + \delta s \int_{h_b(s_0)}^{h_t(s_0)} \kappa(s_0) \sigma_s(s_0, r) \quad dr. \end{aligned}$$

Thus, our balance equation becomes

$$\frac{d}{ds} \int_{h_b}^{h_t} \tau \quad dr = -\kappa \int_{h_b}^{h_t} \sigma_s \quad dr + \left(h_t' \tau_t - \sigma_r(h_t) \nu_t \right) - \left(h_b' \tau_b - \sigma_r(h_b) \nu_b \right).$$

3.3 Rotational balance

Suppose that the force at an arbitrary point in the slab, $\mathbf{c}(s)+r\mathbf{e}_r(s)$, is $\mathbf{F}(s,r)$. Then, the moment of this force about $\mathbf{c}(s_0)$ is

$$\left[\big(\mathbf{c}(s)+r\mathbf{e}_r(s)-\mathbf{c}(s_0)\big)\times\mathbf{F}(s,r)\right]_3.$$

Thus, by rotational equilibrium, we have

$$\int_{s,r\in\Omega}\left[\big(\mathbf{c}(s)+r\mathbf{e}_r(s)-\mathbf{c}(s_0)\big)\times\mathbf{F}(s,r)\right]_3 d\Omega=0$$

where Ω is the boundary of the slab. We will now compute this integral. We have

$$\begin{aligned}
&\int_{s,r\in\Omega}\left[\big(\mathbf{c}(s)+r\mathbf{e}_r(s)-\mathbf{c}(s_0)\big)\times\mathbf{F}(s,r)\right]_3 d\Omega=\\
&\int_{h_b(s_0)}^{h_t(s_0)}\left[\big(\mathbf{c}(s_0)+r\mathbf{e}_r(s_0)-\mathbf{c}(s_0)\big)\times\begin{pmatrix}-\sigma_s(s_0,r)\\-\tau(s_0,r)\end{pmatrix}\right]_3 dr\\
&+\int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)}\left[\big(\mathbf{c}(s_0+\delta s)+r\mathbf{e}_r(s_0+\delta s)-\mathbf{c}(s_0)\big)\times\begin{pmatrix}\sigma_s(s_0,r)+\delta s(\partial_s\sigma_s(s_0,r)-\kappa(s_0)\tau(s_0,r))\\\tau(s_0,r)+\delta s(\partial_s\tau(s_0,r)+\kappa(s_0)\sigma_s(s_0,r))\end{pmatrix}\right]_3 dr\\
&+\int_{s=s_0}^{s=s_0+\delta s}\left[\big(\mathbf{c}(s)+h_t(s)\mathbf{e}_r(s)-\mathbf{c}(s_0)\big)\times\eta_t(s)\begin{pmatrix}-h_t'(s)\sigma_s(s,h_t(s))+\tau(s,h_t(s))\nu_t(s)\\-h_t'(s)\tau(s,h_t(s))+\sigma_r(s,h_t(s))\nu_t(s)\end{pmatrix}\right]_3 dl_t\\
&+\int_{s=s_0}^{s=s_0+\delta s}\left[\big(\mathbf{c}(s)+h_b(s)\mathbf{e}_r(s)-\mathbf{c}(s_0)\big)\times\eta_b(s)\begin{pmatrix}h_b'(s)\sigma_s(s,h_b(s))-\tau(s,h_b(s))\nu_b(s)\\h_b'(s)\tau(s,h_b(s))-\sigma_r(s,h_b(s))\nu_b(s)\end{pmatrix}\right]_3 dl_b.
\end{aligned}$$

The first integral may be computed as follows:

$$\begin{aligned}
&\int_{h_b(s_0)}^{h_t(s_0)}\left[\big(\mathbf{c}(s_0)+r\mathbf{e}_r(s_0)-\mathbf{c}(s_0)\big)\times\begin{pmatrix}-\sigma_s(s_0,r)\\-\tau(s_0,r)\end{pmatrix}\right]_3 dr\\
&=\int_{h_b(s_0)}^{h_t(s_0)}\left[\begin{pmatrix}0\\r\end{pmatrix}\times\begin{pmatrix}-\sigma_s(s_0,r)\\-\tau(s_0,r)\end{pmatrix}\right]_3 dr\\
&=\int_{h_b(s_0)}^{h_t(s_0)} r\sigma_s(s_0,r)\quad dr.
\end{aligned}$$

To compute the second integral, we first find an expression for $\mathbf{c}(s_0+\delta s)+r\mathbf{e}_r(s_0+\delta s)-\mathbf{c}(s_0)$ up to first order in δs. We have

$$\begin{aligned}
&\mathbf{c}(s_0+\delta s)+r\mathbf{e}_r(s_0+\delta s)-\mathbf{c}(s_0)\\
&=\mathbf{c}(s_0)+\delta s\frac{d\mathbf{c}}{ds}(s_0)+O(\delta s^2)+r\Big(\mathbf{e}_r(s_0)+\delta s\frac{d\mathbf{e}_r}{ds}(s_0)+O(\delta s^2)\Big)-\mathbf{c}(s_0)\\
&=\delta s\mathbf{e}_s(s_0)-\delta s\kappa(s_0)r\mathbf{e}_s(s_0)+r\mathbf{e}_r(s_0)\\
&=\begin{pmatrix}\delta s\big(1-\kappa(s_0)r\big)\\r\end{pmatrix}.
\end{aligned}$$

Thus, up to first order

$$\int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)} \left[\big(\mathbf{c}(s_0+\delta s) + r\mathbf{e}_r(s_0+\delta s) - \mathbf{c}(s_0)\big) \times \begin{pmatrix} \sigma_s(s_0,r) + \delta s(\partial_s \sigma_s(s_0,r) - \kappa(s_0)\tau(s_0,r)) \\ \tau(s_0,r) + \delta s(\partial_s \tau(s_0,r) + \kappa(s_0)\sigma_s(s_0,r)) \end{pmatrix} \right]_3 dr$$
$$= \int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)} \left[\begin{pmatrix} \delta s\big(1-\kappa(s_0)r\big) \\ r \end{pmatrix} \times \begin{pmatrix} \sigma_s(s_0,r) + \delta s(\partial_s \sigma_s(s_0,r) - \kappa(s_0)\tau(s_0,r)) \\ \tau(s_0,r) + \delta s(\partial_s \tau(s_0,r) + \kappa(s_0)\sigma_s(s_0,r)) \end{pmatrix} \right]_3 dr$$
$$= \int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)} \delta s \tau(s_0,r) - r\sigma_s(s_0,r) - \delta s r \partial_s \sigma_s(s_0,r) \quad dr$$
$$= -\int_{h_b(s_0)}^{h_t(s_0)} r\sigma_s(s_0+\delta s, r) \quad dr + \delta s \int_{h_b(s_0)}^{h_t(s_0)} \tau(s_0,r) \quad dr$$

Finally, using the fact that δs is small, we may compute the final two integrals in our rotational balance equation as follows:

$$\int_{s=s_0}^{s=s_0+\delta s} \left[\big(\mathbf{c}(s) + h_t(s)\mathbf{e}_r(s) - \mathbf{c}(s_0)\big) \times \eta_t(s) \begin{pmatrix} -h_t'(s)\sigma_s(s,h_t(s)) + \tau(s,h_t(s))\nu_t(s) \\ -h_t'(s)\tau(s,h_t(s)) + \sigma_r(s,h_t(s))\nu_t(s) \end{pmatrix} \right]_3 dl_t$$
$$+ \int_{s=s_0}^{s=s_0+\delta s} \left[\big(\mathbf{c}(s) + h_b(s)\mathbf{e}_r(s) - \mathbf{c}(s_0)\big) \times \eta_b(s) \begin{pmatrix} h_b'(s)\sigma_s(s,h_b(s)) - \tau(s,h_b(s))\nu_b(s) \\ h_b'(s)\tau(s,h_b(s)) - \sigma_r(s,h_b(s))\nu_b(s) \end{pmatrix} \right]_3 dl_b$$
$$= \delta s \Bigg(\big(\mathbf{c}(s_0) + h_t(s_0)\mathbf{e}_r(s_0) - \mathbf{c}(s_0)\big) \times \begin{pmatrix} -h_t'(s_0)\sigma_s(s_0,h_t(s_0)) + \tau(s_0,h_t(s_0))\nu_t(s_0) \\ -h_t'(s_0)\tau(s_0,h_t(s_0)) + \sigma_r(s_0,h_t(s_0))\nu_t(s_0) \end{pmatrix} \Bigg)$$
$$+ \delta s \Bigg(\big(\mathbf{c}(s_0) + h_b(s_0)\mathbf{e}_r(s_0) - \mathbf{c}(s_0)\big) \times \begin{pmatrix} h_b'(s_0)\sigma_s(s_0,h_b(s_0)) - \tau(s_0,h_b(s_0))\nu_b(s_0) \\ h_b'(s_0)\tau(s_0,h_b(s_0)) - \sigma_r(s_0,h_b(s_0))\nu_b(s_0) \end{pmatrix} \Bigg)$$
$$= \delta s \big(h_t(s_0)h_t'(s_0)\sigma_s(s_0,h_t(s_0)) - h_t(s_0)\tau(s_0,h_t(s_0))\nu_t(s_0) \big)$$
$$- \delta s \big(h_b(s_0)h_b'(s_0)\sigma_s(s_0,h_b(s_0)) - h_b(s_0)\tau(s_0,h_b(s_0))\nu_b(s_0) \big).$$

Thus, our rotational balance equation becomes

$$\begin{aligned} 0 &= \delta s \int_{h_b(s_0)}^{h_t(s_0)} \tau(s_0,r) \quad dr \\ &- \left[\int_{h_b(s_0+\delta s)}^{h_t(s_0+\delta s)} r\sigma_s(s_0+\delta s, r) \quad dr - \int_{h_b(s_0)}^{h_t(s_0)} r\sigma_s(s_0,r) \quad dr \right] \\ &+ \delta s \big(h_t(s_0)h_t'(s_0)\sigma_s(s_0,h_t(s_0)) - h_t(s_0)\tau(s_0,h_t(s_0))\nu_t(s_0) \big) \\ &- \delta s \big(h_b(s_0)h_b'(s_0)\sigma_s(s_0,h_b(s_0)) - h_b(s_0)\tau(s_0,h_b(s_0))\nu_b(s_0) \big). \end{aligned}$$

Using similar reasoning and notation as earlier, we obtain the following differential equation:

$$\frac{d}{ds} \int_{h_b}^{h_t} r\sigma_s \quad dr = \int_{h_b}^{h_t} \tau \quad dr + \big(h_t h_t' \sigma_s(h_t) - h_t \tau_t \nu_t \big) - \big(h_b h_b' \sigma_s(h_b) - h_b \tau_b \nu_b \big).$$

4 The Governing Differential Equations

Thus, from the above, we have obtained the following three differential equations:

$$\frac{d}{ds}\int_{h_b}^{h_t} \sigma_s \quad dr = \kappa \int_{h_b}^{h_t} \tau \quad dr + (h_t'\sigma_s(h_t) - \tau_t\nu_t) - (h_b'\sigma_s(h_b) - \tau_b\nu_b) \tag{1}$$

$$\frac{d}{ds}\int_{h_b}^{h_t} \tau \quad dr = -\kappa \int_{h_b}^{h_t} \sigma_s \quad dr + (h_t'\tau_t - \sigma_r(h_t)\nu_t) - (h_b'\tau_b - \sigma_r(h_b)\nu_b) \tag{2}$$

$$\frac{d}{ds}\int_{h_b}^{h_t} r\sigma_s \quad dr = \int_{h_b}^{h_t} \tau \quad dr + \big(h_t h_t'\sigma_s(h_t) - h_t\tau_t\nu_t\big) - \big(h_b h_b'\sigma_s(h_b) - h_b\tau_b\nu_b\big). \tag{3}$$

Note well that setting the curvature to zero, we obtain the differential equations derived in the RSG report [1, p12]. This shows that we have successfully generalised their equations for arbitray ring rolling configurations.

Now, notice that we have six unknowns in the form of stresses at the top and bottom rollers but only three equations. We will need to make modelling assumptions to reduce the number of unknowns to three. The modelling assumptions that we will make are

- A friction law for the workpiece
- The workpiece within the roll gap is at yield
- The workpiece does not have normal stresses outside of the roll gap

We begin with the friction law.

5 Friction Law

We use Coulomb friction to derive some equations. However, before doing this, we need some preliminary definitions.

Assume the workpiece is going into the rolls at a velocity smaller than the velocity of both rollers. By conservation of mass and making the approximate assumption that the workpiece behaves like a fluid when going through the roll gap, it may be shown that the velocity of the work piece increases as it goes through the roll gap. We assume that there is a point for both rollers at which the velocity of the workpiece and the roller is the same. These are the neutral points and the corresponding arc length parameters are labelled s_t^n and s_b^n for the top and bottom rollers respectively on figure 6. By considering friction and the relative velocities between the workpiece and the rollers, it may be concluded that before a neutral point, the roller is pushing the work piece into the roll gap. After a neutral point, the roller is pulling the workpiece back into the roll gap. Thus, to ensure we have the right signs in our equations, we define

$$\text{sign}_t(s) := \text{sign}(s_t^n - s)\text{H}(s - s_t^{in})\text{H}(s_t^{out} - s)$$

and

$$\text{sign}_b(s) := \text{sign}(s_b^n - s)\text{H}(s - s_b^{in})\text{H}(s_b^{out} - s)$$

where H denotes the Heaviside step function. We are now ready to derive our equations.

Recall the Coulomb friction law

$$F = \mu N.$$

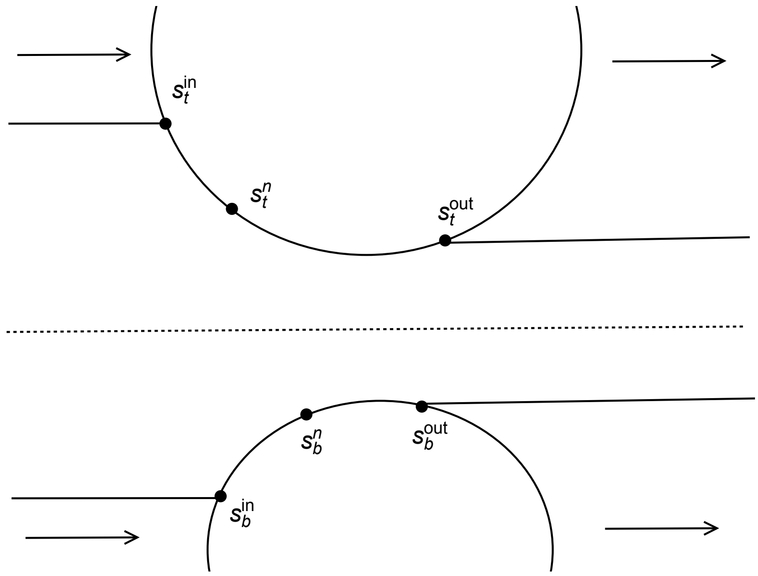

Figure 6: A diagram to show the significant points within the roll gap when considering our friction law

The idea is to derive the tangential and normal components of the traction vector associated with the rollers and conclude they are related by this friction law. Recall that

$$\mathbf{n}_t = \eta_t \begin{pmatrix} -h_t' \\ \nu_t \end{pmatrix}$$

and

$$\mathbf{n}_b = \eta_b \begin{pmatrix} h_b' \\ -\nu_b \end{pmatrix}.$$

Furthermore, note that

$$\begin{aligned} \mathbf{T}^{\mathbf{n}} &= \mathbf{n} \cdot \boldsymbol{\sigma} \\ &= \big(\mathbf{t} \cdot (\mathbf{n} \cdot \boldsymbol{\sigma})\big)\mathbf{t} + \big(\mathbf{n} \cdot (\mathbf{n} \cdot \boldsymbol{\sigma})\big)\mathbf{n}. \end{aligned}$$

Now, we have

$$\begin{aligned} \mathbf{n}_t \cdot (\mathbf{n}_t \cdot \boldsymbol{\sigma}) &= \eta_t \begin{pmatrix} -h_t' \\ \nu_t \end{pmatrix} \cdot \eta_t \begin{pmatrix} -h_t'\sigma_s + \tau_t\nu_t \\ -h_t'\tau_t + \sigma_r\nu_t \end{pmatrix} \\ &= \eta_t^2 (h_t'^2\sigma_s - 2h_t'\tau_t\nu_t + \sigma_r\nu_t^2) \end{aligned}$$

and

$$\begin{aligned} \mathbf{t}_t \cdot (\mathbf{n}_t \cdot \boldsymbol{\sigma}) &= \eta_t \begin{pmatrix} \nu_t \\ h_t' \end{pmatrix} \cdot \eta_t \begin{pmatrix} -h_t'\sigma_s + \tau_t\nu_t \\ -h_t'\tau_t + \sigma_r\nu_t \end{pmatrix} \\ &= \eta_t^2 (-h_t'\sigma_s\nu_t + \tau_t\nu_t^2 - h_t'^2\tau_t + h_t'\sigma_r\nu_t). \end{aligned}$$

Hence, within the roll gap, we have

$$\mu(h_t'^2\sigma_s - 2h_t'\tau_t\nu_t + \sigma_r\nu_t^2)\mathrm{sign}_t(s) = -h_t'\sigma_s\nu_t + \tau_t\nu_t^2 - h_t'^2\tau_t + h_t'\sigma_r\nu_t$$
$$= \tau_t(\nu_t^2 - h_t'^2) + h_t'\nu_t(\sigma_r - \sigma_s).$$

Outside of the roll gap, as there is virtually no friction experienced by the workpiece, $\mu = 0$ and it follows

$$\tau_t(\nu_t^2 - h_t'^2) + h_t'\nu_t(\sigma_r - \sigma_s) = 0.$$

Applying the exact same reasoning to the bottom roller we have in the roll gap

$$-\mu(h_b'^2\sigma_s - 2h_b'\tau_b\nu_b + \sigma_r\nu_b^2)\mathrm{sign}_b(s) = \tau_b(\nu_b^2 - h_b'^2) + h_b'\nu_b(\sigma_r - \sigma_s).$$

and outside of the roll gap, we have

$$\tau_b(\nu_b^2 - h_b'^2) + h_b'\nu_b(\sigma_r - \sigma_s) = 0.$$

Notice that if we set the curvature to zero, we obtain the exact same equations as in [1, p 13].

6 Constitutive laws

Having calculated the stresses, we may calculate the subsequent strains. This relation is a constitutive law. We model the workpiece into three regions:

- An isotropic linearly elastic region outside the roll gap
- A rigid perfectly plastic region between the roll gap
- A region where elastic and plastic deformation occurs. This corresponds to the "boundary" between the roll gap and the outer region.

Having calculated the strains, we may calculate the displacements and thus predict how our workpiece bends.

6.1 Constitutive law for the isotropic linearly elastic region

The constitutive law we will use in this case is

$$\epsilon_{ij} = \frac{1}{E}\big((1+\psi)\sigma_{ij} - \psi\sigma_{kk}\delta_{ij}\big)$$

where ψ is the Poisson ratio and E is Young's modulus (normally ν is used for Poisson's ratio but we have already used that letter !). The justification for the use of this Constitutive law is omitted here but is given in Lubliner [2, pp 51-52].

6.2 Constitutive law for the rigid perfectly plastic region

Firstly, when considering plastic regions, we must consider *incremental* strains rather than usual strains. This is because in general, in a plastic region, a specific stress may correspond to more than one strain and vice versa. This may be seen from a stress - strain curve.

Secondly, we assume that the workpiece in the roll gap satisfies Drucker stability criterion. This criterion say that the work done by tractions Δt_i through a displacement Δu_i is positive or zero for all i. This seems a reasonable assumption for our problem.

With this assumption made and denoting the incremental strain due to plastic deformation as $d\epsilon_{ij}^p$, we use the flow rule as stated in [4, p 11]

$$d\epsilon_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}}$$

where f is the yield function

$$f(\boldsymbol{\sigma}, \boldsymbol{\xi}) = \frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \tau^2 - k^2$$

as described in [1, p5] and $d\lambda$ is called the consistency parameter. Applying this rule and remembering $\tau = \sigma_{12} = \sigma_{21}$, we find that

$$\begin{aligned} d\epsilon_{11}^p &= \frac{d\lambda}{2}(\sigma_{11} - \sigma_{22}) \\ d\epsilon_{22}^p &= \frac{d\lambda}{2}(\sigma_{22} - \sigma_{11}) \\ d\epsilon_{12}^p &= d\epsilon_{21}^p = 2d\lambda\tau. \end{aligned}$$

These equations define the constitutive law for the rigid perfectly plastic region.

6.3 Constitutive law for the elastic - plastic region

We make the ansatz that the incremental strain splits linearly into an elastic part and an plastic part. That is to say, if the incremental strain due to elastic deformation is denoted as $d\epsilon_{ij}^e$, we assume that

$$d\epsilon_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p.$$

This is by no means true all the time but we believe it is reasonable to assume this in our problem. Then, the constitutive law for this region will be a sum for the analogous incremental constitutive law for the isotropic linearly elastic region and the rigid perfectly plastic region.

7 A closed system of equations for the workpiece

We will now derive a closed system of three differential equations which in principle may be solved numerically by a computer. As stated before, we will need to make some assumptions to achieve this. We will comment on the validity of the assumptions when we make them.

The idea is to manipulate our system of equations so that we are solving for three quantities: $\sigma_s(h_t)$, $\sigma_s(h_b)$ and $\overline{\tau}$ (which we shall define shortly). To do this, we will have to make some modelling assumptions about the distributions of the stresses within the workpiece. Then, using additional modelling assumptions (such as our friction law), we will be able to solve our system of equations (using a computer) for the three unknowns stated above. This will allow to fully determine the distribution of the stresses within the workpiece.

To begin with, we introduce the following lemma.

Lemma 1. *We have*

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y) \quad dy = \frac{db}{dx}f(x,b(x)) - \frac{da}{dx}f(x,a(x)) + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,y) \quad dy.$$

Proof. Let $F(x,y)$ be the antiderivative of $f(x,y)$ with respect to y. Then, using the FTC, we have

$$\begin{aligned}
\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y) \quad dy &= \frac{d}{dx}\left(F(x,b(x)) - F(x,a(x))\right) \\
&= \frac{d}{dx}\left(F(x,b(x))\right) - \frac{d}{dx}\left(F(x,a(x))\right) \\
&= \frac{\partial F}{\partial y}(x,b(x))\frac{db}{dx} - \frac{\partial F}{\partial y}(x,a(x))\frac{da}{dx} + \underbrace{\frac{\partial F}{\partial x}(x,b(x)) - \frac{\partial F}{\partial x}(x,a(x))}_{\text{differentiate w.r.t xth position!}}.
\end{aligned}$$

The result is then obtained using the fact that $F(x,y)$ is the antiderivative of $f(x,y)$ with respect to y and $\frac{\partial F}{\partial x}(x,y)$ is the antiderivative of $\frac{\partial f}{\partial x}(x,y)$ with respect to x and the FTC. □

We will need this lemma to calculate the partial derivatives of integrals as can be seen from our PDEs. We begin by deriving a system of closed equations for the workpiece outside of the roll gap.

7.1 Outside the roll gap

Firstly, we assume that there is no normal stress acting through the workpiece outside of the roll gap. This is likely to be a good approximation as there are no compressive forces acting on the workpiece outside of the roll gap. Thus, our equations become

$$\begin{aligned}
\frac{d}{ds}\int_{h_b}^{h_t} \sigma_s \quad dr &= \kappa\int_{h_b}^{h_t} \tau \quad dr + (h_t'\sigma_s(h_t) - \tau_t\nu_t) - (h_b'\sigma_s(h_b) - \tau_b\nu_b) \\
\frac{d}{ds}\int_{h_b}^{h_t} \tau \quad dr &= -\kappa\int_{h_b}^{h_t} \sigma_s \quad dr + h_t'\tau_t - h_b'\tau_b \\
\frac{d}{ds}\int_{h_b}^{h_t} r\sigma_s \quad dr &= \int_{h_b}^{h_t} \tau \quad dr + \big(h_t h_t'\sigma_s(h_t) - h_t\tau_t\nu_t\big) - \big(h_b h_b'\sigma_s(h_b) - h_b\tau_b\nu_b\big).
\end{aligned}$$

We define our average shear stress as

$$\overline{\tau} := \frac{1}{\Delta h}\int_{h_b}^{h_t} \tau \quad dr$$

where

$$\Delta h := h_t - h_b$$

as was done in [1]. This gives us the following system of equations:

$$\begin{aligned}
\frac{d}{ds}\int_{h_b}^{h_t} \sigma_s \quad dr &= \Delta h\kappa\overline{\tau} + (h_t'\sigma_s(h_t) - \tau_t\nu_t) - (h_b'\sigma_s(h_b) - \tau_b\nu_b) \\
\frac{d}{ds}\overline{\tau} &= \frac{1}{\Delta h}\left(-\kappa\int_{h_b}^{h_t} \sigma_s \quad dr + h_t'\tau_t - h_b'\tau_b - \overline{\tau}\frac{d\Delta h}{ds}\right) \\
\frac{d}{ds}\int_{h_b}^{h_t} r\sigma_s \quad dr &= \Delta h\overline{\tau} + \big(h_t h_t'\sigma_s(h_t) - h_t\tau_t\nu_t\big) - \big(h_b h_b'\sigma_s(h_b) - h_b\tau_b\nu_b\big).
\end{aligned}$$

We will now evaluate the remaining integrals by introducing a distribution for σ_s. We will assume that σ_s varies linearly through the workpiece. That is to say, we assume

$$\sigma_s(r) = ar + b$$

for some constants a and b. This is not the most accurate assumption we can make as Timoshenko in [5, p 217] derives a hyperbolic distribution for a beam in bending. The reason we have not assumed a hyperbolic distribution is because the computations become horrendously messy and we believe using a linear assumption will lead to sufficiently accurate results. If we are wrong, it is our hope that the hyperbolic case is explored in the future.

Using the values $\sigma_s(h_t)$ and $\sigma_s(h_b)$, we conclude that

$$\sigma_s(r) = \frac{\sigma_s(h_t) - \sigma_s(h_b)}{\Delta h} r + \frac{\sigma_s(h_b)h_t - \sigma_s(h_t)h_b}{\Delta h}. \tag{4}$$

We may use this to calculate

$$\frac{d}{ds}\int_{h_b}^{h_t} \sigma_s \quad dr,$$

$$-\kappa \int_{h_b}^{h_t} \sigma_s \quad dr$$

and

$$\frac{d}{ds}\int_{h_b}^{h_t} r\sigma_s \quad dr.$$

Firstly, a simple computation shows that

$$-\kappa \int_{h_b}^{h_t} \sigma_s \quad dr = -\frac{\kappa \Delta h}{2}\left(\sigma_s(h_t) + \sigma_s(h_b)\right).$$

Next, the integral

$$\frac{d}{ds}\int_{h_b}^{h_t} \sigma_s \quad dr$$

was computed in [1, p15] (with a slight change of notation) using lemma 1 and our linear assumption. The details are omitted for the sake of brevity but the result is

$$\frac{d}{ds}\int_{h_b}^{h_t} \sigma_s \quad dr = (h_t' - h_b')(\sigma_s(h_t) + \sigma_s(h_b)) + \left(\frac{d}{ds}\sigma_s(h_t) + \frac{d}{ds}\sigma_s(h_b)\right)\frac{\Delta h}{2} - \frac{\sigma_s(h_t) + \sigma_s(h_b)}{2}\frac{d\Delta h}{ds}.$$

Similarly, the remaining integral was also calculated in [1, p14] with a slight change in notation. The result is

$$\frac{d}{ds}\int_{h_b}^{h_t} r\sigma_s \quad dr = h_t' h_t \sigma_s(h_t) - h_b' h_b \sigma_s(h_b) - \frac{\rho_t}{6}\frac{d}{ds}\sigma_s(h_t) + \frac{\rho_b}{6}\frac{d}{ds}\sigma_s(h_b)$$
$$+ \left(\frac{\rho_t}{6\Delta h}\sigma_s(h_t) - \frac{\rho_b}{6\Delta h}\sigma_s(h_b)\right)\frac{d\Delta h}{ds} + \frac{\left(h_t'\sigma_s(h_b) - h_b'\sigma_s(h_t)\right)(h_t + h_b)}{2}.$$

Thus, after substituting our remaining expressions into our system of equations and rearranging, we have the following system of equations:

$$\frac{d}{ds}\sigma_s(h_t) = \frac{2}{\Delta h}\left(-\tau_t\nu_t + \tau_b\nu_b - h_t'\sigma_s(h_b) + h_b'\sigma_s(h_t) + \Delta h\kappa\overline{\tau}\right) + \frac{\sigma_s(h_t)+\sigma_s(h_b)}{\Delta h}\frac{d\Delta h}{ds} - \frac{d}{ds}\sigma_s(h_b)$$
$$\frac{d}{ds}\overline{\tau} = \frac{1}{\Delta h}\left(-\frac{\kappa\Delta h}{2}\left(\sigma_s(h_t)+\sigma_s(h_b)\right) + h_t'\tau_t - h_b'\tau_b - \overline{\tau}\frac{d\Delta h}{ds}\right)$$
$$\frac{d}{ds}\sigma_s(h_b) = \frac{6}{\rho_b}\Big(\Delta h\overline{\tau} - h_t\tau_t\nu_t + h_b\tau_b\nu_b\Big) + \left(-\frac{\rho_t}{\rho_b}\sigma_s(h_t)+\sigma_s(h_b)\right)\frac{1}{\Delta h}\frac{d\Delta h}{ds} - \frac{3}{\rho_b}\big(h_t'\sigma_s(h_b) - h_b'\sigma_s(h_t)\big)(h_t+h_b)$$
$$+\frac{\rho_t}{\rho_b}\frac{d}{ds}\sigma_s(h_t).$$

Note well that setting the curvature to zero, we obtain the exact same differential equations derived in [1] with the normal stresses set to zero.

Finally, we may express τ_t and τ_b in terms of $\sigma_s(h_t)$ and $\sigma_s(h_b)$ by using our friction law outside of the roll gap, remembering that the normal stress is assumed to be zero. In this way, we have a closed system of 3 differential equations which may be solved on a computer. Having solved for the three unknowns, the function that describes the distribution of σ_s may be determined using equation 4. For the shear stress, we assume that τ varies quadratically through the workpiece. Timoshenko derives this quadratic variation in [5, pp 63-64] for a straight beam but states on page 220 that the same distribution may also be taken for a curved beam (as we have here). Justification for this is empirical. This assumption may be used to determine the distribution of τ .

7.2 Inside the roll gap

Because of the large forces exerted on the workpiece within the roll gap by the rolls, the distributions of the stresses within the workpiece are likely to be very complicated. Indeed, the distributions we will use here are probably inaccurate. It is our hope that research will be conducted to determine the most accurate distributions for modelling the stresses within the roll gap.

Having made this assumptions, we will use our yield condition and the friction law to make our system of equations closed. This approach is taken from [1].

We assume that the distributions for the shear stress and σ_s are the same as previously. That is to say, we assume the shear stress varies quadratically and σ_s is determined by equation 4. We emphasise that this is most likely inaccurate.

Having made these assumptions, it is easy to see that the working of the previous subsection follows through to this section but we must now remember to include the end normal stresses because they cannot be assumed to be zero (due to the force applied onto the workpiece by the rollers). We obtain the following system of equations:

$$\frac{d}{ds}\sigma_s(h_t) = \frac{2}{\Delta h}\left(-\tau_t\nu_t + \tau_b\nu_b - h_t'\sigma_s(h_b) + h_b'\sigma_s(h_t) + \Delta h\kappa\overline{\tau}\right) + \frac{\sigma_s(h_t)+\sigma_s(h_b)}{\Delta h}\frac{d\Delta h}{ds} - \frac{d}{ds}\sigma_s(h_b)$$
$$\frac{d}{ds}\overline{\tau} = \frac{1}{\Delta h}\left(-\frac{\kappa\Delta h}{2}\left(\sigma_s(h_t)+\sigma_s(h_b)\right) + (h_t'\tau_t - \sigma_r(h_t)\nu_t) - (h_b'\tau_b - \sigma_r(h_b)\nu_b) - \overline{\tau}\frac{d\Delta h}{ds}\right)$$
$$\frac{d}{ds}\sigma_s(h_b) = \frac{6}{\rho_b}\Big(\Delta h\overline{\tau} - h_t\tau_t\nu_t + h_b\tau_b\nu_b\Big) + \left(-\frac{\rho_t}{\rho_b}\sigma_s(h_t)+\sigma_s(h_b)\right)\frac{1}{\Delta h}\frac{d\Delta h}{ds} - \frac{3}{\rho_b}\big(h_t'\sigma_s(h_b) - h_b'\sigma_s(h_t)\big)(h_t+h_b)$$
$$+\frac{\rho_t}{\rho_b}\frac{d}{ds}\sigma_s(h_t).$$

Setting the curvature to zero, we get the same equations derived by the RSG in [1]. We may express τ_t and τ_b in terms of $\sigma_r(h_t), \sigma_r(h_b), \sigma_s(h_t)$ and $\sigma_s(h_b)$ using our friction law. Then, we may express $\sigma_r(h_t)$ and $\sigma_r(h_b)$ in terms of $\sigma_s(h_t)$ and $\sigma_s(h_b)$ by using the yield criterion as was done in [1, pp24-25]. This will give us a closed system of equations which may be solved using a computer.

8 Concluding remarks

To conclude, we have developed a mathematical model for metal ring rolling. To do this, we assumed that the system was in a quasi-static state and considered equilibrium of an arbitrary slab to derive 3 differential equations. We developed a friction law and proposed a set of constitutive laws. However, there are many things that we were unable to do in this report. It still remains to

- Test our model against simple examples and see whether existing theory predicts the same behaviour
- Computerise our model on software such as MATLAB or Mathematica
- Investigate the distribution of the stresses within the roll gap.

We hope that someone will be able to continue from where we finished and hopefully complete our work!

9 Appendix

In this appendix, we will calculate the displacement field of a straight beam as predicted by our equations for outside the roll gap. We consider a straight beam as our linear variation assumption is likely to be true for straight beams outside the roll gap and the neutral axis coincides with the centreline of the beam making the problem symmetric. By reading this section, the future researcher may be more prepared to calculate the displacements of other beams such as semi circular rings.

For the straight beam case, the curvature will be zero and as we are assuming the beam to be outside the roll gap, there are no normal stresses through the beam. In addition, as we are considering a straight beam, it follows

$$h_t' = h_b' = \frac{d\Delta h}{ds} = 0.$$

Furthermore, using the above, our friction law predicts

$$\tau_t = \tau_b = 0.$$

Thus, we obtain the following system of equations:

$$\begin{aligned}
&\frac{d}{ds}\sigma_s(h_t) = -\frac{d}{ds}\sigma_s(h_b)\\
&\frac{d}{ds}\overline{\tau} = 0\\
&\frac{d}{ds}\sigma_s(h_b) = \frac{6}{\rho_b}\Delta h\overline{\tau} + \frac{\rho_t}{\rho_b}\frac{d}{ds}\sigma_s(h_t).
\end{aligned}$$

After some algebra, we obtain

$$\begin{aligned}
&\frac{d}{ds}\sigma_s(h_t) = -\frac{6\Delta h}{\rho_t + \rho_b}\overline{\tau} \\
&\frac{d}{ds}\overline{\tau} = 0 \\
&\frac{d}{ds}\sigma_s(h_b) = \frac{6\Delta h}{\rho_t + \rho_b}\overline{\tau}.
\end{aligned}$$

Now, we know from Timoshenko that the shear stress varies quadratically through the workpiece and we know that $\tau_t = \tau_b = 0$. Furthermore, as the mean shear stress does not change with respect to s, we guess that $\tau(s, r) = \tau(r)$ i.e. is independent of s. This leads us to guess that

$$\tau(r) = -\tau_{max}(r - h_t)(r - h_b)$$

where τ_{max} is the maximum shear stress within the beam (this maximum will be achieved at the neutral axis). Then, using the fact that

$$\overline{\tau} = \frac{1}{\Delta h}\int_{h_b}^{ht} \tau(r) \quad dr,$$

it follows

$$\overline{\tau} = \frac{\Delta h^2}{6}\tau_{max}.$$

Hence,

$$\begin{aligned}
&\frac{d}{ds}\sigma_s(h_t) = -\frac{\Delta h^3}{\rho_t + \rho_b}\tau_{max} \\
&\frac{d}{ds}\sigma_s(h_b) = \frac{\Delta h^3}{\rho_t + \rho_b}\tau_{max}.
\end{aligned}$$

Therefore, solving these differential equations and using the fact that

$$\sigma_s(r) = \frac{\sigma_s(h_t) - \sigma_s(h_b)}{\Delta h}r + \frac{\sigma_s(h_b)h_t - \sigma_s(h_t)h_b}{\Delta h},$$

we obtain the following expression for $\sigma_s(s, r)$:

$$\sigma_s(s, r) = \left(-\frac{2\Delta h^2}{\rho_t + \rho_b}\tau_{max}s + k_1\right)r + \frac{\Delta h^2(h_t + h_b)}{\rho_t + \rho_b}\tau_{max}s + k_2$$

where k_1 and k_2 are constants which appear when solving the differential equations. Thus, for a straight beam outside the roll gap, we have

$$\begin{aligned}
\sigma_s(s, r) &= \left(-\frac{2\Delta h^2}{\rho_t + \rho_b}\tau_{max}s + k_1\right)r + \frac{\Delta h^2(h_t + h_b)}{\rho_t + \rho_b}\tau_{max}s + k_2 \\
\tau(r) &= -\tau_{max}(r - h_t)(r - h_b) \\
\sigma_r(s, r) &= 0.
\end{aligned}$$

We could proceed further in generality, but we know consider the case where $h_t = 1$ and $h_b = -1$. This is simply a straight beam of depth 2. In this case, the equations of stress become

$$\begin{aligned}\sigma_s(s,r) &= (2\tau_{max}s + k_1)\,r + k_2\\ \tau(r) &= -\tau_{max}(r-1)(r+1)\\ \sigma_r(s,r) &= 0.\end{aligned}$$

Now, recall our constitutive law outside the roll gap

$$\epsilon_{ij} = \frac{1}{E}\big((1+\psi)\sigma_{ij} - \psi\sigma_{kk}\delta_{ij}\big).$$

Remembering that $\sigma_{11} = \sigma_s$, $\sigma_{22} = \sigma_r$ and $\sigma_{12} = \sigma_{21} = \tau$, we have

$$\begin{aligned}\epsilon_{11}(s,r) &= \frac{1}{E}\left[(2\tau_{max}s + k_1)\,r + k_2\right]\\ \epsilon_{22}(s,r) &= \frac{-\psi}{E}\left[(2\tau_{max}s + k_1)\,r + k_2\right]\\ \epsilon_{12}(r) = \epsilon_{21}(r) &= \frac{-\tau_{max}(1+\psi)}{E}(r-1)(r+1).\end{aligned}$$

Now, recall that

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}\right)$$

where $u_i(s,r)$ is the displacement field in the i direction. Thus, we have the following 3 PDEs:

$$\begin{aligned}\frac{\partial u_s}{\partial s} &= \frac{1}{E}\left[(2\tau_{max}s + k_1)\,r + k_2\right]\\ \frac{\partial u_r}{\partial r} &= \frac{-\psi}{E}\left[(2\tau_{max}s + k_1)\,r + k_2\right]\\ \frac{1}{2}\left(\frac{\partial u_s}{\partial r} + \frac{\partial u_r}{\partial s}\right) &= \frac{-\tau_{max}(1+\psi)}{E}(r-1)(r+1).\end{aligned}$$

From the first equation, we conclude

$$u_s(s,r) = \frac{1}{E}\left[\left(\tau_{max}s^2 + k_1 s\right)r + k_2 s\right] + f(r)$$

for some function $f(r)$. Similarly, from the second equation, we conclude

$$u_r(s,r) = \frac{-\psi}{E}\left[\left(\tau_{max}s + \frac{k_1}{2}\right)r^2 + k_2 r\right] + g(s)$$

for some function $g(s)$. If we substitute both these equations in the third PDE and rearrange, we find that

$$\frac{1}{E}\left(\tau_{max}s^2 + k_1 s\right) + g'(s) = -\frac{2\tau_{max}(1+\psi)}{E}(r^2-1) + \frac{\psi\tau_{max}}{E}r^2 - f'(r).$$

Noting the separation of variables, it must follow

$$\begin{aligned}g'(s) + \frac{1}{E}\left(\tau_{max}s^2 + k_1 s\right) &= \Phi\\ -f'(r) - \frac{2\tau_{max}(1+\psi)}{E}(r^2-1) + \frac{\psi\tau_{max}}{E}r^2 &= \Phi\end{aligned}$$

for some separation constant Φ. Solving these ODEs, we find that

$$g(s) = \Phi s - \frac{1}{E}\left(\frac{\tau_{max}}{3}s^3 + \frac{k_1}{2}s^2\right) + A$$

for some constant A and

$$f(r) = -\frac{(2+\psi)\tau_{max}}{3E}r^3 + \frac{2\tau_{max}(1+\psi)}{E}r - \Phi r + B$$

for some constant B. Thus, we have

$$u_s(s,r) = \frac{1}{E}\left[\left(\tau_{max}s^2 + k_1 s\right)r + k_2 s\right] - \frac{(2+\psi)\tau_{max}}{3E}r^3 + \frac{2\tau_{max}(1+\psi)}{E}r - \Phi r + B$$
$$u_r(s,r) = \frac{-\psi}{E}\left[\left(\tau_{max}s + \frac{k_1}{2}\right)r^2 + k_2 r\right] + \Phi s - \frac{1}{E}\left(\frac{\tau_{max}}{3}s^3 + \frac{k_1}{2}s^2\right) + A.$$

This describes the displacement filed for a straight beam outside of the roll gap as predicted by our model. A computer package may be used to visual the bending.

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