# On the Homological Stability of Orthogonal and Spin Groups

by

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*Dedicated to my parents, Asha Rani Sood and Chander Shakhar Sood, for being the best parents in the world; and to my Helper, who is always with me.*

# Contents

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This thesis is dedicated to my parents, Asha Rani Sood and Chander Shakhar Sood, for being the best parents in the world; and to my Helper, who is always with me.

# Declarations

I declare that, to the best of my knowledge, the material in this thesis is original and my own work, conducted under the supervision of Dr Marco Schlichting, unless otherwise indicated. The material in this thesis has not been submitted for any other degree either at the University of Warwick or any other University.

# Abstract

We prove homological stability results for the orthogonal group, special orthogonal group, elementary orthogonal group and the spin group with respect to the hyperbolic form. We prove homological stability over a commutative local ring $R$ with infinite residue field such that $2 \in R^*$.

In the orthogonal case, this improves the range for homological stability given by Mirzaii by 1 and generalises the result obtained by Sprehn and Wahl to the case of local rings. In the special orthogonal case, this generalises the result obtained by Essert for infinite fields to the case of local rings, and is the first homological stability result for the special orthogonal group over a local ring. For the elementary orthogonal group, this is the first known homological stability result. For the spin group, this coincides with $H_1$-stability and $H_2$-stability results stated in Hahn-O'Meara, and is the first homological stability result that accounts for all homology groups.

# Chapter 1

# Introduction

In this thesis, we prove homological stability results for the orthogonal group, special orthogonal group, elementary orthogonal group and the spin group with respect to the hyperbolic form.

For the orthogonal group, this improves the current best homological stability range and generalises the analogous result for fields to the case of local rings. For the special orthogonal group, this is the first homological stability result over a local ring, and generalises the analogous result for infinite fields. For the elementary orthogonal group, this is the first known homological stability result. For the spin group, this coincides with known $H_1$-stability and $H_2$-stability results, and is the first homological stability result that accounts for all homology groups.

Recall, for a ring $R$, the *(split) orthogonal group* $O_{n,n}(R) \subseteq GL_{2n}(R)$, is the subgroup

$$O_{n,n}(R) := \{A \in GL_{2n}(R) |\ {}^tA\psi_{2n}A = \psi_{2n}\}.$$

of $R$-linear automorphisms preserving the form

$$\psi_{2n} = \begin{pmatrix} \psi_2 & & & \\ & \psi_2 & & \\ & & \ddots & \\ & & & \psi_2 \end{pmatrix} = \bigoplus_1^n \psi_2, \quad \psi_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where ${}^tA$ denotes the transpose matrix of $A$. Define $SO_{n,n}(R)$ to be the subgroup of $O_{n,n}(R)$ consisting of all matrices with determinant 1. We will always consider

$O_{n,n}(R)$ as a subgroup of $O_{n+1,n+1}(R)$ via the embedding

$$O_{n,n}(R) \subseteq O_{n+1,n+1}(R) : A \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A \end{pmatrix}.$$

We will be interested in studying homological stability under this embedding.

The homology of the orthogonal group $O_{n,n}$ has long been known to stabilise, in quite large generality; see, e.g., [Vog81], [Bet90], [Cha87]. Recently, Sprehn and Wahl in [SW20] have shown that for every field $\mathbb{F}$ other than the field $\mathbb{F}_2$, $H_k(O_{n,n}(\mathbb{F});\mathbb{Z}) \to H_k(O_{n+1,n+1}(\mathbb{F});\mathbb{Z})$ is an isomorphism for $k \leq n-1$ and surjective for $k \leq n$. In the context of fields, this is currently the best known range of stability. However, they were unable to extend their results to local rings, essentially because the framework that they use is only applicable to *vector spaces*, rather than modules over local rings. In the context of local rings, the first precise range of stability was given by Mirzaii in [Mir04]. Specfically, he proved that for $R$ commutative local ring with infinite residue field, $H_k(O_{n,n}(R);\mathbb{Z}) \to H_k(O_{n+1,n+1}(R);\mathbb{Z})$ is an isomorphism for $k \leq n-2$ and surjective for $k \leq n-1$.

Our first main result is an improvement on the known stability range for $O_{n,n}$ over local rings with infinite residue field, with the additional assumption that we require 2 to be invertible. Specifically, we prove that:

**Theorem 1.0.1.** *Let $R$ be a commutative local ring with infinite residue field such that $2 \in R^*$. Then, the natural homomorphism*

$$H_k(O_{n,n}(R)) \longrightarrow H_k(O_{n+1,n+1}(R))$$

*is an isomorphism for $k \leq n-1$ and surjective for $k \leq n$.*

The proof is modelled on the homological stability proofs given in [NS89] and [Sch17]. Specifically, we consider a highly acyclic chain complex on which $O_{n,n}$ acts, and analyse the resulting hyperhomology spectral sequences. This is a standard method of proving such results, but the main innovation that gives us the improvement in stability is the use of the technique of *localising homology groups*. This technique was first introduced in [Sch17]. It is this technique that makes the hyperhomology spectral sequences easy to analyse.

In addition, the methods we use to prove homological stability for $O_{n,n}(R)$ may be used to prove homological stability for $SO_{n,n}(R)$, which gives our second main result:

**Theorem 1.0.2.** *Let $R$ be a commutative local ring with infinite residue field such that $2 \in R^*$. Then, the natural homomorphism*

$$H_k(SO_{n,n}(R)) \longrightarrow H_k(SO_{n+1,n+1}(R))$$

*is an isomorphism for $k \leq n-1$ and surjective for $k \leq n$.*

This is the first homological stability result for the special orthogonal group over a local ring, and generalises the analogous result for infinite fields obtained by Essert [Ess13].

Next, the elementary orthogonal group $EO_{n,n}(R)$ may be defined in terms of generators and should be viewed as the orthogonal analogue of the elementary linear group $E_n(R)$.

For $r \in R$ and $1 \leq k \neq l \leq n$, define $\gamma_{kl}(r)$ to be the $n \times n$ matrix with $r$ in the $(k,l)$ position, $-r$ in the $(l,k)$ position, and 0 elsewhere. Define $\gamma_{kk}(r)$ to be the zero matrix. In addition, for $1 \leq i \neq j \leq n$, define $e_{ij}(r)$ to be the $n \times n$ elementary linear matrix with 1 along the diagonal and $r$ in the $(i,j)$ position. We then define the family of *elementary orthogonal matrices* as

$$E_{2k,2l}(r) := \begin{pmatrix} I_n & \\ \gamma_{kl}(r) & I_n \end{pmatrix}, \tag{1.1}$$

$$E_{2k-1,2l-1}(r) := \begin{pmatrix} I_n & \gamma_{kl}(r) \\ & I_n \end{pmatrix}, \tag{1.2}$$

and for $k \neq l$,

$$E_{2k-1,2l}(r) := \begin{pmatrix} e_{kl}(r) & \\ & e_{lk}(-r) \end{pmatrix}, \tag{1.3}$$

$$E_{2k,2l-1}(r) := \begin{pmatrix} e_{lk}(-r) & \\ & e_{kl}(r) \end{pmatrix}. \tag{1.4}$$

We define the *elementary orthogonal group* $EO_{n,n}(R)$ as the subgroup of $O_{n,n}(R)$ generated by the elementary orthogonal matrices. We refer the reader to [HO89, Sections 5.3A and 5.3B] for more information about $EO_{n,n}(R)$, including a list of relations amongst these generators.

**Remark 1.** For the sake of notation, we have in the above definitions used the convention that the hyperbolic form on $R^{2n}$ is taken with respect to matrix $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. This convention therefore differs from the standard convention used in this thesis up to conjugation by a suitable permutation matrix, and we will always tacitly assume this whenever working with $EO_{n,n}(R)$.

Our third main result:

**Theorem 1.0.3.** *Let $R$ be a commutative local ring with infinite residue field such that $2 \in R^*$. Then, the natural homomorphism*

$$H_k(EO_{n,n}(R)) \longrightarrow H_k(EO_{n+1,n+1}(R))$$

*is an isomorphism for $k \leq n-1$ and surjective for $k \leq n$.*

To our best knowledge, this is the first ever homological stability result given for $EO_{n,n}$.

Finally, let $R$ be a commutative ring, which for the purposes of this article is such that $2 \in R^*$. We define $\mathrm{Spin}_{n,n}(R)$ to be the Spin group of the quadratic module $(R^{2n}, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is the symmetric bilinear form associated to the matrix $\psi_{2n}$ as above. We refer the reader to the preliminaries for more information about Spin groups. The reader may also want to look at [HO89], [Sch12] and [LM16] as alternative references.

In the case $R$ is a commutative local ring with infinite residue field such that $2 \in R^*$, homological stability for $\mathrm{Spin}_{n,n}$ will follow immediately from homological stability of $EO_{n,n}$ via the relative Hochschild-Serre spectral sequence applied to short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_{n,n} \longrightarrow EO_{n,n} \longrightarrow 1,$$

see Theorem 2.4.21 in the Preliminaries. Indeed, for the purposes of this thesis, it is perhaps best to think of $EO_{n,n}(R)$ as being *defined* in terms of this short exact sequence. This is the perspective that we will adopt.

This gives us our fourth main result:

**Theorem 1.0.4.** *Let $R$ be a commutative local ring with infinite residue field such that $2 \in R^*$. Then, the natural homomorphism*

$$H_k(\mathrm{Spin}_{n,n}(R)) \longrightarrow H_k(\mathrm{Spin}_{n+1,n+1}(R))$$

*is an isomorphism for $k \leq n-1$ and surjective for $k \leq n$.*

This coincides with known $H_1$ and $H_2$-stability results for $\mathrm{Spin}_{n,n}$ given in [HO89], and is the first such homological stability result that accounts for all homology groups.

# Chapter 2

# Preliminaries

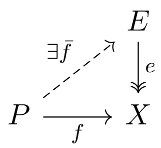
## 2.1 Group homology

In this section, we give a quick review of group homology.

Group homology is defined in terms of *left derived functors*, so we quickly introduce this concept.

### 2.1.1 Left derived functors

Let $\mathcal{A}$ be an *abelian category.* See for example [Wei94, Definition 1.2.2] or [Sri07, Appendix B] for equivalent definitions. An object $P$ in $\mathcal{A}$ is called *projective* if it satisfies the following lifting property: Given an epimorphism $e : E \twoheadrightarrow X$ and morphism $f : P \to X$, there exists a morphism $\bar{f} : P \to E$ such that $e \circ \bar{f} = f$. That is to say, the diagram




commutes. As an example, when $\mathcal{A}$ is the the category of (left) $R$-modules for some ring $R$, the projective objects are precisely the direct summands of free $R$-modules.

We say that $\mathcal{A}$ has *enough projectives* if for every object $A$ of $\mathcal{A}$ there is an epimorphism $P \to A$ with $P$ projective.

Abelian categories with enough projectives are important because every object has a *projective resolution.*

**Definition 2.1.1.** Let $M$ be an object of $\mathcal{A}$. A *left resolution* of $M$ is a chain complex $P.$ with $P_i = 0$ for $i < 0$, together with a map $\varepsilon : P_0 \to M$ so that the augmented complex

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0$$

is *exact*. It is a *projective resolution* if moreover, each $P_i$ is *projective*.

**Lemma 2.1.2.** *Let $\mathcal{A}$ be an abelian category with enough projectives. Then, every object $M$ in $\mathcal{A}$ has a projective resolution.*

*Proof.* See [Wei94, Lemma 2.2.5]. □

We say that an (additive) functor $F : \mathcal{A} \to \mathcal{B}$ is *exact* if whenever

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathcal{A}$, then

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is a short exact sequence in $\mathcal{B}$. Moreover, we say that $F : \mathcal{A} \to \mathcal{B}$ is *right exact* if whenever

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathcal{A}$, then

$$F(A) \to F(B) \to F(C) \to 0$$

is an exact sequence in $\mathcal{B}$.

It is the right exact functors which admit left derived functors.

**Definition 2.1.3.** Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor between two abelian categories, and suppose $\mathcal{A}$ has enough projectives. For $i \geq 0$, we define the *left derived functors* $L_iF$ as follows: For $A$ an object of $\mathcal{A}$, let $P. \to A$ be a projective resolution of $A$. We define

$$L_iF(A) := H_i(F(P.)).$$

**Remark 2.** The above definition does not depend in the choice of projective resolution $P. \to A$ by [Wei94, Lemma 2.4.1].

**Remark 3.** The left derived functors $L_*F$ are indeed *functorial*. That is to say, if $f : A' \to A$ is any map in $\mathcal{A}$, there is a natural map $L_iF(f) : L_iF(A') \to L_iF(A)$ for each $i$. We refer to [Wei94, Lemma 2.4.4] for the details.

**Remark 4.** The definition is cooked up so that the left derived functors $L_*F$ form a *universal homological δ-functor*. We refer the reader to [Wei94, Definition 2.1.4] for this definition and [Wei94, Theorem 2.4.7] for a proof of this fact. But *intuitively*, this says that for every short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, the left derived functors $L_*(F)$ assemble into a long exact sequence

$$\begin{array}{ccccccc} & & & & \cdots & & \\ L_2F(A) & \longrightarrow & L_2F(B) & \longrightarrow & L_2F(C) & & \\ L_1F(A) & \longrightarrow & L_1F(B) & \longrightarrow & L_1F(C) & & \\ F(A) & \longrightarrow & F(B) & \longrightarrow & F(C) & \longrightarrow & 0, \end{array}$$

and are 'universal' with this property in some sense. This compensates for the fact that $F : \mathcal{A} \to \mathcal{B}$ is *right exact*, rather than being exact.

**Example 2.1.4.** Let $R$ be a ring, $B$ a left $R$-module, so that

$$T : \mathrm{Mod} - R \to \mathrm{Ab}$$
$$A \mapsto A \otimes_R B$$

is a functor from right $R$-modules to abelian groups. The tensor-hom adjunction states that this functor is left-adjoint and therefore by [Wei94, Theorem 2.6.1], $T$ is right exact, so that it has left derived functors. Specfically, we define the *Tor functors* as

$$\mathrm{Tor}_n^R(A, B) := L_n(T)(A).$$

We refer the reader to [Wei94] for more information about these functors. Perhaps one property that should be mentioned explixitly is that $L_n(-\otimes_R B)(A) \cong L_n(A\otimes_R -)(B)$. See [Wei94, Theorem 2.7.2].

### 2.1.2 Definition of group homology

Let $G$ be a group. Group homology is defined in terms of *G-modules*, so we first need to define this concept.

**Definition 2.1.5.** Let $G$ be a group. A (left) *$G$-module* is an abelian group $A$ together with a group action $\cdot : G \times A \to A$ such that

$$g \cdot (a + b) = g \cdot a + g \cdot b.$$

A *morphism* of $G$-modules $\varphi : A \to B$ is a morphism of abelian groups such that $\varphi(g \cdot a) = g \cdot \varphi(a)$ for every $g \in G$, $a \in A$.

We thus obtain a category $G$-mod of (left) $G$-modules. Note that this category may be identified with the category of left $\mathbb{Z}[G]$ modules, where $\mathbb{Z}[G]$ denotes the group ring of $G$. This identification is frequently used without mention. In particular, the category of $G$-modules is an abelian category.

Group homology is defined as the left derived functors of the so called *coinvariants* functor, which we shall now define.

**Definition 2.1.6.** Let $G$ be a group and $A$ be a $G$-module. Then, the coinvariants $A_G$ of a $G$-module $A$ are

$$A_G := A/\langle g \cdot a - a | g \in G, a \in A\rangle,$$

where $\langle g \cdot a - a | g \in G, a \in A\rangle$ means the smallest submodule generated by $\{g \cdot a - a | g \in G, a \in A\}$.

Note that taking coinvariants is a functor $-_G : G - \text{mod} \to \text{Ab}$ from the category of $G$-modules to the category of Abelian groups. By [Wei94, Exercise 6.1.1], $-_G$ is a right exact functor, so that we can make the following definition.

**Definition 2.1.7.** Let $G$ be a group and $A$ be a $G$-module. Then, the *homology groups of $G$ with coefficients in $A$*, denoted $H_*(G, A)$, are defined as the left derived functors

$$H_*(G, A) := L_*(-_G)(A).$$

**Remark 5.** It is standard convention to write $H_*(G) := H_*(G, \mathbb{Z})$, where $\mathbb{Z}$ denotes the integers given the *trivial* $G$-module structure i.e. $g \cdot n = n$ for every $g \in G, n \in \mathbb{Z}$.

The trivial $G$-module $\mathbb{Z}$ is universal amongst trivial $G$-modules in the following sense: if $A$ is an abelian group with trivial $G$-action, we have for every $n$ a short exact sequence of abelian groups

$$0 \to H_n(G) \otimes A \to H_n(G, A) \to \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(G), A) \to 0.$$

This is known as the *universal coefficient sequence for group homology.* See [Bro94, §III Exercise 1.3] for more details. In particular, if $A$ is moreover assumed to be *torsion-free*, then by [Wei94, Corollary 3.1.5], we have

$$H_n(G, A) \cong H_n(G) \otimes A.$$

**Remark 6.** Note that for any $G$-module $A$, $A_G \cong \mathbb{Z} \otimes_G A$, so that $H_*(G, A) \cong \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, A)$.

**Example 2.1.8.** By definition, $H_0(G) = \mathbb{Z}$. By [Wei94, Theorem 6.1.11], $H_1(G) \cong G/[G, G]$, the abelianisation of $G$. By [Wei94, Theorem 6.8.8], $H_2(G) \cong \frac{R \cap [F,F]}{[F,R]}$, where $G = \langle F | R \rangle$ is a presentation of $G$. This is known as Hopf's theorem.

**Remark 7.** The definition we present here is perhaps best suited for computations, but there exists equivalent definitions. For example, for a group $G$, if we let $BG := K(G, 1)$ denote the *Eilenberg-Maclane* space of $G$ (see for example [Hat01]), by [Wei94, Theorem 6.10.5], we have

$$H_*(BG, \mathbb{Z}) \cong H_*(G, \mathbb{Z}),$$

where the left-hand side denotes singular homology. This result may be seen as the start of homological algebra (see the remark after [Wei94, Theorem 6.10.5]).

**Remark 8** (Functoriality of Group Homology)**.** By definition, for a fixed group $G$, group homology $H_*(G, A)$ is functorial in $G$-modules $A$. However, it is also true that group homology $H_*(G, A)$ is functorial in the *pair* $(G, A)$.

More specifically, let $G, G'$ be groups and $A, A'$ be $G, G'$-modules respectively. Let $(\varphi, f) : (G, A) \to (G', A')$ a pair of maps where $\varphi : G \to G'$ is a group homomorphism and $f : A \to A'$ is a homomorphism of abelian groups such that $f(g \cdot a) = \varphi(g) \cdot f(a)$ for every $g \in G, a \in A$. Then, $(\varphi, f)$ induces a map on group homology $(\varphi, f)_* : H_*(G, A) \to H_*(G', A')$. We refer the reader to [Bro94, §III.8] for more details.

**Remark 9** (Shapiro's lemma)**.** Shapiro's lemma relates the homology of a subgroup to the homology of its group. Unsurprisingly therefore, it is very useful when performing computations.

In this thesis, we will use Shapiro's lemma stated as follows:

**Lemma 2.1.9.** *Let $G$ be a group. Let $H \leq G$ be a subgroup. Then, the map of pairs*

$$(i, 1 \otimes 1) : (H, \mathbb{Z}) \to (G, \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z})$$

*given by the inclusion* $i : H \hookrightarrow G$ *and* $1 \otimes 1 : 1 \mapsto 1 \otimes 1$ *induces an isomorphism on homology groups*

$$(i, 1 \otimes 1)_* : H_*(H, \mathbb{Z}) \xrightarrow{\cong} H_*(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}).$$

We refer the reader to [Bro94, Chapter III, Exercise 8.2] for more details.

#### 2.1.2.1 Relative homology

As in topology, we can define a notion of relative group homology.

Specifically, let $G$ be a group and let $H \leq G$ be a subgroup. Let $A$ be a $G$-module. We want to define the relative homology groups $H_*(G, H; A)$.

Let $P. \to \mathbb{Z}$ be a (right) projective $G$-module resolution of $\mathbb{Z}$. Note, $P. \to \mathbb{Z}$ is also a projective $H$-mod resolution.

Furthermore, note that $P. \otimes_G A$ can be used to compute $H_*(G, A)$; $P. \otimes_H A$ can be used to compute $H_*(H, A)$ and there is a canonical map $P. \otimes_H A \to P. \otimes_G A$. This motivates the following definition:

**Definition 2.1.10.** In the above notation, define

$$H_*(G, H; A) := H_*(\mathrm{Cone}(P. \otimes_H A \to P. \otimes_G A)).$$

Here, recall that the cone of a map $f : B. \to C.$ between chain complexes is a chain complex $\mathrm{Cone}(f)$ whose degree $n$ part is $B_{n-1} \oplus C_n$ and whose differential is given by $d(b, c) = (-d(b), d(c) - f(b))$. In particular, by definition of the cone construction, we have a long exact sequence

$$\cdots \to H_n(H, A) \to H_n(G, A) \to H_n(G, H; A) \to H_{n-1}(H, A) \to \cdots .$$

(See [Wei94, 1.5.2] for more details.) Thus, by the 5-lemma, we see that this definition is well-defined, as usual group homology is well defined.

In addition, this definition is entirely analogous to the situation in topology. Indeed, for a map of topological space $f : X \to Y$, we have

$$\begin{aligned} H_*(Y, X) &\cong H_*(\mathrm{Cyl}(f), X \times \{0\}) \\ &\cong \tilde{H}_*(\mathrm{Cone}(f)), \end{aligned}$$

where the first isomorphism follows from the fact that the mapping cylinder $\mathrm{Cyl}(f)$ deformation retracts onto $Y$, and the second isomorphism follows from the fact that $(\mathrm{Cyl}(f), X \times \{0\})$ is a good pair, in the sense of [Hat01].

There is also a universal coefficient theorem for relative group homology which will be used to prove homological stability for the Spin groups. We outline a proof, as there does not seem to be a proof written in the literature.

**Theorem 2.1.11.** *Let $G$ be a group, $H \leq G$ subgroup and $A$ a trivial $G$-module. Then, for all $n \geq 1$, there are (noncanonical) isomorphisms*

$$H_n(G, H; A) \cong (H_n(G, H) \otimes A) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(G, H) \otimes A).$$

*Proof.* Let $P. \to \mathbb{Z}$ be a free mod-$G$ resolution of $\mathbb{Z}$. As $A$ is a trivial $G$-module, we have $P. \otimes_H A \cong (P.)_H \otimes A$ and $P. \otimes_G A \cong (P.)_G \otimes A$. Therefore,

$$\begin{aligned}\mathrm{Cone}(P. \otimes_H A \to P. \otimes_G A) &\cong \mathrm{Cone}((P.)_H \otimes A \to (P.)_G \otimes A)\\ &\cong \mathrm{Cone}((P.)_H \to (P.)_G) \otimes A\\ &\cong \mathrm{Cone}(P. \otimes_H \mathbb{Z} \to P. \otimes_G \mathbb{Z}) \otimes A.\end{aligned}$$

Thus, by [Wei94, Theorem 3.6.2]

$$\begin{aligned}H_n(G, H; A) &\cong H_n(\mathrm{Cone}(P. \otimes_H \mathbb{Z} \to P. \otimes_G \mathbb{Z}) \otimes A)\\ &\cong (H_n(G, H) \otimes A) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(G, H), A).\end{aligned}$$

□

### 2.1.3 The homological stability problem

Let $\{G_n\}_{n \geq 0}$ be a sequence of groups, equipped with inclusions $G_n \hookrightarrow G_{n+1}$. By remark 8, these maps induce maps on group homology $H_k(G_n) \to H_k(G_{n+1})$. The homological stability problem seeks to understand the behaviour of these maps. This problem had been historically motivated by Algebraic K-Theory, and the idea is that for a certain a collection of groups, the maps $H_k(G_n) \to H_k(G_{n+1})$ eventually (depending on $k$) become *isomorphisms.* Note that by the above discussion about relative homology groups, this problem is equivalent to the relative homology $H_k(G_n, G_{n+1})$ vanishing in a certain range.

For example, for a ring $R$, its Algebraic K-Theory

$$K_i(R) = \pi_i(BGL(R)^+)$$

comes equipped with the *hurewicz map* into homology

$$\begin{aligned}\pi_i(BGL(R)^+) &\longrightarrow H_i(BGL(R)^+)\\ &\xrightarrow{\cong} H_i(GL(R)).\end{aligned}$$

It is therefore interesting to understand when does the homology of $GL_n(R)$ stabilise? Indeed, this problem goes back to Quillen and has been studied by others, for example [NS89]. We refer the reader to Wahl's survey article [Wah22] for more historical detail and motivation. The motivation for this thesis came from the analogous situation in *Hermitian K-Theory*, where for a ring $R$ with $2 \in R^*$, we have a hurewicz map

$$\begin{aligned}GW_i(R) \cong \pi_i(BO_{\infty,\infty}(R)^+) &\longrightarrow H_i(BO_{\infty,\infty}(R)^+)\\ &\xrightarrow{\cong} H_i(O_{\infty,\infty}(R)),\end{aligned}$$

where $O_{\infty,\infty}(R) := \varinjlim O_{n,n}(R)$ is the infinite orthogonal group.

The strategy to prove such homological stability results is standard and goes back to Quillen. Specifically, the idea is to construct a *highly acyclic chain complex* on which the groups $G_n$ act on *transitively*, and analyse the resulting *hyperhomology spectral sequences*. Hard work is usually necessary in proving the acyclicity of the chain complex, and the analysis of the spectral sequences.

In particular, one needs to have a solid command of spectral sequences, so we review this theory in the next section.

## 2.2 Spectral sequences

In this section, we give a quick review of the theory of spectral sequences, leading to the hyperhomology spectral sequences that are needed to prove our homological stability results.

**Definition 2.2.1.** A (homological) *spectral sequence* (starting with $E^a$) in an abelian category $\mathcal{A}$ consists of the following data:

- A family of $\{E^r_{p,q}\}$ of objects in $\mathcal{A}$ for every $p, q \in \mathbb{Z}$, $r \geq a$. We refer to $E^r$ as the *$r$'th page* of the spectral sequence, and we say the *total degree* of the term $E^r_{p,q}$ is $n := p + q$.
- A famlily of maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$, called *differentials*, such that $d^r \circ d^r = 0$. (We say that the differentials $d^r$ have *bidegree* $(-r, r-1)$).

- Isomorphisms between $E^{r+1}_{p,q}$ and the homology of $E^r_{**}$ at the spot $E^r_{p,q}$:

$$E^{r+1}_{p,q} \cong \ker(d^r_{p,q}) / \operatorname{Im}(d^r_{p+r,q-r+1}).$$

Moreover, there is a *category* of homological spectral sequences: a morphism $f : E' \to E$ is a family of maps $f^r_{p,q} : E'^r_{p,q} \to E^r_{p,q}$ in $\mathcal{A}$ with $d^r f^r = f^r d'^r$ such that $f^{r+1}_{p,q}$ is the map induced by $f^r_{p,q}$ on homology.

See figure 2.1 for an example of what a homological spectral sequence looks like.

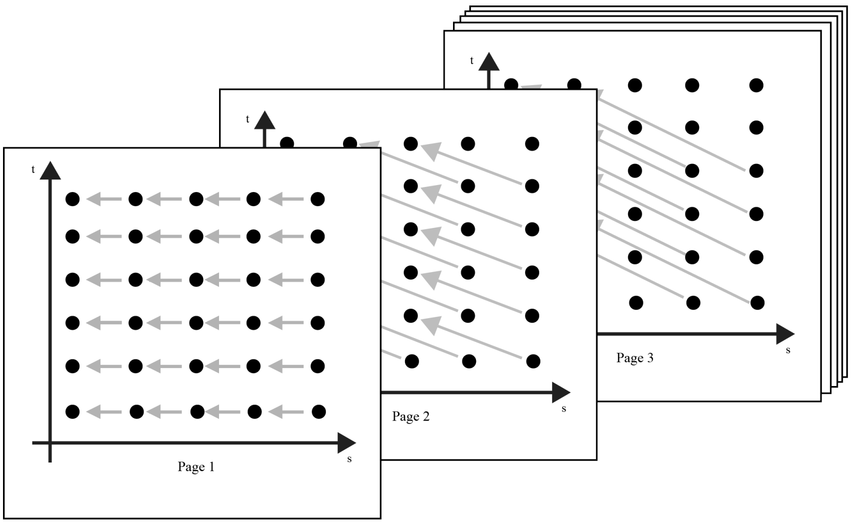



Figure 2.1: An example of a homological spectral sequence. Image taken from https://picturethismaths.wordpress.com/2016/02/04/spectral-sequences/

**Example 2.2.2.** We will usually be dealing with a specific type of spectral sequence, namely a *first quadrant* spectral sequence. This is a spectral sequence where $E^a_{p,q} = 0$ unless $p, q \geq 0$. Note, if this condition holds for a given page $E^{r_0}$, then its holds for every $r \geq r_0$. The spectral sequences shown in figure 2.1 shows the non-zero part of a first quadrant spectral sequence. A first quadrant spectral sequence is itself an example of a *bounded* spectral sequence: a spectral sequence $\{E^r_{p,q}\}$ that for every $n \in \mathbb{Z}$ has only *finitely many non-zero terms* of total degree $n$ on every page $E^r$.

**Remark 10.** For a first quadrant spectral sequence, note that $E^r_{p,q} = E^{r+1}_{p,q}$ for every large $r$ (taking $r > \max\{p, q+1\}$ will do). We say that the spectral sequence *stabilises*, and denote the stable value $E^\infty_{p,q}$. More generally, a bounded spectral sequence $\{E^r_{p,q}\}$ will have stable values $E^\infty_{p,q}$ of $E^r_{p,q}$ for every $(p, q)$, for similar reasons as above.

The spectral sequences that actually buy you computations are the *convergent* spectral sequences.

**Definition 2.2.3.** Let $\{E^r_{p,q}\}$ be a (bounded) spectral sequence. We say that this spectral sequence *converges* to a $\mathbb{Z}$-graded object $H_*$ (in the abelian category $\mathcal{A}$) if for every $H_n$, we have a *finite filtration*

$$0 = F_sH_n \subseteq \cdots \subseteq F_{p-1}H_n \subseteq F_pH_n \subseteq F_{p+1}H_n \subseteq \cdots \subseteq F_tH_n = H_n$$

such that for every $(p,q)$ with $p+q=n$, we are given isomorphisms

$$E^\infty_{p,q} \cong F_pH_n/F_{p-1}H_n.$$

The traditional way to write such convergence is

$$E^a_{p,q} \Rightarrow H_{p+q},$$

where $E^a$ is the starting page of the spectral sequence. It is common to call $H_* = \bigoplus_n H_n$ the *abutment* of the spectral sequence.

Moreover, there is a sensible notion of a *morphism* between convergent spectral sequences. Indeed, let $H_*$ and $H'_*$ be the abutments of convergent spectral sequences $\{E^r_{p,q}\}$ and $\{E'^r_{p,q}\}$ respectively, and let $f: E \to E'$ be a morphism of spectral sequences. We say a map $h: H_* \to H'_*$ is *compatible* with morphism $f: E \to E'$ if $h$ maps $F_pH_n$ to $F_pH'_n$ and the diagrams

$$\begin{array}{ccc} E^\infty_{p,q} & \xrightarrow{\quad f \quad} & E'^\infty_{p,q} \\ \downarrow{\cong} & & \downarrow{\cong} \\ F_pH_n/F_{p-1}H_n & \xrightarrow[h]{} & F_pH'_n/F_{p-1}H'_n \end{array}$$

commute. We then refer to the pair $(f,h)$ as the *morphism* between the convergent spectral sequences $E$ and $E'$.

Let us see some examples of convergent spectral sequences.

**Example 2.2.4** (Serre spectral sequence)**.** Let $F \to X \to B$ be a fibration with $B$ path connected. If $\pi_1(B)$ acts trivially on $H_*(F)$, we have a spectral sequence

$$E^2_{p,q} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(X).$$

We refer to [SS89, §9] for more details. Note that the trivial action hypothesis may be removed if one is willing to work with local coefficents.

**Example 2.2.5** (Group homology spectral sequence)**.** Let

$$1 \to N \to G \to G/N \to 1$$

be a short exact sequence of groups, and let $A$ be a $G$-module. Then, there is a convergent first quadrant spectral sequence

$$E^2_{p,q} = H_p(G/N, H_q(N, A)) \Rightarrow H_{p+q}(G, A),$$

called the *Hochschild-Serre Spectral Sequence.* We refer the reader to [Bro94, §VII.6] or [Wei94, §6.8] for more details. This spectral sequence will used frequently in this thesis. Moreover, there is relative version of this spectral sequence. Specifically, let

$$1 \to N \to G_1 \to H_1 \to 1$$

and

$$1 \to N \to G_2 \to H_2 \to 1$$

be a pair of short exact sequences of groups, equipped with inclusions $G_1 \hookrightarrow G_2$, $H_1 \hookrightarrow H_2$ such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G_1 & \longrightarrow & H_1 & \longrightarrow & 1 \\ & & \downarrow = & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & N & \longrightarrow & G_2 & \longrightarrow & H_2 & \longrightarrow & 1 \end{array}$$

commutes. Let $A$ be a $G_2$-module. Then, there is a spectral sequence

$$E^2_{p,q} = H_p(H_2, H_1; H_q(N, A)) \Rightarrow H_{p+q}(G_2, G_1; A),$$

called the *relative Hochschild-Serre Spectral Sequence.* We refer to [McC01, Exercise 5.5] for more details.

This spectral sequence will be used to immediately deduce homological stability of $\mathrm{Spin}_{n,n}$ from homological stability of $EO_{n,n}$.

**Example 2.2.6** (Spectral sequence of a filtration)**.** let

$$0 = F_sC \subseteq \cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq F_{p+1}C \subseteq \cdots \subseteq F_tC = C$$

be a bounded filtration of a chain complex $C$. Then, there is convergent spectral sequence

$$E^1_{p,q} = H_{p+q}(F_pC/F_{p-1}C) \Rightarrow H_{p+q}(C).$$

We refer the reader to [Wei94, §5.5] for a proof of this fact. This is an important example, as this in particular allows you to prove the existence of many convergent spectral sequences associated to a *double complex*, such as the Lyndon-Hochschild-Serre Spectral Sequence above (see for example [Bro94, §VII.6]) and the *hyperhomology spectral* sequences that we are now about to introduce. We give some details in the next section, but we refer the reader to [Wei94, §5.6] for more details about how a double complex gives rise to convergent spectral sequences via the above example.

**Example 2.2.7** (Spectral sequences associated to a double complex)**.** An important application of the above example are the spectral sequences obtained from a double complex.

Let $C = C_{**}$ be a double complex. This is simply a family of object $\{C_{p,q}\}$ in the abelian category $\mathcal{A}$, together with maps

$$d^h : C_{p,q} \to C_{p-1,q} \quad \text{and} \quad d^v : C_{p,q} \to C_{p,q-1}$$

such that $d^h \circ d^h = d^v \circ d^v = 0$ and $d^v \circ d^h = -d^h \circ d^v$. (Note, some authors require the condition $d^v \circ d^h = d^h \circ d^v$ instead of $d^v \circ d^h = -d^h \circ d^v$, but the two definitions give equivalent theories provided one keeps track of the signs).

For our purposes, we may assume $C$ is *bounded* i.e. $C_{p,q} = 0$ along the diagonal $p + q = n$ for all but finitely many $(p, q)$. Let $\text{Tot}(C)$ be the *total complex* of $C$. This is a chain complex with $\text{Tot}(C)_n := \bigoplus_{p+q=n} C_{p,q}$, with differential $d = d^h + d^v$. There are two natural filtrations that one can consider on $\text{Tot}(C)$:

- The *column filtration*, given by

$$\cdots \subseteq {}^I F_{n-1}\text{Tot}(C) \subseteq {}^I F_n\text{Tot}(C) \subseteq {}^I F_{n+1}\text{Tot}(C) \subseteq \cdots ,$$

  where

$${}^I F_n\text{Tot}(C) := \text{Tot}({}^I \tau_{\leq n}(C)),$$

$${}^I \tau_{\leq n}(C)_{p,q} := \begin{cases} C_{p,q} & \text{if} \quad p \leq n, \\ 0 \quad \text{if} & p > n. \end{cases}$$

- The *row filtration*, given by

$$\cdots \subseteq {}^{II} F_{n-1}\text{Tot}(C) \subseteq {}^{II} F_n\text{Tot}(C) \subseteq {}^{II} F_{n+1}\text{Tot}(C) \subseteq \cdots ,$$

where

$$^{II}F_n\mathrm{Tot}(C) := \mathrm{Tot}(^{II}\tau_{\leq n}(C)),$$

$$^{II}\tau_{\leq n}(C)_{p,q} := \begin{cases} C_{p,q} & \text{if} \quad q \leq n, \\ 0 & \text{if} \quad q > n. \end{cases}$$

The reader should draw some pictures to visualise these filtrations. If $C$ is a *first quadrant double complex*, these two filtrations give rise to spectral sequences

$$^{I}E^2_{p,q} = H^h_p H^v_q(C) \Rightarrow H_{p+q}(\mathrm{Tot}(C))$$

and

$$^{II}E^2_{p,q} = H^v_p H^h_q(C) \Rightarrow H_{p+q}(\mathrm{Tot}(C))$$

respectively, where the superscripts $H^v$ and $H^h$ denote the direction in which homology is taken. We refer to [Wei94, §5.6] for more details.

### 2.2.1 Hyperhomology

Let $G$ be a group. For a (left) $G$-module $M$, we have

$$\begin{aligned} H_*(G, M) &\cong \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, M) \\ &\cong H_*(F. \otimes_G M), \end{aligned}$$

where $F. \to \mathbb{Z}$ is a (right) $\mathbb{Z}G$-projective resolution of $\mathbb{Z}$.

It is useful to generalise this, allowing coefficients to take values in *complexes of (non-negative) $G$-modules.* (Here, a complex of $G$-modules means a chain complex consisting of $G$-modules, such that the actions commute with the differentials).

**Definition 2.2.8.** Let $G$ be a group, and let $C. = (C_n)_{n\geq 0}$ be a complex of $G$-modules. Define the *homology of $G$ with coefficients in $C.$* as

$$H_*(G, C.) := H_*(\mathrm{Tot}(F. \otimes_G C.)),$$

where $F. \to \mathbb{Z}$ is a (right) $\mathbb{Z}G$-projective resolution of $\mathbb{Z}$.

This is also known as the *hyperhomology* of $G$ with coefficients in $C.$, or simply the *hyperhomology* of $G$ when the complex of $G$-modules is understood.

**Remark 11.** This definition is well-defined up to canonical isomorphism, although it is not obvious to see this. One way to show this definition is well-defined is to identify the hyperhomology of $G$ with coefficients in $C.$ with a certain type of *left hyper-derived* functor $\mathbb{L}_*F(C.)$, and use the fact that these functors are well-defined. We will not go into the details, but we refer the reader to [Wei94, §5.7] for the definition of left hyper-derived functor, and leave the connection between these two concepts as an interesting exercise!

**Remark 12.** If $C.$ is the single module $M$ concentrated in degree 0, we have $H_*(G, C.) \cong H_*(G, M)$. Thus, hyperhomology indeed generalizes usual group homology.

Since $\mathrm{Tot}(F. \otimes C.)$ is the total complex of double complex $D_{p,q} := F_p \otimes_G C_q$, by the discussion in example 2.2.6 applied to row filtration ${}^{II}F_p\mathrm{Tot}(D)$, noting that

$$H_{p+q}({}^{II}F_p\mathrm{Tot}(D)/{}^{II}F_{p-1}\mathrm{Tot}(D)) \cong H_q(F. \otimes_G C_p) \cong H_q(G, C_p),$$

we have the spectral sequence

$$E^1_{p,q} = H_q(G, C_p) \Rightarrow H_{p+q}(G, C.).$$

In addition, by the discussion in example 2.2.7 applied to the column filtration ${}^{I}F_p\mathrm{Tot}(D)$, we have a convergent spectral sequence

$$E^2_{p,q} = H_p(G, H_qC.) \Rightarrow H_{p+q}(G, C.).$$

We call the spectral sequences

$$\begin{aligned} E^2_{p,q} &= H_p(G, H_qC.) \Rightarrow H_{p+q}(G, C.) \\ E^1_{p,q} &= H_q(G, C_p) \Rightarrow H_{p+q}(G, C.) \end{aligned}$$

the *hyperhomology* spectral sequences. We will use these spectral sequences to prove our homological stability results.

For now, note that the first of these spectral sequences gives a slick proof that quasi-isomorphic chain complexes have the same hyperhomology.

**Proposition 2.2.9.** *Let $\tau : C \to C'$ be a quasi-isomorphism of $G$-chain complexes i.e. a $G$-equivariant chain map that induces an isomorphism on homology groups. Then, $\tau$ induces an isomorphism in hyperhomology $H_*(G, C) \xrightarrow{\cong} H_*(G, C')$.*

*Proof.* Note that $\tau$ induces a map of spectral sequences which is an isomorphism at the $E^2$-level, hence $\tau$ induces an isomorphism on the abutments. □

### 2.2.2 Exact couples

Exact couples are a general source of spectral sequences. The role that they will play in this thesis is they will allow us to define *actions* on our hyperhomology spectral sequences, as we will be able to act on their associated exact couples.

We briefly review the theory of exact couples and give some examples.

**Definition 2.2.10.** Let $\mathcal{A}$ be an abelian category. Then, an *exact couple* in $\mathcal{A}$ is a commutative diagram of the form

$$(E, D) = \begin{array}{ccc} E & \xrightarrow{k} & D \\ & \nwarrow_{j} & \downarrow i \\ & & D \end{array}$$

which is exact at each vertex. A *morphism* of exact couples $(E, D) \to (\hat{E}, \hat{D})$ is a pair of maps $(E \to \hat{E}, D \to \hat{D})$ that commute with the structure maps defining the exact couples.

For our purposes, we will need to extend this definition to include an *abutment.*

**Definition 2.2.11.** An *exact couple with (homological) abutment* $A$ is an an exact couple as above with a map $\sigma : D \to A$ such that the diagram

$$(E, D, A) = \begin{array}{ccccc} E & \xrightarrow{k} & D & \xrightarrow{\sigma} & A \\ & \nwarrow_{j} & \downarrow i & \nearrow_{\sigma} & \\ & & D & & \end{array}$$

commutes. A *morphism* of exact couples with abutment $(E, D, A) \to (\hat{E}, \hat{D}, \hat{A})$ is a triple of maps $(E \to \hat{E}, D \to \hat{D}, A \to \hat{A})$ that commute with the structure maps defining the exact couples with abutment.

Before introducing some examples, it is perhaps beneficial to see how exact couples give rise to spectral sequences. This is done via the *derived exact couple* of an exact couple.

**Definition 2.2.12.** Let

$$(E, D, A) = \begin{array}{ccccc} E & \xrightarrow{k} & D & \xrightarrow{\sigma} & A \\ & \nwarrow_{j} & \downarrow i & \nearrow_{\sigma} & \\ & & D & & \end{array}$$

be an exact couple with abutment. We define its *derived exact couple with abutment* to be the diagram

$$(E', D', A) = \begin{array}{ccccc} E' & \xrightarrow{k'} & D' & \xrightarrow{\sigma'} & A \\ & \nwarrow^{j'} & \downarrow{\scriptstyle i'} & \nearrow_{\sigma'} & \\ & & D' & & \end{array}$$

where

$$E' := \frac{\ker jk}{\operatorname{Im} jk}, \qquad D' := \operatorname{Im} i$$

and

$$k'([x]) := k(x), \qquad i'(y) := i(y), \qquad j'(iz) := [j(z)], \qquad \sigma'(y) := \sigma(y).$$

It is bookwork to check that the above definition is well-defined and results in an exact couple with abutment. In particular, we can iterate the above construction $r$-times to obtain an exact couple with abutment $(E^r, D^r, A)$. We call this the *$r$th derived exact couple with abutment of* $(E, D, A)$. We will denote the structure maps in the $r$th derived exact couple with abutment by $k^r, i^r, j^r, \sigma$. Sometimes, we may omit the superscripts if the context is clear.

The idea then is that $\{E^r\}$ will assemble into a spectral sequence, converging to something associated with the abutment $A$. This happens in a particular situation, which we shall now briefly describe.

In an exact couple, we typically have

$$E = \bigoplus_{p,q \in \mathbb{Z}} E_{p,q}, \qquad D = \bigoplus_{p,q \in \mathbb{Z}} D_{p,q}, \qquad A = \bigoplus_{n \in \mathbb{Z}} A_n,$$

with $k, i, j$ of bidegrees $(0, -1), (1, -1), (0, 0)$ respectively and $\sigma$ homogeneous. In other words, $k, i, j, \sigma$ restrict to maps

$$\begin{aligned} k &: E_{p,q} \to D_{p,q-1} \\ i &: D_{p,q} \to D_{p+1,q-1} \\ j &: D_{p,q} \to E_{p,q} \\ \sigma &: D_{p,q} \to A_{p+q}. \end{aligned}$$

In the derived exact couple, by definition of $i'$ and $\sigma'$, bidegree($i'$) = bidegree($i$)= $(1, -1)$, and $\sigma'$ is homogeneous. Furthermore, note that $D'_{p,q} := \operatorname{Im}(D_{p,q} \xrightarrow{i} D_{p+1,q-1})$. Therefore, by definition of $j'$, we deduce bidgree($j'$) = bidegree($j$)= $(0, 0)$. It remains to work out the bidegree of $k'$.

Let $[x] \in E'_{p+1,q}$. Then, by definition,

$$\begin{aligned} x \in \ker(jk) &= \{x \in E_{p+1,q} | jkx = 0\} \\ &= \{x \in E_{p+1,q} | kx \in \ker(j) = \operatorname{Im}(i)\} \\ &= k^{-1}(iD_{p,q}). \end{aligned}$$

Therefore, $k(x) \in \operatorname{Im}(D_{p,q} \xrightarrow{i} D_{p+1,q-1}) = D'_{p,q}$. Thus, we deduce

$$k' : E_{p+1,q} \to D'_{p,q}.$$

That is to say, $\text{bidegree}(k') = (-1, 0) = \text{bidegree}(k) - \text{bidegree}(i)$.

Iterating, we deduce that for an $r$th derived exact couple $(E^r, D^r, A)$, the bidegree of $k^r$ is $= (-r, r-1)$. More generally, if $k$ starts of with bidegree $(-a, a-1)$, then the bidegree of $k^r$ is $= (-a - r, r + a - 1)$.

Defining differentials $d^r_{p,q}$ as

$$d^{r+a}_{p,q} : E^{r+a}_{p,q} \xrightarrow{k^r} D^{r+a}_{p-r-a,q+r+a-1} \xrightarrow{j^r} E^{r+a}_{p-r-a,q+r+a-1}$$

and putting the above together, we have established the following proposition:

**Proposition 2.2.13.** *Let*

$$(E, D) = \begin{array}{ccc} \bigoplus_{p,q} E_{p,q} & \xrightarrow{k} & \bigoplus_{p,q} D_{p,q} \\ & \nwarrow^{j} & \downarrow i \\ & & \bigoplus_{p,q} D_{p,q} \end{array}$$

*be an exact couple in which $k, i, j$ have bidegrees $(-a, a-1), (1, -1)$ and $(0, 0)$ respectively. Then, the derived exact couples $\{E^r_{p,q}\}$ assemble into a spectral sequence, starting at the $a$th page with differentials*

$$d^{r+a}_{p,q} : E^{r+a}_{p,q} \xrightarrow{k^r} D^{r+a}_{p-r-a,q+r+a-1} \xrightarrow{j^r} E^{r+a}_{p-r-a,q+r+a-1}.$$

□

Moreover, note that a morphism of exact couples of the above type induce a morphism of the corresponding spectral sequences. Thus, a map between spectral sequences can be cooked up by defining a map on their corresponding exact couples.

Of course, what we are actually interested in are spectral sequence that *converge*. This is where the abutment comes in.

We begin by defining a filtration on $A_n$. Specifically, for $n = p + q$, define

$$F_p A_{p+q} := \operatorname{Im}(D_{p,q} \xrightarrow{\sigma} A_{p+q}).$$

Note that the map $D_{p,q} \xrightarrow{\sigma} A_{p+q}$ factors as $D_{p-1,q+1} \xrightarrow{i} D_{p,q} \xrightarrow{\sigma} A_{p+q}$. Therefore, we obtain a filtration

$$\cdots \subseteq F_{p-1}A_{p+q} \subseteq F_p A_{p+q} \subseteq F_{p+1}A_{p+q} \subseteq \cdots \subseteq A_{p+q}.$$

In addition, note that

$$\begin{aligned} \operatorname{Im}(D^1_{p,q} \xrightarrow{\sigma} A_{p+q}) &= \operatorname{Im}(i(D_{p,q}) \xrightarrow{\sigma} A_{p+q}) \\ &= \operatorname{Im}(D_{p,q} \xrightarrow{\sigma} A_{p+q}). \end{aligned}$$

Therefore, the filtration does not change when taking derived exact couples.

**Proposition 2.2.14.** *Let $\{E^r_{p,q}\}$ be a spectral sequence constructed from an exact couple with abutment as above, with starting page $E^1$. Suppose*

- *For every $p < 0, D^1_{p,q} = 0$.*
- *For every $n$, there exists $p_0(n)$ such that for every $p \geq p_0(n)$, $D^1_{p,n-p} \xrightarrow{\sigma} A_n$ is an isomorphism.*

*Then, we have a convergent spectral sequence*

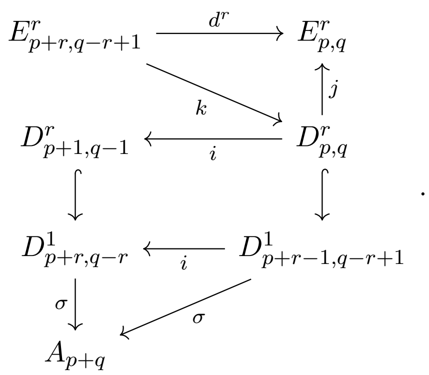
$$E^1_{p,q} \Rightarrow A_{p+q}.$$

*Proof.* By definition, $E^{r+1}_{p,q}$ is the homology of

$$E^r_{p+r,q-r+1} \xrightarrow{d^r} E^r_{p,q} \xrightarrow{d^r} E^r_{p-r,q+r-1}.$$

To show that these terms stabilize, we will show each differential is zero for $r$ sufficiently large.

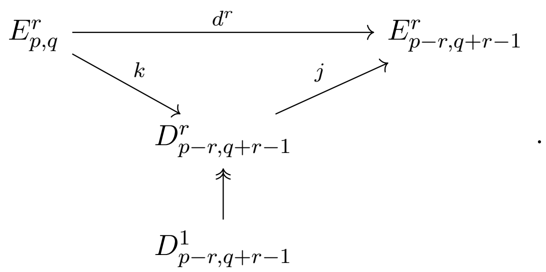
For the first differential, consider the following commutative diagram



The map $D^1_{p+r,q-r} \xrightarrow{\sigma} A_{p+q}$ is an isomorphism for every $p + r \geq p_{p+q}$, and the map $D^1_{p+r-1,q-r+1} \xrightarrow{\sigma} A_{p+q}$ is an isomorphism for every $p+r-1 \geq p_{p+q}$. Therefore, both maps are isomorphisms when $r > p_0 - p$, where $p_0 := p_0(p+q)$. Therefore, the map $i : D^1_{p+r-1,q-r+1} \xrightarrow{i} D^1_{p+r,q-r}$ is an isomorphism when $r > p_0 - p$.

Then, as the maps $D^r_{p,q} \hookrightarrow D^1_{p+r-1,q-r+1}$ and $D^r_{p+1,q-1} \hookrightarrow D^1_{p+r,q-r}$ are injective, it follows that the map $D^r_{p,q} \xrightarrow{i} D^r_{p+1,q-1}$ is injective when $r > p_0 - p$. Therefore, by exactness, we deduce that $k = 0$ when $r > p_0 - p$. Thus, for every $r > p_0 - p$, the differential $E^r_{p+r,q-r+1} \xrightarrow{d^r} E^r_{p,q}$ is zero.

For the second differential, consider the commutative diagram



Here, the bottom map $D^1_{p-r,q+r-1} \twoheadrightarrow D^r_{p-r,q+r-1}$ is the canonical projection.

By assumption, we have $D^1_{p-r,q+r-1} = 0$ for every $p-r < 0$. Therefore, $k = 0$ for every $r > p$.

Thus, for every $r > \max(p_0 - p, p)$, we have

$$E^r_{p+r,q-r+1} \xrightarrow{d^r=0} E^r_{p,q} \xrightarrow{d^r=0} E^r_{p-r,q+r-1}.$$

We have shown that the terms of the spectral sequence stabilize. It remains to show $E^1_{p,q} \Rightarrow A_{p+q}$.

Recall, we have a filtration given by

$$F_pA_{p+q} := \mathrm{Im}(D^1_{p,q} \xrightarrow{\sigma} A_{p+q}).$$

Note that this is a finite filtration as $D^1_{p,q} = 0$ for every $p < 0$ and $D^1_{p,n-p} \xrightarrow{\sigma} A_n$ is an isomorphism for $p$ sufficiently large.

Let $r > \max(p_0 - p, p) + 1$. From the exact couple, we obtain a commutative diagram with the top row exact and the bottom row exact by commutativity:

$$\begin{array}{ccccccccc} & & D^r_{p-1,q+1} & \xrightarrow{i} & D^r_{p,q} & \xrightarrow{j} & E^r_{p,q} & \xrightarrow{k} & D^r_{p-r,q+r-1} \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow= & & \downarrow= \\ 0 & \longrightarrow & F_{p-1}A_{p+q} & \longrightarrow & F_pA_{p+q} & \longrightarrow & E^\infty_{p,q} & \longrightarrow & 0 \end{array} .$$

Here, the map $D^r_{p,q} \to F_pA_{p+q}$ is an isomorphism for every $r > p_0 - p$ as $D^1_{p+r-1,q-r+1} \to A_{p+q}$ is an isomorphism in this range. Similarly, $D^r_{p-1,q+1} \to F_{p-1}A_{p+q}$ is an isomorphism for every $r > p_0 - p + 1$.

Therefore, by exactness, we have

$$E^\infty_{p,q} \cong F_pA_{p+q}/F_{p-1}A_{p+q}$$

i.e. $E^1_{p,q} \Rightarrow A_{p+q}$. □

**Example 2.2.15** (Exact couple of a filtration)**.** Let $C.$ be a filtered chain complex, with filtration

$$\cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq F_{p+1}C \subseteq \cdots ,$$

which for our purposes we may assume to be bounded. The short exact sequences of chain complexes

$$0 \to F_{p-1}C \to F_pC \to F_pC/F_{p-1}C \to 0$$

gives a long exact sequence of homology groups

$$\cdots \to H_{p+q}(F_{p-1}C) \xrightarrow{i} H_{p+q}(F_pC) \xrightarrow{j} H_{p+q}(F_pC/F_{p-1}C) \xrightarrow{k} H_{p+q-1}(F_{p-1}C) \to \cdots ,$$

which may be rolled up into an exact couple

$$(E^1, D^1) = \begin{array}{ccc} \bigoplus_{p,q} H_{p+q}(F_pC/F_{p-1}C) & \xrightarrow{k} & \bigoplus_{p,q} H_{p+q}(F_pC) \\ & \nwarrow^{j} & \downarrow i \\ & & \bigoplus_{p,q} H_{p+q}(F_pC) \end{array}$$

satisfying the hypothesis of proposition 2.2.13 (with $a = 1$). Thus, by proposition 2.2.13, the derived couples assemble into a spectral sequence.

Moreover, if we assume further that $C. = (C_n)_{n\geq 0}$ is bounded from above and our filtration is of the form

$$0 = F_{-1}C \subseteq \cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq F_{p+1}C \subseteq \cdots \subseteq F_tC = C$$

it would follow that the conditions of proposition 2.2.14 are satisfied, with $A_n = H_n(C)$ and $\sigma : H_n(F_pC) \to H_n(C)$ induced by the inclusion. This gives us a convergent spectral sequence

$$E^1_{p,q} = H_{p+q}(F_pC/F_{p-1}C) \Rightarrow A_{p+q} = H_{p+q}(C)$$

associated to the exact couple with abutment

$$(E^1, D^1, A) = \begin{array}{ccccc} \bigoplus_{p,q} H_{p+q}(F_pC/F_{p-1}C) & \xrightarrow{k} & \bigoplus_{p,q} H_{p+q}(F_pC) & \xrightarrow{\sigma} & \bigoplus_{p,q} H_{p+q}(C) \\ & \nwarrow^{j} & \downarrow i & \nearrow^{\sigma} & \\ & & \bigoplus_{p,q} H_{p+q}(F_pC) & & \end{array}.$$

One can show that this spectral sequence is naturally isomorphic to spectral sequence cited in example 2.2.6, see for example [Wei94, Theorem 5.9.4].

As an important example of this, let $G$ a group; $C. = (C_n)_{n\geq 0}$ a bounded chain complex of $G$-modules and $F. \to \mathbb{Z}$ be a (right) $\mathbb{Z}G$-projective resolution. Then, for the double complex $D := F \otimes_G C$, we have the row filtration

$$0 = {}^{II}F_{-1}\mathrm{Tot}(D) \cdots \subseteq {}^{II}F_p\mathrm{Tot}(D) \subseteq \cdots \subseteq {}^{II}F_t\mathrm{Tot}(D) = \mathrm{Tot}(D),$$

where one can show

$$H_{p+q}({}^{II}F_p\mathrm{Tot}(D)/{}^{II}F_{p-1}\mathrm{Tot}(D)) \cong H_q(F. \otimes_G C_p) \cong H_q(G, C_p).$$

Thus, we obtain the hyperhomology spectral sequence

$$E^1_{p,q} = H_q(G, C_p) \Rightarrow H_{p+q}(G, C.)$$

from the exact couple with abutment

$$\begin{array}{ccccc}
\bigoplus_{p,q} H_{p+q}(G, C_{\leq p}/C_{\leq p-1}) & \xrightarrow{k} & \bigoplus_{p,q} H_{p+q}(G, C_{\leq p}) & \xrightarrow{\sigma} & \bigoplus_{p,q} H_{p+q}(G, C.) \\
 & {\scriptstyle j}\nwarrow & \downarrow{\scriptstyle i} & \nearrow{\scriptstyle \sigma} & \\
 & & \bigoplus_{p,q} H_{p+q}(G, C_{\leq p}) & &
\end{array} .$$

**Remark 13.** It is possible (but not obvious!) to identify the maps $i, j, k$ with the maps in the long exact sequence in hyperhomology associated to the short exact sequence of complexes

$$0 \to C_{\leq p-1} \to C_{\leq p} \to C_{\leq p}/C_{\leq p-1} \to 0.$$

We do not need to use this in the thesis, so will not spell out the details.

## 2.3 The derived tensor product

In this thesis, it will be at times convenient to use the derived tensor product $-\otimes^{\mathbb{L}}-$. A complete exposition of the derived tensor product will take us too far afield. We will only recall the main idea and state the properties that we will need to use in this thesis. Precise references will be given throughout.

Annoyingly, almost all references about this material use cohomological indexing convention (to keep the algebraic geometers happy), whereas we work exclusively with homological indexing convention. Rather than reproduce all the proofs in terms of homological indexing, we will just state the definitions and theorems using this convention. The proofs we cite prove the analogous statements in the other convention.

### 2.3.1 Derived category

Let $\mathcal{A}$ be an abelian category. Let $\mathrm{Ch}(\mathcal{A})$ denote the category of chain complexes of $\mathcal{A}$. Define $K(\mathcal{A})$ to be the category given by datum:

- Objects of $K(\mathcal{A})$ are the objects of $\mathrm{Ch}(\mathcal{A})$
- Morphisms of $K(\mathcal{A})$ are the chain homotopy equivalence classes of chain maps in $\mathrm{Ch}(\mathcal{A})$.

An important collection morphism in $K(\mathcal{A})$ are the so called *quasi-isomorphisms*: those morphisms which induce isomorphisms on homology groups. Denote the collection of quasi-isomorphisms by $Q$. We then define the *derived category* $D(\mathcal{A})$ to be the *localization* of $K(\mathcal{A})$ at the quasi-isomorphisms, i.e.

$$D(\mathcal{A}) := K(\mathcal{A})[Q^{-1}].$$

Intuitively, this means that the objects of $D(\mathcal{A})$ are the same as $K(\mathcal{A})$, but the morphisms of $D(A)$ have been changed so that every quasi-isomorphism becomes an *isomorphism*. For more details, we refer the reader to [Wei94, Section 10]. We will write $q : K(\mathcal{A}) \to D(\mathcal{A})$ for the localization functor (so that in particular $q(f)$ is an isomorphism for every $f$ in $Q$), and we will think of $D(\mathcal{A})$ in terms of its universal property corresponding to its definition as a localization.

Note that $K(\mathcal{A})$ and $D(\mathcal{A})$ are examples of *triangulated categories*. Intuitively, a triangulated category $\mathcal{T}$ is an additive category equipped with an auto-equivalence $\Sigma : \mathcal{T} \to \mathcal{T}$, called the *suspension functor* and *exact triangles*

$$X \to Y \to Z \to \Sigma X$$

which are meant to mimic *short exact sequences* in abelian categories, and satisfy various axioms. An *exact functor* is then a functor $F : \mathcal{T} \to \mathcal{T}'$ between triangulated categories which commute the suspension functors up to natural isomorphism and map exact triangles to exact triangles. For example, $q : K(\mathcal{A}) \to D(\mathcal{A})$ is an exact functor. Thus, the collection of all triangulated categories form a *category* in this sense. We refer the reader to [Wei94] for more details about triangulated categories. Many important features of $K(\mathcal{A})$ and $D(\mathcal{A})$ are proven using this triangulated structure.

Denoting $\mathrm{Ch}^b(\mathcal{A})$, $\mathrm{Ch}^-(\mathcal{A})$,$\mathrm{Ch}^+(\mathcal{A})$ to be the full subcategories of $\mathrm{Ch}(\mathcal{A})$ consisting of bounded, bounded above and bounded below chain complexes respectively, one can analogously define $K^b(\mathcal{A})$, $K^-(\mathcal{A})$, $K^+(\mathcal{A})$ and $D^b(\mathcal{A})$, $D^-(\mathcal{A})$, $D^+(\mathcal{A})$ respectively.

### 2.3.2 Total derived functors

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between two abelian categories. Since $F$ preserves chain homotopy equivalences, $F$ extends to a functor $F : K(\mathcal{A}) \to K(\mathcal{B})$. This is even an exact functor. However, $F$ *may not preserve quasi-isomorphisms*, so that it may not extend to a functor $D(\mathcal{A}) \to D(\mathcal{B})$. Derived functors are meant to 'fix' this issue.

We state the following definition in terms of *Kan extensions*. We refer the reader to [ML71] for more details about this important concept. Intuitively, Kan extensions are maps that will make the forthcoming diagrams commute up to natural transformation, and is the universal functor that does this in some sense.

**Definition 2.3.1.** Let $F : K(\mathcal{A}) \to K(\mathcal{B})$ as above. We define the *total left derived functor of* $F$, denoted $LF$, to be the exact functor $LF : D(\mathcal{A}) \to D(\mathcal{B})$ which is the *right Kan extension* of $qF$ along $q$:

$$\begin{array}{ccccc} K(\mathcal{A}) & \xrightarrow{F} & K(\mathcal{B}) & \xrightarrow{q} & D(\mathcal{B}) \\ \downarrow q & & & \nearrow_{LF:=\mathrm{Ran}(qF)} & \\ D(\mathcal{A}). & & & & \end{array} .$$

One similarly defines the *total right derived functor* as a left Kan extension, but we do not need this in this thesis.

When the domain of $F$ is $K^*(\mathcal{A})$, the total left derived functor is denoted by $L^*(F)$, where $* = b, -, +$.

**Example 2.3.2.** If $F : \mathcal{A} \to \mathcal{B}$ is an exact functor, then $F$ preserves quasi-isomorphisms and therefore extends to a functor $F : D(\mathcal{A}) \to D(\mathcal{B})$. One can check $F$ is its own left and right total derived functor.

The following theorem gives us a sufficient condition for the total left-derived functor to exist, and a particular instance where we can compute the total left-derived functor.

**Theorem 2.3.3.** *Let $F : K^+(\mathcal{A}) \to K(\mathcal{B})$ be an exact functor of triangulated categories, and suppose $\mathcal{A}$ has enough projectives. Then, the total left derived functor $L^+F$ on $D^+(\mathcal{A})$ exists. Moreover, if $P$ is a bounded below chain complex of projectives, then*

$$L^+F(P) \cong qF(P),$$

*where $q : K(\mathcal{B}) \to D(\mathcal{B})$.*

*Proof.* See [Wei94, Theorem 10.5.6]. □

**Remark 14.** With this existence theorem at hand, one is able to make a precise connection between the total left-derived functor and the usual left-derived functors defined at the beginning of the thesis. Namely, if $\mathcal{A}$ has enough projectives, then for any $X \in \mathcal{A}$,

$$L_iF(X) \cong H_iL^+F(X),$$

where $X$ is viewed as a chain complex concentrated in degree 0. This follows from the more general statement that for any $X \in \mathrm{Ch}(\mathcal{A})$,

$$\mathbb{L}_i F(X) \cong H_i L^+ F(X),$$

where $\mathbb{L}_i F$ is the *left hyper-derived functors* mentioned earlier in remark 11. One deduces the former from the latter by noting that for every $X \in \mathcal{A}$, $\mathbb{L}_i F(X) \cong L_i F(X)$, where $X$ is viewed as a chain complex concentrated in degree 0. We refer to [Wei94, Corollary 10.5.7] for more details.

### 2.3.3 The derived tensor product

We are now in a position to define the derived tensor product.

Let $R$ be a ring. Let $A \in K(\mathrm{Mod}-R)$ and consider the functor

$$\begin{aligned} \mathrm{Tot}(A \otimes_R -) :& K^-(R-\mathrm{Mod}) \to K(\mathrm{Ab}) \\ & B \mapsto \mathrm{Tot}(A \otimes_R B), \end{aligned}$$

where $A \otimes_R B$ is the double complex $\{A_p \otimes_R B_q\}$ with horizontal differentials $d \otimes 1$ and vertical differentials $(-1)^p \otimes d$.

Observe that $R$-Mod has enough projectives, so that $\mathrm{Tot}(A \otimes_R -)$ has a total left derived functor.

**Definition 2.3.4.** In the above notation, the *derived tensor product* $A \otimes_R^{\mathbb{L}} B$ is defined as

$$A \otimes_R^{\mathbb{L}} B := L^-(\mathrm{Tot}(A \otimes_R -))(B).$$

**Remark 15.** By [Wei94, Exercise 10.6.1], $L^-(\mathrm{Tot}(A \otimes_R -))(B) \cong L^-(\mathrm{Tot}(- \otimes_R B))(A)$.

We list the properties of the derived tensor product that we will need in this thesis.

**Lemma 2.3.5.** *If $A, A'$ and $B$ are bounded below chain complexes and $A \to A'$ is a quasi-isomorphism, then $A \otimes_R^{\mathbb{L}} B \cong A' \otimes_R^{\mathbb{L}} B$.*

*Proof.* See [Wei94, Lemma 10.6.2]. □

**Lemma 2.3.6** (Shapiro's lemma)**.** *Let $1 \to N \to G \to G/N \to 1$ be a short exact sequence of groups. Let $M$ be a left $G$-module. Then,*

$$\mathbb{Z}[G/N] \otimes_G^{\mathbb{L}} M \cong \mathbb{Z} \otimes_N^{\mathbb{L}} M.$$

*Proof.* Note that

$$\begin{aligned}\mathbb{Z}[G/N] &\cong \mathbb{Z} \otimes_N \mathbb{Z}G \\ &\cong \mathbb{Z} \otimes_N^{\mathbb{L}} \mathbb{Z}G,\end{aligned}$$

where the second isomorphism uses the fact that $\mathbb{Z}G$ is a free $N$-module (combine lemma 2.3.5 and Theorem 2.3.3).

Therefore,

$$\begin{aligned}\mathbb{Z}[G/N] \otimes_G^{\mathbb{L}} M &\cong (\mathbb{Z} \otimes_N^{\mathbb{L}} \mathbb{Z}G) \otimes_G^{\mathbb{L}} M \\ &\cong \mathbb{Z} \otimes_N^{\mathbb{L}} (\mathbb{Z}G \otimes_G^{\mathbb{L}} M) \\ &\cong \mathbb{Z} \otimes_N^{\mathbb{L}} M,\end{aligned}$$

where the last two isomorphisms follow from the corresponding isomorphisms at the level of usual tensor products. □

## 2.4 Clifford algebras and spin groups

This section is needed to prove Homological stability for $EO_{n,n}$ and $\mathrm{Spin}_{n,n}$. Although we do not claim any material in this section is new, it is perhaps the first time anybody needed to prove theorem 2.4.21 in the case of *local rings*. In the author's opinion, this theorem is an important structural result in the context of orthogonal groups, so may be of independent interest. This section can be safely skipped until one needs to read chapter 5.

### 2.4.1 Definitions, existence and basic properties

To begin with, let $M = (M, q)$ be a non-singular quadratic module over a commutative ring $R$, which for the purposes of this thesis, is such that $2 \in R^*$. We will call an element $x \in M$ *anisotropic* if $q(x) \in R^*$. We define $b(x, y) = b_q(x, y) := \frac{1}{2}(q(x+y) - q(x) - q(y))$ to be the symmetric bilinear form associated to $q$. We will say that $x, y \in M$ are *orthogonal* if $b(x, y) = 0$.

**Definition 2.4.1.** A pair $(A, f)$ consisting of an $R$-algebra $A$ and a homomorphism of $R$-modules $f : M \to A$ is said to be *compatible* with $M$ if for every $x \in M$,

$$f(x)^2 = q(x)1_A.$$

**Definition 2.4.2.** A Clifford algebra of $M$ is a compatible pair $(\mathrm{Cl}(M), i)$ which satisfies the following universal property:

If $(A, f)$ is any pair which is compatible with $M$, then there exists a unique homomorphism of $R$-algebras $g : \mathrm{Cl}(M) \to A$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{i} & \mathrm{Cl}(M) \\ & {\scriptstyle f}\searrow & \big\downarrow {\scriptstyle g} \\ & & A \end{array}$$

commutes.

We establish that any quadratic module $M$ has a Clifford algebra:

**Theorem 2.4.3.** *Let $M$ be a quadratic module over $R$. Then, $M$ has a Clifford algebra* $(\mathrm{Cl}(M), i)$*, which is unique up to unique isomorphism.*

*Proof.* The uniqueness statement follows from the universal property of the Clifford algebra, so it suffices to prove existence. We define

$$\begin{aligned} M^{\otimes n} &:= M \otimes_R \cdots \otimes_R M \quad (n \text{ times}) \quad \text{for } n > 0, \\ M^{\otimes 0} &:= R, \\ M^{\otimes n} &:= 0 \quad \text{for } n < 0, \end{aligned}$$

and define

$$T(M) := \bigoplus_{n \in \mathbb{Z}} M^{\otimes n},$$

the tensor algebra of $M$. Let $i : M \to T(M)$ denote the inclusion.

Note that the tensor algebra $T(M)$ is a $\mathbb{Z}$-graded $R$-algebra, with product

$$(x_1 \otimes \cdots \otimes x_m)(x_{m+1} \otimes \cdots \otimes x_n) := (x_1 \otimes \cdots \otimes x_m \otimes x_{m+1} \otimes \cdots \otimes x_n).$$

Also note that $T(M)$ has the following universal property: If $A$ is a $R$-algebra and $f : M \to A$ is a $R$-module homomorphism, then there exists a unique $R$-algebra homomorphism $g : T(M) \to A$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{i} & T(M) \\ & {\scriptstyle f}\searrow & \big\downarrow {\scriptstyle g} \\ & & A \end{array}$$

commutes. Of course, $g$ is defined by

$$g(x_1 \otimes \cdots \otimes x_n) := f(x_1) \dots f(x_n).$$

Continuing with the construction, define $I(q)$ to be the two-sided ideal of $T(M)$ generated by the set

$$\{x \otimes x - q(x) | x \in M\}.$$

We then define the quotient $R$-algebra

$$\mathrm{Cl}(M) := T(M)/I(q),$$

and define $i : M \to \mathrm{Cl}(M)$ to be the canonical map. By construction, it is clear that $(\mathrm{Cl}(M), i)$ is a compatible pair, so it remains to check the universal property.

Let $(A, f)$ be a pair compatible with $M$. By the universal property of $T(M)$, there exists an unique $R$-algebra homomorphism $g : T(M) \to A$ such that $gi = f$. Furthermore, note that

$$g(x \otimes x - q(x)) = g(x)^2 - q(x) = f(x)^2 - q(x) = q(x) - q(x) = 0.$$

Thus, $g$ factors through the quotient, to give a map $g : \mathrm{Cl}(M) \to A$. □

**Remark 16.** If $x, y \in M$ are orthogonal, then in $\mathrm{Cl}(M)$, $xy = -yx$, as $0 = b(x, y) = q(x + y) - q(x) - q(y) = (x + y)^2 - x^2 - y^2 = xy + yx$.

**Remark 17.** The identity of $\mathrm{Cl}(M)$, denoted $1_{\mathrm{Cl}(M)}$, together with the elements $\{i(x)|x \in M\}$, generate $\mathrm{Cl}(M)$ as an $R$-algebra.

**Remark 18.** The Clifford algebra $\mathrm{Cl}(M)$ is canonically a $\mathbb{Z}_2$-graded algebra, with the grading defined as follows: We define $\mathrm{Cl}_0(M)$ be the submodule of $\mathrm{Cl}(M)$ spanned by $1_{\mathrm{Cl}(M)}$ and $\{i(x_{i_1}) \dots i(x_{i_k})|k \text{ even}\}$; and we define $\mathrm{Cl}_1(M)$ be the submodule of $\mathrm{Cl}(M)$ spanned by $\{i(x_{i_1}) \dots i(x_{i_k})|k \text{ odd}\}$. Clearly, $\mathrm{Cl}_0(M)$ is a subalgebra of $\mathrm{Cl}(M)$.

**Remark 19.** Consider the graded centre $Z_{gr}(\mathrm{Cl}(M))$ of the Clifford algebra $\mathrm{Cl}(M)$. This is defined to be the graded subspace of the Clifford algebra $\mathrm{Cl}(M)$ whose homogeneous elements $h(Z_{gr}(\mathrm{Cl}(M)))$ are determined by

$$c \in h(Z_{gr}(\mathrm{Cl}(M))) \iff cs = -(1)^{\partial s \partial c} sc \quad \forall s \in h(\mathrm{Cl}(M)),$$

where $h(\mathrm{Cl}(M))$ denotes the homogeneous elements of $\mathrm{Cl}(M)$ and $\partial$ denotes the degree of the homogeneous element.

When $M$ is free of finite rank, we cite the following important structural result:

**Lemma 2.4.4.** *Let $M$ be a free non-singular quadratic module of finite rank. Then*

$$Z_{gr}(\mathrm{Cl}(M)) = R.$$

*Proof.* See [HO89, Theorem 7.1.11.]. □

**Remark 20.** By the universal property of the Clifford algebra, every $\sigma \in O(M)$ uniquely determines an automorphism of $R$-algebras $\mathrm{Cl}(\sigma) : \mathrm{Cl}(M) \to \mathrm{Cl}(M)$. This association gives rise to a group homomorphism

$$\mathrm{Cl} : O(M) \to \mathrm{Aut}(\mathrm{Cl}(M)).$$

In particular, taking $\sigma := -1_M$ provides a unique automorphism $\mathrm{Cl}(-1_M) : \mathrm{Cl}(M) \to \mathrm{Cl}(M)$ such that $\mathrm{Cl}(-1_M)(i(x)) = -i(x)$ for all $x \in M$. Observe that $\mathrm{Cl}(-1_M)|_{\mathrm{Cl}_0(M)} = 1_{\mathrm{Cl}_0(M)}$ and $\mathrm{Cl}(-1_M)|_{\mathrm{Cl}_1(M)} = -1_{\mathrm{Cl}_1(M)}$.

The map $\mathrm{Cl}(-1_M)$ is used to define the so called 'canonical involution' $^-$ on $\mathrm{Cl}(M)$.

But first, let $\mathrm{Cl}(M)^{op}$ denote the opposite algebra of $\mathrm{Cl}(M)$. By the universal property of the Clifford algebra, there exists an unique algebra homomorphism $\sim$: $\mathrm{Cl}(M) \to \mathrm{Cl}(M)^{op}$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ i\ } & \mathrm{Cl}(M) \\ & {}_{i}\searrow & \Big\downarrow{\scriptstyle \sim} \\ & & \mathrm{Cl}(M)^{op} \end{array}$$

commutes. We will consider $\sim$ as a map $\sim$: $\mathrm{Cl}(M) \to \mathrm{Cl}(M)$. Note that $\widetilde{cd} = \tilde{d}\tilde{c}$ for every $c, d \in \mathrm{Cl}(M)$, and $\widetilde{i(x)} = i(x)$ for every $x \in M$, so that $\tilde{\tilde{c}} = c$ for every $c \in \mathrm{Cl}(M)$ and $\sim$ is therefore an involution on $\mathrm{Cl}(M)$.

We then define the *canonical involution* $^- : \mathrm{Cl}(M) \to \mathrm{Cl}(M)$ to be the composite

$$\mathrm{Cl}(M) \xrightarrow{\mathrm{Cl}(-1)} \mathrm{Cl}(M) \xrightarrow{\sim} \mathrm{Cl}(M).$$

One easily checks that this does indeed define an involution on $\mathrm{Cl}(M)$. Observe that $^-$ is the unique $R$-linear anti-automorphism of $\mathrm{Cl}(M)$ which satisfies $\overline{i(x)} = -i(x)$

for every $x \in M$. We will use the canonical involution in our definition of the Spin group.

### 2.4.2 The groups $\Gamma(M), S\Gamma(M)$, $\mathrm{Spin}(M)$ and the Spinor Norm

We define the groups $\Gamma(M), S\Gamma(M)$ and $\mathrm{Spin}(M)$. We also define the Spinor norm and study some of its basic properties, as needed in this thesis. Unless stated otherwise, our exposition will closely follow [HO89, Chapter 7].

#### 2.4.2.1 The Groups $\Gamma(M), S\Gamma(M)$, $\mathrm{Spin}(M)$

**Definition 2.4.5.** We define the *Clifford group* $\Gamma(M)$ to be the group

$$\Gamma(M) := \{c \in \mathrm{Cl}(M)^* | cMc^{-1} = M\}.$$

Note that for every $c \in \Gamma(M)$, we canonically obtain a map

$$\pi c : M \to M$$
$$(\pi c)(x) := cxc^{-1}.$$

Furthermore, note that $\pi c$ preserves the quadratic form $q$ as $q(\pi c(x)) = q(cxc^{-1}) = cxc^{-1} \otimes cxc^{-1} = cx^2c^{-1} = q(x)$. Thus, the assignment $c \mapsto \pi c$ defines a group homomorphism

$$\pi : \Gamma(M) \to O(M).$$

**Definition 2.4.6.** We define the *Special Clifford group* $S\Gamma(M)$ to be the group

$$S\Gamma(M) := \{c \in \mathrm{Cl}(M)_0^* | cMc^{-1} = M\}.$$

Note that $S\Gamma(M) = \Gamma(M) \cap \mathrm{Cl}_0^*$.

**Definition 2.4.7.** We define the *Spin group* $\mathrm{Spin}(M)$ to be the group

$$\mathrm{Spin}(M) := \{c \in S\Gamma(M) | c\overline{c} = 1\}.$$

Thus, by construction, we have a chain of inclusions $\mathrm{Spin}(M) \subseteq S\Gamma(M) \subseteq \Gamma(M)$.

Later, it will be important for us to understand $\ker(\pi|_{S\Gamma(M)})$ and $\ker(\pi|_{\mathrm{Spin}(M)})$.

**Proposition 2.4.8.** $\ker(\pi : S\Gamma(M) \to O(M)) = R^*$.

*Proof.* Let $c \in \ker(\pi|_{S\Gamma(M)})$. Then, $c \in \mathrm{Cl}_0^*(M)$ and $cxc^{-1} = x$ for every $x \in M$. Therefore, as $M$ generates $\mathrm{Cl}(M)$ as an $R$-algebra, we use lemma 2.4.4 to conclude that $c \in Z_{gr}(\mathrm{Cl}(M)) = R$. Similarly, $c^{-1} \in R$, so that $\ker(\pi|_{S\Gamma(M)}) \subseteq R^*$. The other inclusion is trivial. □

**Corollary 2.4.9.** $\ker(\pi : \mathrm{Spin}(M) \to O(M)) \cong \mathbb{Z}_2$.

*Proof.* From proposition 2.4.8, it is clear that

$$\ker(\pi : \mathrm{Spin}(M) \to O(M)) = \{r \in R^* | r^2 = 1\}.$$

Passing to the residue field, we deduce that the square roots of 1 are of the form $r = \varepsilon \pm 1$ for some $\varepsilon$ in the maximal ideal. Using the equation $r^2 = 1$, we obtain equation $\varepsilon(\varepsilon \pm 2) = 0$. As 2 is a unit, we deduce $\varepsilon \pm 2$ is a unit, so that $\varepsilon = 0$. □

**Definition 2.4.10.** We define the *spinorial kernel*

$$O'(M)$$

to be the image of the homomorphism $\pi : \mathrm{Spin}(M) \to O(M)$.

When $R$ is a local ring with $2 \in R^*$, we show that $O'(M)$ is precisely the kernel of the *spinor map* $\theta : SO(M) \to R^*/R^{*2}$, see definition 2.4.16 and proposition 2.4.20.

In addition, we cite the following theorem, which says that when $R^{2n}$ is a free hyperbolic module over a (semi-)local ring $R$, the spinorial kernel is *precisely* the elementary orthogonal group $EO_{n,n}(R)$ when $n \geq 2$.

**Theorem 2.4.11.** *Let $R$ be a commutative semi-local ring. Let $R^{2n}$ be the free hyperbolic module. Denote $O'_{n,n}(R) := O'(R^{2n})$. Then, for every $n \geq 2$, $O'_{n,n}(R) = EO_{n,n}(R)$.*

*Proof.* See [HO89, Theorem 9.2.8.]. □

Thus, when $R$ is a (semi-)local ring with $2 \in R^*$ and $n \geq 2$, we have the short exact sequences

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_{n,n}(R) \xrightarrow{\pi} EO_{n,n}(R) \to 1.$$

*From now on, we will assume that $R$ is a local ring with $2 \in R^*$, and all modules over $R$ are finitely generated projective, so that they are free of finite rank.*

#### 2.4.2.2 The Spinor Norm

In order to define the spinor norm, we first need to define an important class of isometries.

**Definition 2.4.12.** Let $x \in M$ anisotropic and define $N := \langle x \rangle^{\perp}$. Then, the linear map

$$\begin{aligned} \tau_x : M &\to M \\ y &\mapsto y - 2\frac{b(x,y)}{b(x,x)}x \end{aligned}$$

is called a *reflection* in hyperplane $N$ orthogonal to $x$.

This name is suggested by the following lemma:

**Lemma 2.4.13.** *1.* $\tau_x(x) = -x$, $\tau_x|_N = 1_N$.

*2.* $\tau_x$ *is an isometry of* $(M, b)$.

*3.* $\tau_x \circ \tau_x = 1_M$.

*4.* $\det \tau_x = -1$.

*Proof.* The first three statements follow from direct computations. For the last statement, note that $M = \langle x \rangle \oplus N$. Therefore, by Witt's Cancellation Theorem [MH73, Chapter I, Theorem 4.4], we may choose a basis $e_2, \ldots, e_n$ of $N$ and complete it to a basis of $M$ by setting $e_1 = x$. The matrix of $\tau_x$ with respect to this basis shows that $\det \tau_x = -1$. □

**Proposition 2.4.14.** *For every* $x \in M$ *anisotropic, we have* $\pi(x) = -\tau_x$.

*Proof.* For every $y \in M$, we have

$$\begin{aligned} \tau_x(y) &= y - 2\frac{b(x,y)}{b(x,x)}x \\ &= y - \frac{q(x+y) - q(x) - q(y)}{q(x)}x \\ &= y - (xy + yx)x^{-1} \\ &= -xyx^{-1} \\ &= -\pi(x)(y). \end{aligned}$$

□

The next proposition will be used to show that our definition of the spinor norm is well-defined.

**Proposition 2.4.15.** *Let $u_1, \ldots, u_r$ be anisotropic elements in $M$. If the product $\tau_{u_1}\tau_{u_2}\cdots\tau_{u_r}$ is the identity in $O(M)$, then the product $q(u_1)\cdots q(u_r)$ belongs to $R^{*2}$.*

*Proof.* Similar to [Lam05, Proposition 1.12.V]. By proposition 2.4.14, $\pi(u)|_M = -\tau_u$, so that $1_M = (-1)^r\pi(u_1\cdots u_r)|_M$. But,

$$(-1)^r = \det(\tau_{u_1}\tau_{u_2}\cdots\tau_{u_r}) = \det(1_M) = 1.$$

Thus, we deduce that $r$ must be even. Therefore, we have that

$$c := u_1\cdots u_r \in \mathrm{Cl}_0(M) \cap Z(\mathrm{Cl}(M)) \subset Z_{gr}(\mathrm{Cl}(M)) = R.$$

Similarly, we have that $c^{-1} \in R$, so that $c \in R^*$. We conclude that

$$R^{*2} \ni c^2 = c\bar{c} = u_1\cdots u_r u_r\cdots u_1 = q(u_1)\cdots q(u_r).$$

□

Consider any isometry $\sigma \in O(M)$, where the rank of $M$ is at least 2. By the Cartan-Dieudonné theorem for local rings, see for example [Kli61, Theorem 2], there exists a factorisation $\sigma = \tau_{u_1}\tau_{u_2}\cdots\tau_{u_r}$, where the $u_i$ are anisotropic vectors. We define

$$\theta(\sigma) := q(u_1)\cdots q(u_r) \in R^*/R^{*2}.$$

By proposition 2.4.15, $\theta(\sigma)$ does not depend on the choice of factorisation chosen to represent $\sigma$.

**Definition 2.4.16.** The map $\theta : O(M) \to R^*/R^{*2}$ is called the *spinor norm.*

The spinor norm is the unique group homomorphism satisfying the property $\theta(\tau_u) = q(u)R^{*2}$ for every anisotropic element $u \in M$.

For $R$ a local ring with $2 \in R^*$, we want to establish the existence of short exact sequences

$$1 \to EO_{n,n}(R) \to SO_{n,n}(R) \xrightarrow{\theta} R^*/R^{*2} \to 1$$
$$1 \to EO_{n,n}(R) \to O_{n,n}(R) \xrightarrow{\theta\times\det} R^*/R^{*2} \times \mathbb{Z}_2 \to 1.$$

We begin with the following proposition, which is useful when computing with the Spinor norm.

**Proposition 2.4.17.** *Let $(M, q_M)$ and $(N, q_N)$ be free non-singular quadratic modules of finite rank over $R$. Let $A \in O(M)$ and let $B \in O(N)$, considered as matrices. Let $A \oplus B \in O(M \perp N)$ denote the block sum of matrices $A \oplus B = \begin{pmatrix} A & \\ & B \end{pmatrix}$. Then, $\theta(A \oplus B) = \theta(A)\theta(B)$.*

*Proof.* Suppose $A \in O(M)$ is represented by $A = \tau_{v_1} \cdots \tau_{v_k}$ and suppose $B \in O(N)$ is represented by $B = \tau_{w_1} \cdots \tau_{w_l}$. Then, $A \oplus B \in O(M \perp N)$ is represented by $\tau_{\bar{v}_1} \cdots \tau_{\bar{v}_k} \tau_{\bar{w}_1} \cdots \tau_{\bar{w}_l}$, where $\bar{v}_i, \bar{w}_j \in M \perp N$ are the images of of the vectors $v_i$ and $w_j$ under the canonical embeddings $M \hookrightarrow M \perp N$ and $N \hookrightarrow M \perp N$ respectively.

Therefore,

$$\begin{aligned}\theta(A \oplus B) &= q_{M\perp N}(\bar{v}_1) \cdots q_{M\perp N}(\bar{v}_k) q_{M\perp N}(\bar{w}_1) \cdots q_{M\perp N}(\bar{w}_l) \\ &= q_M(v_1) \cdots q_M(v_1) q_N(w_1) \cdots q_N(w_l) \\ &= \theta(A)\theta(B).\end{aligned}$$

□

The above proposition is used to prove that the spinor norm $\theta : SO_{n,n}(R) \to R^*/R^{*2}$ is *surjective.*

**Proposition 2.4.18.** *The spinor norm $\theta : SO_{n,n}(R) \to R^*/R^{*2}$ is surjective.*

*Proof.* Let $r \in R^*$, and consider the matrix $\sigma = \begin{pmatrix} r & & \\ & r^{-1} & \\ & & 1 \end{pmatrix}$. Note that $\sigma \in SO_{n,n}(R)$. By proposition 2.4.17, we have that $\theta(\sigma) = \theta\left(\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}\right)$. As $\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} = \begin{pmatrix} 0 & r \\ r^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a product of reflections defined by vectors $(r, -1)$ and $(1, -1)$, we compute that

$$\theta\left(\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}\right) = q(r, -1)q(1, -1) = 4r = r \pmod{R^{*2}}.$$

□

Finally, we want to show that the spinorial kernel $O'(M)$ is precisely the kernel of the spinor map $\theta : SO(M) \to R^*/R^{*2}$. This is done in by the following two propositions.

**Proposition 2.4.19.** $\mathrm{Im}(\pi : S\Gamma(M) \to O(M)) = SO(M)$.

*Proof.* Firstly, note that $SO(M) \subseteq \mathrm{Im}(\pi)$. Indeed, if $\tau_v \tau_w$ are a product of any two reflections, then $\pi(vw) = \tau_v \tau_w$.

Suppose that $SO(M) \subsetneq \mathrm{Im}(\pi)$. Then, there exists a $\sigma \in O(M) \setminus SO(M)$ such that $\sigma \in \mathrm{Im}(\pi)$. As $\sigma \in O(M) \setminus SO(M)$, we have that $\sigma = \tau_{v_1} \cdots \tau_{v_k}$ for $v_i$ anisotropic and $k$ odd. Furthermore, as $\sigma \in \mathrm{Im}(\pi)$, there exists $c \in S\Gamma(M)$ such that $\pi(c) = \sigma$. Note, $\pi(v_1 \cdots v_k) = -\tau_{v_1} \cdots \tau_{v_k} = -\sigma$. Defining $d := v_1 \cdots v_k$, we deduce $\pi(cd^{-1}) = -1_M$. This means that $cd^{-1}x(cd^{-1})^{-1} = -x$ for every $x \in M$.

Note that $c \in \mathrm{Cl}_0(M)$ and $d^{-1} \in \mathrm{Cl}_1(M)$. Therefore, $cd^{-1} \in \mathrm{Cl}_1(M)$. As $cd^{-1}x(cd^{-1})^{-1} = -x$ for every $x \in M$ and $M$ generates $\mathrm{Cl}(M)$ as an $R$-algebra, we use lemma 2.4.4 to conclude that $cd^{-1} \in Z_{gr}(\mathrm{Cl}(M)) = R$. Thus, $c = dr$ for some $r \in R$, and it therefore follows that $c \in \mathrm{Cl}_1(M)$. Thus, $c \in \mathrm{Cl}_0(M) \cap \mathrm{Cl}_1(M) = 0$, which is a contradiction as $c$ is invertible. □

**Proposition 2.4.20.** *We have* $O'(M) = \ker(\theta : SO(M) \to R^*/R^{*2})$.

*Proof.* Let $\sigma \in \ker(\theta|_{SO(M)})$. We want to show $\sigma \in O'_{n,n}(M)$. Suppose $\sigma = \tau_{v_1} \cdots \tau_{v_k}$. Note that $k$ is even and each $v_i$ is anisotropic.

By definition, $1 = \theta(\sigma) = q(v_1) \cdots q(v_k)$. Therefore, $r := q(v_1) \cdots q(v_k) \in R^{*2}$. Suppose that $r = s^2$. As $\tau_{v_1} = \tau_{s^{-1}v_1}$, we may replace $v_1$ with $s^{-1}v_1$ to obtain $\sigma = \tau_{v_1} \cdots \tau_{v_k}$ such that $q(v_1) \cdots q(v_k) = 1$. Therefore, in $\mathrm{Cl}(M)$, $v_1 \cdots v_k \overline{v_1 \cdots v_k} = 1$. Define $c := v_1 \dots v_k$. As all $v_i \in \mathrm{Cl}(M)^*$ and $k$ is even, we deduce $c \in \mathrm{Spin}(M)$. By construction, $\pi(c) = \tau_{v_1} \cdots \tau_{v_k} = \sigma$, so that $\sigma \in O'(M)$.

Now let $c \in \mathrm{Spin}(M)$ and consider $\pi(c) \in O'(M)$. We want to show $\pi(c) \in \ker(\theta|_{SO(M)})$. By proposition 2.4.19, $\pi(c) \in SO(M)$. Therefore, $\pi(c) = \tau_{v_1} \cdots \tau_{v_k}$ for $v_i$ anisotropic and $k$ even. As $k$ is even, we deduce $c^{-1}v_1 \cdots v_k \in S\Gamma(M)$. Furthermore, by definition, $\pi(c^{-1}v_1 \cdots v_k) = 1$. Therefore, by proposition 2.4.8 $c^{-1}v_1 \dots v_k \in \ker(\pi|_{S\Gamma}) = R^*$. Thus, $c = rv_1 \cdots v_k$ for some $r \in R^*$. As $c \in \mathrm{Spin}(M)$, we obtain $1 = c\bar{c} = r^2 q(v_1) \cdots q(v_k)$, so that $q(v_1) \cdots q(v_k) \in R^{*2}$. Thus, by definition, $\theta(\pi(c)) = 1$. □

Thus, for $R$ a local ring with $2 \in R^*$, we have established the following theorem:

**Theorem 2.4.21.** *Let $R$ be a commutative local ring with $2 \in R^*$, and let $n \geq 2$. Then, we have short exact sequences*

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_{n,n}(R) \xrightarrow{\pi} EO_{n,n}(R) \to 1$$
$$1 \to EO_{n,n}(R) \to SO_{n,n}(R) \xrightarrow{\theta} R^*/R^{*2} \to 1$$
$$1 \to EO_{n,n}(R) \to O_{n,n}(R) \xrightarrow{\theta \times \det} R^*/R^{*2} \times \mathbb{Z}_2 \to 1.$$

*Proof.* Combine corollary 2.4.9; theorem 2.4.11; proposition 2.4.20 and proposition 2.4.18. □

These short exact sequences are used to prove homological stability for $EO_{n,n}(R)$ and $\mathrm{Spin}_{n,n}(R)$, when $R$ is a local ring with infinite residue field such that $2 \in R^*$.

#### 2.4.2.3 Homological stability for $\mathrm{Spin}_n(\mathbb{F})$

As a fun aside, we can use the spinor norm to immediately prove a homological stability result for $\mathrm{Spin}_n(\mathbb{F})$, using a homological stability result for $SO_n(\mathbb{F})$ due to Nakada [Nak15]. Here, $\mathbb{F}$ is a Pythagorean field of characteristic $\neq 2$, and the quadratic form is the Euclidean form $q(x) = \sum_i^n x_i^2$. (A Pythagorean field is a field where the sum of two squares is always a square, for example the real numbers.)

**Theorem 2.4.22.** *Let $\mathbb{F}$ be a Pythagorean field of characteristic $\neq 2$. Then, the natural homomorphism*

$$H_k(\mathrm{Spin}_n(\mathbb{F})) \to H_k(\mathrm{Spin}_{n+1}(\mathbb{F}))$$

*is an isomorphism for $2k \leq n-1$ and surjective for $2k \leq n$.*

*Proof.* We want to prove

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_n(\mathbb{F}) \xrightarrow{\pi} SO_n(\mathbb{F}) \to 1$$

is a short exact sequence. The theorem then immediately follows from the relative Serre spectral sequence

$$E^2_{p,q} = H_p(SO_n(\mathbb{F}), SO_{n-1}(\mathbb{F}); H_q(\mathbb{Z}_2)) \Rightarrow H_{p+q}(\mathrm{Spin}_n(\mathbb{F}), \mathrm{Spin}_{n-1}(\mathbb{F})),$$

and Nakada's result [Nak15]. What needs to be proved is $O_n'(\mathbb{F}) = SO_n(\mathbb{F})$.

By proposition 2.4.20, $O_n'(\mathbb{F}) = \ker(\theta : SO_n(\mathbb{F}) \to \mathbb{F}^*/\mathbb{F}^{2*})$, so that it suffices to prove the spinor norm in this case is the zero map.

For $\sigma = \tau_{v_1} \cdots \tau_{v_k} \in SO_n(\mathbb{F})$, $\theta(\sigma) = q(v_1) \cdots q(v_k)$. But, $\mathbb{F}$ is *Pythagorean* and $q$ is the *Euclidean form*, so that $q(v_i) \in \mathbb{F}^{2*}$ for every $v_i$. Thus, $\theta(\sigma) = 0$. $\square$

# Chapter 3

# Homological Stability for $O_{n,n}$

From now on, unless stated otherwise, $R$ will be a commutative local ring with infinite residue field such that 2 invertible.

## 3.1 The complex of totally isotropic unimodular sequences

### 3.1.1 Notation and conventions

Let $M_n(R)$ denote the collection of $n \times n$ matrices with coefficients in $R$. Let $GL_n(R) \subseteq M_n(R)$ denote group of all invertible $n \times n$ matrices in $M_n(R)$.

Let $\psi := \psi_{2n} := \psi_2 \oplus \cdots \oplus \psi_2$ be the standard hyperbolic form of rank $2n$

$$\psi_{2n} = \begin{pmatrix} \psi_2 & & & \\ & \psi_2 & & \\ & & \ddots & \\ & & & \psi_2 \end{pmatrix} = \bigoplus_1^n \psi_2, \quad \psi_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For $u, v \in R^{2n}$, we will always use the notation

$$\langle u, v \rangle := {}^t u \psi_{2n} v$$

to denote the inner product of $u$ and $v$ with respect to form $\psi_{2n}$. Note that the form $\psi_{2n}$ defines a *symmetric bilinear form*, so that $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in R^{2n}$. In addition, we will always denote the ordered standard basis vectors of $R^{2n}$ as $e_1, f_1, \ldots, e_n, f_n$, so that $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$ and $\langle e_i, f_j \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$. For instance, $e_1 = (1, 0, 0, ...0)$, $f_1 = (0, 1, 0, 0, ..., 0)$, $e_2 = (0, 0, 1, 0, ..., 0)$, $f_2 = (0, 0, 0, 1, 0, ..., 0)$ *etc.* considered as column vectors.

For a ring $R$, the *(split) orthogonal group* $O_{n,n}(R) \subseteq GL_{2n}(R)$, is the subgroup

$$O_{n,n}(R) := \{A \in GL_{2n}(R) |\ {}^t A \psi_{2n} A = \psi_{2n}\}$$

of $R$-linear automorphisms preserving the form $\psi_{2n}$, where ${}^tA$ denotes the transpose matrix of $A$. We define $SO_{n,n}(R)$ to be the subgroup of $O_{n,n}(R)$ consisting of all matrices with determinant 1. We define $EO_{n,n}(R)$ as the subgroup of $O_{n,n}(R)$ generated by matrices of the form (1.1),(1.2), (1.3) and (1.4). For $R$ commutative ring such that $2 \in R^*$, we define $\mathrm{Spin}_{n,n}(R)$ to be the Spin group of quadratic module $(R^{2n}, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is the symmetric bilinear form defined with respect to matrix $\psi_{2n}$. We will use the convention $\mathrm{Spin}_{0,0}(R) = EO_{0,0}(R) = SO_{0,0}(R) = O_{0,0}(R) := 1$ are the trivial groups. We will always consider $R^{2n}$ embedded in $R^{2n+2}$ via $v \mapsto (0, 0, v)$ and $O_{n,n}(R)$ as a subgroup of $O_{n+1,n+1}(R)$ via the embedding

$$O_{n,n}(R) \subseteq O_{n+1,n+1}(R) : A \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A \end{pmatrix}.$$

We will be interested studying the homological stability of $O_{n,n}$, $SO_{n,n}$, $EO_{n,n}$ and $\mathrm{Spin}_{n,n}$ with respect to the above embeddings. We will sometimes write $O_{n,n}$, $SO_{n,n}$, $EO_{n,n}$ and $\mathrm{Spin}_{n,n}$ instead of $O_{n,n}(R)$, $SO_{n,n}(R)$, $EO_{n,n}(R)$ and $\mathrm{Spin}_{n,n}(R)$ when the ring $R$ is understood from context.

### 3.1.2 The chain complex

To define the chain complex we want to consider, we need to make some preliminary definitions.

**Definition 3.1.1.** A *space* over a ring $R$ is a projective $R$-module of finite rank. A submodule $M \subset V$ of a space $V$ is called a subspace if it is a direct factor.

**Definition 3.1.2.** Let $q \geq 0$ be an integer, and $W$ a free $R$-module of rank $n$. A sequence of $q$ vectors $(v_1, \dots, v_q)$ in $W$ will be called *unimodular* if every subsequence of length $r \leq \min\{n, q\}$ generates a subspace of rank $r$. We denote by $\mathcal{U}_q(W)$ the set of unimodular sequences of length $q$ in $W$.

**Remark 21.** For $R$ a local ring, the sequence of vectors $(v_1, \dots, v_q)$ in $R^{2n}$ is unimodular if and only if $(\bar{v}_1, \dots, \bar{v}_q)$ in $k^{2n}$ is unimodular, where $k$ denotes the residue field of $R$ and $\bar{v}_i$ the class of $v_i$ in $k^{2n}$.

**Definition 3.1.3.** A sequence of vectors $(v_1, \dots, v_q)$ in $R^{2n}$ will be called *totally isotropic* if for every $i, j = 1, ...q$ we have $\langle v_i, v_j \rangle = 0$.

We now introduce the chain complex that we want to consider. Specifically, consider chain complex

$$C_*(n) := (C_*(R^{2n}), d) = \cdots \to C_2(R^{2n}) \to C_1(R^{2n}) \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \tag{3.1}$$

where for $k \geq 1$, $C_k(R^{2n})$ is defined as the free abelian group $C_k(R^{2n}) := \mathbb{Z}[\mathcal{IU}_k(R^{2n})]$ generated by the set of unimodular totally isotropic sequences of length $k$ in $R^{2n}$:

$$\mathcal{IU}_k(R^{2n}) := \{(v_1, \ldots, v_k) : v_i \in R^{2n}, (v_1, \ldots, v_k) \text{ totally isotropic and unimodular}\}.$$

We set $C_0(n) := \mathbb{Z}$.

The differential $d$ is defined on basis elements by

$$d(v_1, \ldots, v_k) := \sum_{i=1}^{k} (-1)^{i+1} d_i(v_1, \ldots, v_k),$$

$$d_i(v_1, \ldots, v_k) := (v_1, \ldots, \hat{v}_i, \ldots, v_k).$$

**Remark 22.** The simplicial set that gives rise to chain complex (3.1) has already been studied in [Pan87] and [Mir04]. As we do not need to consider simplicial sets in this article, we stick to chain complex notation.

Note that for $A \in O_{n,n}(R)$, $A$ acts from the left on the chain complex $(C_*(R^{2n}), d)$ by acting on basis elements:

$$A \cdot (v_1, \ldots, v_k) := (Av_1, \ldots, Av_k).$$

For a resolution $P_*$ of the trivial $O_{n,n}(R)$-module $\mathbb{Z}$ by projective right $O_{n,n}(R)$-modules, the bicomplex $P_* \otimes_{O_{n,n}} C_*(n)$ gives rise to two hyperhomology spectral sequences

$$E^2_{p,q}(n) = H_p(O_{n,n}, H_q(C_*(n))) \Rightarrow H_{p+q}(O_{n,n}, C_*(n)) \tag{3.2}$$
$$E^1_{p,q}(n) = H_q(O_{n,n}, C_p(n)) \Rightarrow H_{p+q}(O_{n,n}, C_*(n)). \tag{3.3}$$

Replacing $O_{n,n}$ with $SO_{n,n}$ and $EO_{n,n}$, we similarly obtain hyperhomology spectral sequences

$$E^2_{p,q}(n) = H_p(SO_{n,n}, H_q(C_*(n))) \Rightarrow H_{p+q}(SO_{n,n}, C_*(n))$$
$$E^1_{p,q}(n) = H_q(SO_{n,n}, C_p(n)) \Rightarrow H_{p+q}(SO_{n,n}, C_*(n))$$

and

$$E^2_{p,q}(n) = H_p(EO_{n,n}, H_q(C_*(n))) \Rightarrow H_{p+q}(EO_{n,n}, C_*(n))$$
$$E^1_{p,q}(n) = H_q(EO_{n,n}, C_p(n)) \Rightarrow H_{p+q}(EO_{n,n}, C_*(n)).$$

These spectral sequences will eventually give us our desired homological stability results.

### 3.1.3 Proving acyclicity

We would like to prove that the complex $(C_*(n), d)$ is $(n-1)$-acyclic. This has already been proven by Mirzaii [Mir04], but our proof has the advantage that it does not refer to the simplicial techniques used in [vdK80]. However, we do make use of a concept *general position*. This was first defined in [Pan87], and used in both [Pan87] and [Mir04] to prove their respective acyclicity results. We give the definition as stated in [Mir04].

**Definition 3.1.4.** Let $S = \{v_1, \ldots, v_k\}$ and $T = \{w_1, \ldots, w_{k'}\}$ be basis of two totally isotropic free summands of $R^{2n}$. We say that $T$ is in general position with $S$, if $k \leq k'$ and the $k' \times k$- matrix $(\langle w_i, v_j \rangle)$ has a left inverse.

We may also say that a totally isotropic subspace $W$ is in general position with respect to a totally isotropic subspace $V$ if there is a basis $T$ of $W$ which is in general position with respect to a basis $S$ of $V$ as in definition 3.1.4. The following result, whose proof we refer to [Mir04, Chapter 2, Proposition 4.2], will be used to prove acyclicity.

**Proposition 3.1.5.** *Let $n \geq 2$ be an integer and assume $T_i$, $i = 1, ..., \ell$ are finitely many subsets of $R^{2n}$ such that each $T_i$ is a basis of a free totally isotropic summand of $R^{2n}$ with $k$ elements, where $k \leq n-1$. Then, there is a basis, $T = \{w_1, \ldots, w_n\}$, of a free totally isotropic summand of $R^{2n}$ such that $T$ is in general position with all $T_i$, $i = 1, ..., \ell$. Moreover, dim$(W \cap V_i^{\perp}) = n - k$, where $W$ =Span$(T)$ and $V_i$ =Span$(T_i)$, $i = 1, ..., \ell$.*

□

In addition, the following lemma will be both useful and reassuring.

**Lemma 3.1.6.** *Let $W$ and $V$ be totally isotropic subspaces of $R^{2n}$. Assume that $W$ is in general position with respect to $V$. Then $W \cap V = \{0\}$.*

**Remark 23.** If $R$ is a local ring, lemma 3.1.6 implies that if $W$ is in general position with respect to a unimodular sequence $(u_1, \ldots, u_k)$ for $k < n$, then $(u_1, \ldots, u_k, w)$ is unimodular for every unimodular vector $w \in W$.

*Proof of lemma 3.1.6.* As $W$ is in general position with respect to $V$, the map

$$\begin{aligned} \pi : & W \twoheadrightarrow R^k \\ & w \mapsto (\langle w, v_1 \rangle, \ldots, \langle w, v_k \rangle) \end{aligned}$$

is surjective. Therefore, for every $1 \leq i \leq k$, there exists a $v_i^{\#} \in W$ such that $\pi(v_i^{\#}) = (0, \ldots, 0, 1, 0, \ldots, 0)$, the 1 being in the $i$th position. Now, let $y \in W \cap V$. Note, as $y \in V$ and $V$ is free with basis $v_1, \ldots, v_k$, we may write $y$ uniquely as $y = \sum_i a_i v_i$ for some $a_i \in R$. Evaluating $\langle v_i^{\#}, \cdot \rangle$ on $y$ and noting that $\langle v_i^{\#}, v_j \rangle = \delta_{ij}$, we deduce that $a_i = \langle v_i^{\#}, y \rangle$ for every $i = 1, ., , , k$. But $v_i^{\#}, y \in W$ and $W$ is totally isotropic, so $\langle v_i^{\#}, y \rangle = 0$ for $i = 1, ..., k$. Therefore, $y = 0$. □

For $u = \sum_i m_i u_i \in \mathbb{Z}[\mathcal{IU}_p(R^{2n})]$ and $v = \sum_j n_j v_j \in \mathbb{Z}[\mathcal{IU}_q(R^{2n})]$ such that $(u_i, v_j) \in \mathcal{IU}_{p+q}(R^{2n})$ for all $i, j$, we will write $(u, v)$ for the element

$$(u, v) = \sum_{i,j} m_i n_j (u_i, v_j) \in \mathbb{Z}[\mathcal{IU}_{p+q}(R^{2n})].$$

Using proposition 3.1.5 and lemma 3.1.6, we prove the following.

**Lemma 3.1.7.** *Let $p, q \geq 0$ and $p + q < n$. Let $(u, f) \in \mathbb{Z}[\mathcal{IU}_{p+q}(R^{2n})]$ such that $u \in \mathcal{IU}_p$ and $f \in \mathbb{Z}[\mathcal{U}_q(W)]$, where $W$ is a free totally isotropic summand of $R^{2n}$ of dimension $n$ in general position with respect to $U$ =Span$(u)$. If $df = 0 \in \mathbb{Z}[\mathcal{U}_{q-1}(W)]$, then there exists an element $g \in \mathbb{Z}[\mathcal{U}_{q+1}(W)]$ such that $dg = f$ and $(u, g) \in \mathbb{Z}[\mathcal{IU}_{p+q+1}(R^{2n})]$.*

*Proof.* Since $f \in \mathbb{Z}[\mathcal{U}_q(W)]$ and $(u, f) \in \mathbb{Z}[\mathcal{IU}_{p+q}(R^{2n})]$, we have $f \in \mathbb{Z}[\mathcal{U}_q(L)]$, where $L = W \cap U^{\perp}$. As $W$ is in general position with respect to $U$, $L$ is a finitely generated free $R$-module of rank $n - p$. Write $f = \sum_i n_i(v_1^i, \ldots, v_q^i)$. Then, as $R$ is a local ring with infinite residue field and $L$ is a finitely generated free $R$-module of rank $n - p > q$, we deduce that there exists a $v \in L$, such that $(v, v_1^i, \ldots, v_q^i) \in \mathcal{U}_q(L)$ for every $i$. This is standard; see for instance [Sch17, Lemmas 5.5 and 5.6]. Let $g := \sum_i n_i(v, v_1^i, \ldots, v_q^i)$. Then $dg = f$ by construction. Moreover as $g \in \mathbb{Z}[\mathcal{U}_{q+1}(L)]$, $(u, g)$ defines a totally isotropic sequence of vectors and as $g \in \mathbb{Z}[\mathcal{U}_{q+1}(W)]$, by lemma 3.1.6, $(u, g)$ is a unimodular sequence, so that $(u, g) \in \mathbb{Z}[\mathcal{IU}_{p+q+1}(R^{2n})]$. □

**Corollary 3.1.8.** *Let $k \leq n-1$, and let $z \in C_k(n) = \mathbb{Z}[\mathcal{IU}_k(R^{2n})]$ be a cycle. Then, $z$ is homologous to a cycle $z' \in \mathbb{Z}[\mathcal{U}_k(W)] \subset C_k(n)$ contained within a free totally isotropic summand $W$ of $R^{2n}$ of dimension $n$.*

*Proof.* Suppose $z = \sum_i n_i u_i$ where $n_i \in \mathbb{Z}$ and $u_i \in \mathcal{IU}_k(R^{2n})$. By proposition 3.1.5, there exists a free totally isotropic subspace $W$ of rank $n$ in general position with respect to all $U_i$ =Span($u_i$). Choose unimodular vectors $f_i \in W \cap U_i^{\perp}$ which is possible since $\dim W \cap U_i^{\perp} \geq 1$, by proposition 3.1.5. Note, by lemma 3.1.6, $(u_i, f_i) \in \mathcal{IU}_{k+1}(R^{2n})$ for every $i$. Consider the chain $\xi := \sum_i n_i(u_i, f_i) \in C_{k+1}(n)$. Note that

$$d\xi = \sum_i n_i(du_i, f_i) + (-1)^{k+1} \sum_i n_i u_i = \sum_i n_i(du_i, f_i) + (-1)^{k+1} z,$$

so that $z_1 := (-1)^{k+1} \sum_i n_i(du_i, f_i)$ is homologous to $z$, which we write as $z_1 \sim z$. Now, recursively assume that $z_q \in C_k(n)$ is cycle such that $z_q \sim \sum_i (u_i, f_i)$, where $u_i \in \mathcal{IU}_p$; $f_i \in \mathbb{Z}[\mathcal{U}_q(W)]$, $W$ is a free totally isotropic summand of $R^{2n}$ of dimension $n$ in general position with respect to all $u_i$, $p, q \geq 0$ such that $p + q = k < n$. Then we collect terms so that $u_i \neq u_j$ for every $i \neq j$. By assumption, we have

$$0 = dz_q = \sum_i d(u_i, f_i) = \sum_i \left[(du_i, f_i) + (-1)^{p+1}(u_i, df_i)\right].$$

As $u_i \neq u_j$ and $W$ is in general position with every $u_i$, hence, no column vector of $u_i$ is in $W$, we deduce $df_i = 0$ for every $i$. Therefore, by lemma 3.1.7, for every $i$, there exists $g_i \in \mathbb{Z}[\mathcal{U}_{q+1}(W)]$ such that $dg_i = f_i$ and $(u_i, g_i) \in \mathbb{Z}[\mathcal{IU}_{k+1}(R^{2n})]$. Note that

$$d(u_i, g_i) = (du_i, g_i) + (-1)^{p+1}(u_i, dg_i) = (du_i, g_i) + (-1)^{p+1}(u_i, f_i).$$

We deduce $z_q \sim \sum_i (u_i, f_i) \sim (-1)^p \sum_i (du_i, g_i) = z_{q+1}$. The corollary is the case $q = k$, $p = 0$ setting $z' = z_k$. □

**Theorem 3.1.9.** *The complex $(C_*(n), d)$ is $(n-1)-$acyclic, that is,*

$$H_i(C_*(n), d) = 0 \qquad \textit{for } i \leq n-1.$$

*Proof.* Let $k \leq n-1$ and let $z \in C_k(n)$ a cycle. By corollary 3.1.8, $z$ is homologous to a cycle $z' \in \mathbb{Z}[\mathcal{U}_k(W)]$ contained within a free totally isotropic summand $W$ of $R^{2n}$ of dimension $n$. As $R$ is a local ring with infinite residue field, we deduce that there exists a $\tau \in \mathbb{Z}[\mathcal{U}_{k+1}(W)] \subset C_{k+1}(n)$ such that $d\tau = z'$, by the standard argument recalled in the proof of Lemma 3.1.7. In paricular, $z$ is a boundary. $\square$

## 3.2 Homological stability for $O_{n,n}$

### 3.2.1 Transitivity of the group action

We need to prove that the action of $O_{n,n}$ on $\mathcal{IU}_p(R^{2n})$ is *transitive* for all $p \leq n$. It suffices to prove the following lemma.

**Lemma 3.2.1.** *Let $p \leq n$ and let $(u_1, \ldots, u_p) \in \mathcal{IU}_p(R^{2n})$. Then, $(u_1, \ldots, u_p)$ may be extended to a hyperbolic basis of $R^{2n}$.*

*Proof.* By Witt's Cancelation theorem, which holds when $R$ is a local ring with 2 invertible (cf. [MH73, Chapter I, Theorem 4.4]), it will be sufficient to find $u_1^\#, \ldots, u_p^\#$ such that $(u_1, u_1^\#, \ldots, u_p, u_p^\#)$ has Gram matrix $\psi_{2p}$. (Note that $\mathrm{Span}\{u_1, u_1^\#, \ldots, u_p, u_p^\#\}$ is a non-degenerate subspace).

We have that $(u_1, \ldots, u_p) \in \mathcal{IU}_p(R^{2n})$, so this sequence is in particular a unimodular sequence of vectors in $R^{2n}$. Thus, the matrix $u = (u_1, \ldots, u_p)$ is left invertible. Therefore, the matrix ${}^t u \psi_{2n}$ is right invertible. This is equivalent to saying that the map

$$\begin{aligned} T : & R^{2n} \to R^p \\ & x \mapsto (\langle u_1, x\rangle, \ldots, \langle u_p, x\rangle) \end{aligned}$$

is surjective. Thus, for $i = 1, ..., p$, there exists $u_i^\#$ such that $T(u_i^\#)$ is the $i$-th standard basis vector of $R^p$. Replacing $u_i^\#$ with $u_i^\# - \frac{\langle u_i^\#, u_i^\#\rangle}{2} u_i$, we conclude the Gram matrix of $(u_1, u_1^\#, \ldots, u_p, u_p^\#)$ is $\psi_{2p}$. $\square$

### 3.2.2 Analysis of stabilisers

#### 3.2.2.1 Computation of stabilisers

Let $G$ be a group acting on a set $S$ from the left. Shapiro's lemma gives an isomorphism

$$\bigoplus_{[x]\in S/G} (i_x, x)_* : \bigoplus_{[x]\in S/G} H_*(G_x, \mathbb{Z}) \xrightarrow{\cong} H_*(G, \mathbb{Z}[S])$$

of homology groups, where the direct sum is over a set of representatives $x \in S$ of equivalence classes $[x] \in S/G$; the group $G_x$ is the *stabiliser* of $G$ at $x \in S$; the homomorphism $i_x : G_x \subseteq G$ is the inclusion; and $x$ also denotes the homomorphism of abelian groups $\mathbb{Z} \to \mathbb{Z}[S] : 1 \mapsto x$.

We apply Shapiro's lemma in the case $G = O_{n,n}(R)$ and $S = \mathcal{IU}_p(R^{2n})$. In particular, as the action of $O_{n,n}(R)$ on $\mathcal{IU}_p(R^{2n})$ is *transitive* for all $p \leq n$, Shapiro's lemma gives isomorphisms

$$H_*(St(e_1, \ldots, e_p)) \xrightarrow{\cong} H_*(O_{n,n}, C_p(n)), \tag{3.4}$$

for all $p \leq n$, where $St(e_1, \ldots, e_p)$ denotes the stabiliser of $(e_1, \ldots, e_p) \in \mathcal{IU}_p(R^{2n})$. We compute these stabilisers:

**Proposition 3.2.2.** *Let $1 \leq k \leq n$. Then, in the above notation, the stabilisers $A \in St(e_1, \ldots, e_k)$ are of the form*

$$A = \begin{pmatrix} 1 & c^1_1 & 0 & c^1_2 & \cdots & 0 & c^1_k & {}^t u_1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c^2_1 & 1 & c^2_2 & \cdots & 0 & c^2_k & {}^t u_2 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & c^k_1 & 0 & c^k_2 & \cdots & 1 & c^k_k & {}^t u_k \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & x_1 & 0 & x_2 & \cdots & 0 & x_k & B \end{pmatrix}$$

*where $c^i_j \in R; u_i, x_i \in R^{2(n-k)}$ and $B \in M_{2(n-k)}(R)$, subject to the conditions*

$$u_i + {}^t B \psi_{2(n-k)} x_i = 0, \tag{3.5}$$

$$c^i_j + c^j_i + \langle x_i, x_j \rangle = 0, \tag{3.6}$$

$$B \in O_{n-k,n-k}. \tag{3.7}$$

For example, for $k = 1$, we have

$$St(e_1) = \left\{ \begin{pmatrix} 1 & c & {}^t u \\ 0 & 1 & 0 \\ 0 & x & B \end{pmatrix} \middle| \; u + {}^t B \psi x = 0; \; 2c + \langle x, x \rangle = 0; \; B \in O_{n-1,n-1}. \right\}.$$

*Proof.* Let $A \in St(e_1, \ldots, e_k)$. Then, $Ae_i = e_i$ for all $1 \leq i \leq k$ by definition, which gives the 1st, 3rd, ..., $(2k-1)$st columns of $A$. Moreover, for a fixed $1 \leq i \leq k$ and any $1 \leq j \leq n$, we have

$$\langle e_i, Ae_j \rangle = \langle Ae_i, Ae_j \rangle = \langle e_i, e_j \rangle = 0$$

and

$$\langle e_i, Af_j \rangle = \langle Ae_i, Af_j \rangle = \langle e_i, f_j \rangle = \delta_{ij}.$$

Therefore, as $\langle e_k, e_l \rangle = 0$ and $\langle e_k, f_l \rangle = \delta_{kl}$ for all $1 \leq k, l \leq n$, we deduce that the coefficient of $f_i$ in the expression for $Ae_j$ and $Af_j$ is 0 for all $j \neq i$ and the coefficient of $f_i$ in the expression for $Af_i$ is 1. This gives the 2nd, 4th, ..., $2k$th rows of $A$. The remaining coefficients give the $c^i_j \in R; u_i, x_i \in R^{2(n-k)}$ and $B \in M_{2(n-k)}(R)$. We use the equation ${}^tA\psi A = \psi$ to determine the conditions on these variables. Specifically, one has that for any $A \in St(e_1, \dots, e_k)$,

$$
{}^tA\psi A = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
c^1_1 & 1 & c^2_1 & 0 & \cdots & c^k_1 & 0 & {}^tx_1 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
c^1_2 & 0 & c^2_2 & 1 & \cdots & c^k_2 & 0 & {}^tx_2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
c^1_k & 0 & c^2_k & 0 & \cdots & c^k_k & 1 & {}^tx_k \\
u_1 & 0 & u_2 & 0 & \cdots & u_k & 0 & {}^tB
\end{pmatrix}
\psi
\begin{pmatrix}
1 & c^1_1 & 0 & c^1_2 & \cdots & 0 & c^1_k & {}^tu_1 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & c^2_1 & 1 & c^2_2 & \cdots & 0 & c^2_k & {}^tu_2 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & c^k_1 & 0 & c^k_2 & \cdots & 1 & c^k_k & {}^tu_k \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & x_1 & 0 & x_2 & \cdots & 0 & x_k & B
\end{pmatrix}
$$

$$
= \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & c^1_1 + c^1_1 + \langle x_1, x_1 \rangle & 0 & c^1_2 + c^2_1 + \langle x_1, x_2 \rangle & \cdots & 0 & c^1_k + c^k_1 + \langle x_1, x_k \rangle & {}^tu_1 + {}^tx_1\psi B \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & c^1_2 + c^2_1 + \langle x_2, x_1 \rangle & 1 & c^2_2 + c^2_2 + \langle x_2, x_2 \rangle & \cdots & 0 & c^2_k + c^k_2 + \langle x_2, x_k \rangle & {}^tu_2 + {}^tx_2\psi B \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & c^1_k + c^k_1 + \langle x_k, x_1 \rangle & 0 & c^2_k + c^k_2 + \langle x_k, x_2 \rangle & \cdots & 1 & c^k_k + c^k_k + \langle x_k, x_k \rangle & {}^tu_k + {}^tx_k\psi B \\
0 & u_1 + {}^tB\psi x_1 & 0 & u_2 + {}^tB\psi x_2 & \cdots & 0 & u_k + {}^tB\psi x_k & {}^tB\psi B
\end{pmatrix}
$$

$$= \psi.$$

Whence the equations. □

To ease notation, we will from now on denote $T_k := St(e_1, \dots, e_k)$. We will use the convention that $T_0 = O_{n,n}$. Note, we may see from the structure of the matrices in $T_k$ that the projection map $\rho : T_k \twoheadrightarrow O_{n-k,n-k}$, sending the matrix $A$ in proposition 3.2.2 to $\rho(A) = B$, defines a *group homomorphism*. We denote its kernel by $L_k$, so that we have a short exact sequence of groups

$$1 \to L_k \to T_k \xrightarrow{\rho} O_{n-k,n-k} \to 1. \tag{3.8}$$

The associated Hochschild-Serre sepectral sequence is

$$E^2_{p,q} = H_p(O_{n-k,n-k}; H_q(L_k)) \Rightarrow H_{p+q}(T_k). \tag{3.9}$$

#### 3.2.2.2 The local $R^*$-action

In this section, we will define an $R^*$-action on short exact sequence (3.8) which we call 'local action'. Using spectral sequence (3.9), we will show that, *after localisation*, the homology of $T_k$ and $O_{n-k,n-k}$ coincide. In the next section, we will see that the local actions are induced by a 'global' $R^*$-action on the spectral sequence (3.3).

**Definition 3.2.3** (Local action)**.** Let $a \in R^*$. For $0 \le k \le n$, define a $2n \times 2n$ matrix $D_{a,k}$ by

$$D_{a,k} := \begin{pmatrix} D_a & & & \\ & \ddots & & \\ & & D_a & \\ & & & 1_{2n-2k} \end{pmatrix} = \left( \bigoplus_1^k D_a \right) \bigoplus 1_{2(n-k)}, \quad D_a := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

Note that $D_{a,k} \in O_{n,n}(R)$. The *local action* of $R^*$ on $T_k$ is the conjugation action of $D_{a,k}$ on $T_k$.

The local action preserves $T_k$ because

$$D_{a,k} \begin{pmatrix} 1 & c_1^1 & 0 & c_2^1 & \cdots & 0 & c_k^1 & {}^t u_1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_1^2 & 1 & c_2^2 & \cdots & 0 & c_k^2 & {}^t u_2 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & c_1^k & 0 & c_2^k & \cdots & 1 & c_k^k & {}^t u_k \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & x_1 & 0 & x_2 & \cdots & 0 & x_k & B \end{pmatrix} D_{a,k}^{-1}$$

$$= \begin{pmatrix} 1 & a^2 c_1^1 & 0 & a^2 c_2^1 & \cdots & 0 & a^2 c_k^1 & a^t u_1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a^2 c_1^2 & 1 & a^2 c_2^2 & \cdots & 0 & a^2 c_k^2 & a^t u_2 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & a^2 c_1^k & 0 & a^2 c_2^k & \cdots & 1 & a^2 c_k^k & a^t u_k \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & a x_1 & 0 & a x_2 & \cdots & 0 & a x_k & B \end{pmatrix} \in T_k.$$

Also notice that this conjugation action restricted to $O_{n-k,n-k} \subset T_k$ is *trivial.* Thus, we have an $R^*$-action on short exact sequence (3.8), which will induce an $R^*$-action on spectral sequence (3.9).

We now introduce the idea of *localising homology groups*. Let $m \ge 1$ be an integer. Choose units $a_1, \dots, a_m \in R^*$ such that for every non-empty subset $I \subset \{1, \dots, m\}$, the partial sum $a_I := \sum_{i \in I} a_i$ is a unit in $R$. Call such a sequence $(a_1, \dots, a_m)$ an $S(m)$-sequence. Choosing an $S(m)$-sequence is possible for every $m > 0$ because $R$ has infinite residue field.

Let $s_m \in \mathbb{Z}[R^*]$ be the element

$$s_m = - \sum_{\emptyset \neq I \subset \{1,\dots,m\}} (-1)^{|I|} \langle a_I \rangle \in \mathbb{Z}[R^*],$$

first considered in [Sch17], where $\langle u \rangle \in \mathbb{Z}[R^*]$ denotes the element of the group ring corresponding to $u \in R^*$. Note that

$$1 = - \sum_{\emptyset \neq I \subset \{1,\dots,m\}} (-1)^{|I|},$$

so that a trivial $R^*$-action induces a trivial action by the elements $s_m$. The magic of these elements $s_m$ lie in the following proposition, for the proof of which we refer the reader to [Sch21, Proposition D.4.].

**Proposition 3.2.4.** *Let $R$ be a commutative ring and $u = (u_1, \dots, u_m)$ an $S(m)$-sequence in $R$. Let*

$$N \to G \to A$$

*be a central extension of groups. Assume that the group of units $R^*$ acts on the exact sequence. Assume furthermore that the groups $A$ and $N$ are the underlying abelian groups $(A, +, 0)$ and $(N, +, 0)$ of $R$-modules $(A, +, 0, \cdot)$ and $(N, +, 0, \cdot)$, and the $R^*$-actions on $A$ and $N$ in the exact sequence are given by*

$$R^* \times A \to A : (t, a) \mapsto t \cdot a$$

*and*

$$R^* \times N \to N : (t, y) \mapsto t^2 \cdot y$$

*respectively. Then, for every $1 \leq n < m/2$,*

$$s_m^{-1} H_n(G) = 0.$$

□

We may now *localise* the spectral sequence (3.9) with respect to the elements

$s_m$ to obtain for all $m \geq 1$ the localised spectral sequences

$$\begin{aligned} s_m^{-1}E^2_{pq} &= s_m^{-1}H_p(O_{n-k,n-k}; H_q(L_k)) \\ &\cong H_p(O_{n-k,n-k}; s_m^{-1}H_q(L_k)) \end{aligned} \Rightarrow s_m^{-1}H_{p+q}(T_k), \tag{3.10}$$

the isomorphism coming from the fact that $R^*$ acts trivially on $O_{n-k,n-k}$.

We show that localising with respect to the elements $s_m$ *kills* the non-zero homology groups of $L_k$ when $m$ is taken to infinity.

**Lemma 3.2.5.** *We have $s_m^{-1}H_0(L_k) = \mathbb{Z}$ and for all $1 \leq 2q < m$, $s_m^{-1}H_q(L_k) = 0$.*

*Proof.* We claim there is a short exact sequence of groups

$$1 \to (R^{\binom{k}{2}}, +) \to L_k \to ((R^{2(n-k)})^k, +) \to 1. \tag{3.11}$$

The first arrow maps

$$(c_1, \dots) \mapsto A_{(c_1,\dots)}$$

where $A_{(c_1,\dots)} \in L_k$ is defined by the conditions (3.5), (3.6) and (3.7) subject to $B = 1$, $x_i = 0$ and using equation (3.6) to determine the remaining constants (with some ordering specified beforehand). Note that we have used here that 2 is invertible, as (3.6) implies $2c_i^i = 0$ for all $i$. The second arrow maps

$$\begin{pmatrix} 1 & c_1^1 & 0 & c_2^1 & \cdots & 0 & c_k^1 & {}^tu_1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_1^2 & 1 & c_2^2 & \cdots & 0 & c_k^2 & {}^tu_2 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & c_1^k & 0 & c_2^k & \cdots & 1 & c_k^k & {}^tu_k \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & x_1 & 0 & x_2 & \cdots & 0 & x_k & 1 \end{pmatrix} \mapsto (x_1, \dots, x_k).$$

One may check that these arrows define group homomorphisms, fitting into the short exact sequence (3.11), and (3.11) is actually a central extension.

Furthermore, this central extension is $R^*$-equivariant where $b \in R^*$ acts on $(R^{\binom{k}{2}}, +)$ via pointwise multiplication by $b^2$, the element $b \in R^*$ acts on $L_k$ via conjugation by $D_{b,k}$, and it acts on $((R^{2(n-k)})^k, +)$ via pointwise multiplication by $b$. By Proposition 3.2.4, $s_m^{-1}H_q(L_k) = 0$ for all $1 \leq 2q < m$. The equality $s_m^{-1}H_0(L_k) = \mathbb{Z}$ follows from fact that $R^*$ acts trivially on $H_0(L_k)$. □

**Corollary 3.2.6.** *The inclusion $O_{n-k,n-k} \hookrightarrow T_k$ induces isomorphism*

$$H_t(O_{n-k,n-k}) \xrightarrow{\cong} s_m^{-1} H_t(T_k)$$

*for all $t < m/2$.*

*Proof.* By lemma 3.2.5, the localised Hochschild-Serre spectral sequence degenerates at $E^2$ for $1 \leq 2t < m$ to yield isomorphism

$$\rho : s_m^{-1} H_t(T_k) \xrightarrow{\cong} H_t(O_{n-k,n-k})$$

for all $t < m/2$. Since $\rho$ is a retract of the inclusion, we are done. □

#### 3.2.2.3 A global action on the spectral sequence

Next, we want to realise these 'local actions' as a 'global action' on the spectral sequence

$$E^1_{p,q}(n) = H_q(O_{n,n}, C_p(n)) \Rightarrow H_{p+q}(O_{n,n}, C_*(n)). \tag{3.12}$$

We do this by defining an action on the associated exact couple with abutment.

Recall that for a group $G$ and a chain complex of $G$-modules $C_*$, the spectral sequence

$$E^1_{p,q} = H_q(G, C_p) \Rightarrow H_{p+q}(G, C_*)$$

may be obtained from the exact couple with abutment

$$\begin{array}{ccccc} \bigoplus_{p,q} E^1_{p,q} & \xrightarrow{k} & \bigoplus_{p,q} D^1_{p,q} & \xrightarrow{\sigma} & \bigoplus_{p+q} A_{p+q} \\ & {}^{j}\nwarrow & \big\downarrow{\scriptstyle i} & \nearrow^{\sigma} & \\ & & \bigoplus_{p,q} D^1_{p,q} & & \end{array} \tag{3.13}$$

with $E^1_{p,q} = H_{p+q}(G, C_{\leq p}/C_{\leq p-1})$; $D^1_{p,q} = H_{p+q}(G, C_{\leq p})$; $A_{p+q} = H_{p+q}(G, C_*)$; the maps $i, j, k$ being the maps of the long exact sequence of homology groups associated to the short exact sequence of complexes

$$0 \to C_{\leq p-1} \to C_{\leq p} \to C_{\leq p}/C_{\leq p-1} \to 0,$$

and $\sigma$ is induced by the inclusion.

To define the global action, it will be convenient to introduce the *general* split orthogonal group, which is defined as follows.

**Definition 3.2.7.** For a ring $R$, define $GO_{n,n}(R) \subset GL_{2n}(R)$ as the subgroup

$$GO_{n,n}(R) := \{A \in GL_{2n}(R) |\ {}^tA\psi_{2n}A = a\psi_{2n},\ \text{for some } a \in R^*\}.$$

In the above notation, we will call $a \in R^*$ the *associated unit* of $A$.

For $n \geq 1$ we have short exact sequence of groups

$$1 \to O_{n,n} \to GO_{n,n} \to R^* \to 1 \tag{3.14}$$

where the first arrow is given by the inclusion and the second arrow maps $A \in GO_{n,n}$ to its associated unit. For instance, for $a \in R^*$, the matrix

$$B_a := \begin{pmatrix} 1 & & & & \\ & a & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & a \end{pmatrix}$$

is in $GO_{n,n}$ and has associated unit $a$ which proves exactness at the right.

**Definition 3.2.8** (Global action)**.** The group homomorphism $GO_{n,n}(R) \to R^*$ makes $\mathbb{Z}[R^*]$ into a right $GO_{n,n}(R)$-module and left $R^*$-module, and both actions commute. In particular, for any bounded below complex of $GO_{n,n}$-modules $M.$, the groups

$$\operatorname{Tor}_i^{GO_{n,n}}(\mathbb{Z}[R^*], M.) = H_i(\mathbb{Z}[R^*] \otimes^{\mathbb{L}}_{GO_{n,n}} M.)$$

are left $\mathbb{Z}[R^*]$-modules functorial in $M.$, and the spectral sequence

$$E^1_{p,q} = \operatorname{Tor}_q^{GO_{n,n}}(\mathbb{Z}[R^*], M_p) \Rightarrow \operatorname{Tor}_{p+q}^{GO_{n,n}}(\mathbb{Z}[R^*], M.) \tag{3.15}$$

is a spectral sequence of left $R^*$-modules. This spectral sequence is the spectral sequence of the exact couple (3.13) with $E^1_{p,q} = \operatorname{Tor}_{p+q}^{GO_{n,n}}(\mathbb{Z}[R^*], M_{\leq p}/M_{\leq p-1})$; $D^1_{p,q} = \operatorname{Tor}_{p+q}^{GO_{n,n}}(\mathbb{Z}[R^*], M_{\leq p})$; $A_{p+q} = \operatorname{Tor}_{p+q}^{GO_{n,n}}(\mathbb{Z}[R^*], M.)$. For $n \geq 1$, the inclusions $\mathbb{Z} \subset \mathbb{Z}[R^*] : 1 \mapsto 1$ and $O_{n,n} \subset GO_{n,n}$ yield isomorphisms

$$\mathbb{Z} \otimes^{\mathbb{L}}_{O_{n,n}} M. \xrightarrow{\sim} \mathbb{Z}[R^*] \otimes^{\mathbb{L}}_{GO_{n,n}} M.$$

by Shapiro's lemma. For $M. = C.(n)$, this identifies the spectral sequence (3.15) with (3.12) and makes the latter into a spectral sequence of $R^*$-modules. We use this structure to define the *global action* of $R^*$ on (3.12).

Specifically, we now show that, under the isomorphism (3.4), the local actions corresponding to conjugation with $D_{a,k}$ are induced by the global action corresponding to multiplication with $a^{-2} \in R^*$. For this end, we will need to prove that the appropriate diagrams commute. We will use the following two lemmas.

**Lemma 3.2.9.** *Let $G$ and $K$ be groups. Let $M$ and $N$ be a $G$-module and a $K$-module, respectively. Consider the diagram of morphisms*

$$(G, M) \underset{(f_2,\varphi_2)}{\overset{(f_1,\varphi_1)}{\rightrightarrows}} (K, N)$$

*where $f_1, f_2$ are group homomorphisms and $\varphi_1, \varphi_2$ $G$-module homomorphisms, $N$ is considered a $G$-module via $f_1$ and $f_2$ respectively. Suppose there exists a $\kappa \in K$ such that for all $g \in G$ and for all $m \in M$,*

$$f_2(g) = \kappa f_1(g)\kappa^{-1} \qquad \text{and} \qquad \varphi_2(m) = \kappa\varphi_1(m).$$

*Then*

$$(f_1, \varphi_1)_* = (f_2, \varphi_2)_* : H_*(G, M) \to H_*(K, N).$$

*Proof.* By assumption, we have the following commutative diagram:

$$\begin{array}{ccc} H_*(G,M) & & \\ {\scriptstyle (f_1,\varphi_1)_*}\downarrow & \searrow^{(f_2,\varphi_2)_*} & \\ H_*(K,N) & \xrightarrow[(c_\kappa,\mu_\kappa)_*]{} & H_*(K,N), \end{array}$$

where $(c_\kappa, \mu_\kappa) : (K, N) \to (K, N)$ is the map $(k, n) \mapsto (\kappa k\kappa^{-1}, \kappa n)$. By [Bro94, Chapter III.8], the bottom horizontal map equals the identity. □

We will also need to recall functoriality of Tor. This is given by the following lemma.

**Lemma 3.2.10.** *Let $G$ and $K$ be groups; let $M$ and $P$ be a right $G$-module and right $K$-module respectively; and let $N$ and $Q$ be a left $G$-module and left $K$-module respectively.*

*Consider the diagram of morphisms*

$$(M, G, N) \underset{(f_2,\varphi_2,g_2)}{\overset{(f_1,\varphi_1,g_1)}{\rightrightarrows}} (P, K, Q)$$

*where $\varphi_1, \varphi_2$ are group homomorphisms; $f_1, f_2$ right $G$-module homomorphisms where*

*$P$ is considered a right $G$-module via $\varphi_1$ and $\varphi_2$ respectively and $g_1, g_2$ left $G$-module homomorphisms where $Q$ is considered a left $G$-module via $\varphi_1$ and $\varphi_2$ respectively.*

*Suppose there exists a $\kappa \in K$ such that for all $g \in G$; for all $m \in M$ and for all $n \in N$,*

$$f_2(m) = f_1(m)\kappa^{-1} \quad , \quad \varphi_2(g) = \kappa\varphi_1(g)\kappa^{-1} \quad \text{and} \quad g_2(m) = \kappa g_1(n).$$

*Then*

$$(f_1, \varphi_1, g_1)_* = (f_2, \varphi_2, g_2)_* : \mathrm{Tor}_*^G(M, N) \to \mathrm{Tor}_*^K(P, Q).$$

*Proof.* By assumption, the following diagram commutes:

$$\begin{array}{ccc} H_*(M \otimes_G^{\mathbb{L}} N) & & \\ \downarrow{\scriptstyle (f_1,\varphi_1,g_1)_*} & \searrow^{(f_2,\varphi_2,g_2)_*} & \\ H_*(P \otimes_K^{\mathbb{L}} Q) & \xrightarrow[(\lambda_{\kappa^{-1}}, c_\kappa, \mu_\kappa)_*]{} & H_*(P \otimes_K^{\mathbb{L}} Q), \end{array}$$

where

$$\lambda_{\kappa^{-1}} : P \to P, p \mapsto p\kappa^{-1}, \quad c_\kappa : K \to K, k \mapsto \kappa k \kappa^{-1}, \quad \mu_\kappa : Q \to Q, q \mapsto \kappa q.$$

Therefore, the bottom horizontal map is induced by the map

$$\begin{aligned} &P \otimes_G Q \to P \otimes_G Q \\ &p \otimes q \mapsto p\kappa^{-1} \otimes \kappa q = p \otimes q \end{aligned}$$

which is the identity. □

By definition the $R^*$-action on $\mathrm{Tor}_q^{GO_{n,n}}(\mathbb{Z}[R^*], C_k(n))$ corresponding to left multiplication with $a \in R^*$ is induced by the map

$$(\mathbb{Z}[R^*], GO_{n,n}, C_k(n)) \xrightarrow{(a,id,id)} (\mathbb{Z}[R^*], GO_{n,n}, C_k(n))$$

where $a : \mathbb{Z}[R^*] \to \mathbb{Z}[R^*]$ is the map corresponding to left multiplication with $a$. With this, we have the following proposition, which gives us a model of this action in terms of the groups $\mathrm{Tor}_q^{O_{n,n}}(\mathbb{Z}, C_k(n)) \cong H_q(O_{n,n}, C_k(n))$.

**Proposition 3.2.11.** *Let $k, q \geq 0$ and $n \geq 1$. Then, for all $a \in R^*$, the following* *diagram commutes:*

$$\begin{array}{ccc} \mathrm{Tor}_q^{GO_{n,n}}(\mathbb{Z}[R^*], C_k(n)) & \xrightarrow{(a,id,id)_*} & \mathrm{Tor}_q^{GO_{n,n}}(\mathbb{Z}[R^*], C_k(n)) \\ {\scriptstyle (i,i,id)_*}\uparrow\cong & & \cong\uparrow{\scriptstyle (i,i,id)_*} \\ \mathrm{Tor}_q^{O_{n,n}}(\mathbb{Z}, C_k(n)) & \xrightarrow[(id,C_{B_a},B_a)_*]{} & \mathrm{Tor}_q^{O_{n,n}}(\mathbb{Z}, C_k(n)), \end{array}$$

*where the vertical maps are the isomorphisms given by Shapiro's lemma;* $B_a$ *denotes left multiplication by* $B_a \in GO_{n,n}$ *and* $C_{B_a}$ *denotes the map induced by conjugation with the element* $B_a$ *on* $O_{n,n}$*.*

*Proof.* We use lemma 3.2.10. Specifically, consider the diagram

$$(\mathbb{Z}, O_{n,n}, C_k(n)) \underset{(f_2,\varphi_2,g_2)}{\overset{(f_1,\varphi_1,g_1)}{\rightrightarrows}} (\mathbb{Z}[R^*], GO_{n,n}, C_k(n))$$

where $(f_1, \varphi_1, g_1) := (\mu_a, i, 1)$ and $(f_2, \varphi_2, g_2) := (i, C_{B_a}, \mu_{B_a})$. Here, $\mu_a$ is defined via $\mu_a(1) := a$ and $\mu_{B_a}$ is defined via left multiplication on basis elements by $B_a$. Let $\kappa := B_a \in GO_{n,n}$. Note that from short exact sequence 3.14, we deduce $B_a$ acts on $R^*$ by multiplication with $a$. Thus, $i(1) = 1 = \mu_a(1)\kappa^{-1}$. Furthermore, $C_{B_a} = \kappa i \kappa^{-1}$ and for every $(v_1, \ldots, v_k) \in \mathcal{IU}_k(R^{2n})$, $\mu_{B_a}(v_1, \ldots, v_k) = \kappa(v_1, \ldots, v_k)$ (the case $k = 0$ being trivial). Thus, by lemma 3.2.10, the diagram commutes. □

Next, note that the action on $\mathrm{Tor}_q^{O_{n,n}}(\mathbb{Z}, C_k(n))$ induced by

$$(\mathbb{Z}, O_{n,n}, C_k(n)) \xrightarrow{(id,C_{B_a},B_a)} (\mathbb{Z}, O_{n,n}, C_k(n))$$

is equivalent to the action on $H_q(O_{n,n}, C_k(n))$ induced by

$$(O_{n,n}, C_k(n)) \xrightarrow{(C_{B_a},B_a)} (O_{n,n}, C_k(n)).$$

To make the connection with the action induced by conjugation with $D_{a,k}$, we prove the following intermediate proposition.

**Proposition 3.2.12.** *Let* $k, q \geq 0$ *and* $n \geq 1$*. Then, for all* $a \in R^*$*, the following diagram commutes:*

$$\begin{array}{ccc} H_q(O_{n,n}, C_k(n)) & \xrightarrow{(C_{B_{a^{-2}}},B_{a^{-2}})_*} & H_q(O_{n,n}, C_k(n)) \\ {\scriptstyle id}\uparrow & & \uparrow{\scriptstyle id} \\ H_q(O_{n,n}, C_k(n)) & \xrightarrow[(id,\phi_a)_*]{} & H_q(O_{n,n}, C_k(n)), \end{array}$$

*where for* $a \in R^*$*, the map*

$$(id, \phi_a) : (O_{n,n}, C_k(n)) \to (O_{n,n}, C_k(n))$$

*is defined to be the identity on* $O_{n,n}$ *and on basis elements of* $C_k(n)$ *as*

$$\phi_a : (v_1, \dots, v_k) \mapsto (a^{-1}v_1, \dots, a^{-1}v_k).$$

*Proof.* We use lemma 3.2.9. Specifically, consider the diagram

$$(O_{n,n}, C_k(n)) \overset{(f_1,\varphi_1)}{\underset{(f_2,\varphi_2)}{\rightrightarrows}} (O_{n,n}, C_k(n))$$

where $(f_1, \varphi_1) := (id, \phi_a)$ and $(f_2, \varphi_2) := (C_{B_{a^{-2}}}, B_{a^{-2}})$. Define

$$\kappa := D_{a,n} = \begin{pmatrix} a & & & & \\ & a^{-1} & & & \\ & & \ddots & & \\ & & & a & \\ & & & & a^{-1} \end{pmatrix}.$$

Denoting for $a \in R^*$,

$$\underline{a} := \begin{pmatrix} a & & & & \\ & a & & & \\ & & \ddots & & \\ & & & a & \\ & & & & a \end{pmatrix},$$

note that $B_{a^{-2}} = \kappa\underline{a}^{-1}$, so that $C_{B_{a^{-2}}} = C_\kappa C_{\underline{a}^{-1}}$. But, $C_{\underline{a}^{-1}} = id$, so that $C_{B_{a^{-2}}} = C_\kappa$. Furthermore, note that for every $(v_1, \dots, v_k) \in \mathcal{IU}_k(R^{2n})$, $B_{a^{-2}}(v_1, \dots, v_k) = \kappa\phi_a(v_1, \dots, v_k)$, since $B_{a^{-2}} = \kappa\underline{a}^{-1}$. Thus, by lemma 3.2.9, the diagram commutes. □

Finally, we show that $(id, \phi_a)$ induces the desired local actions.

**Proposition 3.2.13.** *Let* $k, q \geq 0$*. Then, for all* $a \in R^*$*, the following diagram commutes:*

$$\begin{array}{ccc} H_q(O_{n,n}, C_k(n)) & \xrightarrow{(id,\phi_a)_*} & H_q(O_{n,n}, C_k(n)) \\ {\scriptstyle (i,(e_1,\dots,e_k))_*}\uparrow\cong & & \cong\uparrow{\scriptstyle (i,(e_1,\dots,e_k))_*} \\ H_q(T_k, \mathbb{Z}) & \xrightarrow[C_{D_{a,k}}]{} & H_q(T_k, \mathbb{Z}), \end{array}$$

*where the vertical arrows are the isomorphisms given by Shapiro's lemma and the map $C_{D_{a,k}}$ denotes the map induced by conjugation with the element $D_{a,k}$ on the stabiliser $T_k$.*

*Proof.* We use lemma 3.2.9. Specifically, we have to consider the diagram

$$(T_k, \mathbb{Z}) \overset{(f_1,\varphi_1)}{\underset{(f_2,\varphi_2)}{\rightrightarrows}} (O_{n,n}, C_k(n))$$

where $(f_1, \varphi_1) := (i, (a^{-1}e_1, \ldots, a^{-1}e_k))$ and $(f_2, \varphi_2) := (iC_{D_{a,k}}, (e_1, \ldots, e_k))$, and $i : T_k \to O_{n,n}$ is the natural inclusion of groups. Let $\kappa = D_{a,k} \in O_{n,n}$. Then, for every $A \in T_k$,

$$f_2(A) = iC_{D_{a,k}}(A) = D_{a,k}AD_{a,k}^{-1} = D_{a,k}i(A)D_{a,k}^{-1} = \kappa f_1(A)\kappa^{-1}$$

and

$$(e_1 \ldots, e_k) = D_{a,k}(a^{-1}e_1, \ldots, a^{-1}e_k) = \kappa(a^{-1}e_1, \ldots, a^{-1}e_k).$$

By lemma 3.2.9, the diagram commutes. □

Thus, we have shown that there exists an $R^*$-action on the spectral sequence

$$E^1_{p,q}(n) = H_q(O_{n,n}, C_p(n)) \Rightarrow H_{p+q}(O_{n,n}, C_*(n))$$

which induces the desired local actions considered previously. Using corollary 3.2.6, we obtain the following.

**Corollary 3.2.14.** *For every $m \geq 1$, the localised spectral sequence*

$$_mE^1_{p,q}(n) = s_m^{-1}E^1_{p,q}(n) \Rightarrow s_m^{-1}H_{p+q}(O_{n,n}, C_*(n)) \tag{3.16}$$

*has $_mE^1_{p,q}$ terms*

$$_mE^1_{p,q} = s_m^{-1}H_q(O_{n,n}, C_p(n)) \cong H_q(O_{n-p,n-p})$$

*for all $q < m/2$ and for all $p \leq n$.*

□

Under these identifications, the differentials $d^1 : {}_mE^1_{p,q} \to {}_mE^1_{p-1,q}$ take the form

$$d^1 : H_q(O_{n-p,n-p}) \to H_q(O_{n-p+1,n-p+1})$$

whenever $q < m/2$ and $p \leq n$. Our next task is to compute these differentials.

### 3.2.3 Computation of the localised $d^1$ differentials, and proof of homological stability

**Proposition 3.2.15.** *For all $q < m/2$ and $p \leq n$, the homomorphism $d^1_{p,q} : H_q(O_{n-p,n-p}) \to H_q(O_{n-p+1,n-p+1})$ is*

$$d^1_{p,q} = \begin{cases} 0, & p \text{ even} \\ i_*, & p \text{ odd}, \end{cases}$$

*where $i : O_{n-p,n-p} \hookrightarrow O_{n-p+1,n-p+1}$ denotes the inclusion.*

*Proof.* For all $p \leq n$, we want to show that the following diagram commutes:

$$\begin{array}{ccc} H_q(O_{n-p,n-p}) & \xrightarrow{(\varepsilon,(e_1,\ldots,e_p))_*} & H_q(O_{n,n}, C_p(n)) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle (d_i)_*} \\ H_q(O_{n-p+1,n-p+1}) & \xrightarrow[(\varepsilon,(e_1,\ldots,e_{p-1}))_*]{} & H_q(O_{n,n}, C_{p-1}(n)), \end{array} \tag{3.17}$$

where $\varepsilon : O_{n-p,n-p} \hookrightarrow O_{n,n}$ denotes the inclusion map; $(e_1, \ldots, e_p) : 1 \mapsto (e_1, \ldots, e_p)$ and recall that $d_i(v_1, \ldots, v_p) = (v_1, \ldots, \hat{v}_i. \ldots, v_p)$. Again, we will prove this diagram commutes using lemma 3.2.9. Specifically, consider the diagram

$$(O_{n-p,n-p}, \mathbb{Z}) \underset{(\varepsilon \circ i,(e_1,\ldots,e_{p-1}))}{\overset{(\varepsilon,(e_1,\ldots,\hat{e}_i,\ldots,e_p))}{\rightrightarrows}} (O_{n,n}, C_{p-1}(n)).$$

Define $A \in O_{n,n}$ by sending a hyperbolic basis to a hyperbolic basis as follows:

$$\begin{gathered} (e_1, \ldots, \hat{e}_i, \ldots, e_p) \mapsto (e_1, \ldots, e_{p-1}) \\ (f_1, \ldots, \hat{f}_i, \ldots, f_p) \mapsto (f_1, \ldots, f_{p-1}) \\ e_i \mapsto e_p \\ f_i \mapsto f_p \\ e_j \mapsto e_j \text{ and } f_j \mapsto f_j \text{ for all } p+1 \leq j \leq n. \end{gathered}$$

Then, by construction, $(e_1, \ldots, e_{p-1}) = A(e_1, \ldots, \hat{e}_i, \ldots e_p)$. Note the matrix of $A$ has the form

$$A = \begin{pmatrix} \sigma & 0 \\ 0 & 1_{2(n-p)} \end{pmatrix}$$

for some permutation matrix $\sigma$. Therefore, we deduce that for every $B \in O_{n-p,n-p}$,

$$\varepsilon \circ i(B) = A\varepsilon(B)A^{-1}.$$

By lemma 3.2.9, the diagram commutes. The proposition then follows from the fact that the differential $d^1 : H_q(O_{n,n}, C_p(n)) \to H_q(O_{n,n}, C_{p-1}(n))$ is induced by the differential $d = \sum_{i=1}^{p}(-1)^{i+1}d_i : C_p(n) \to C_{p-1}(n)$ and the above remains true after localisation, with the horizontal arrows becoming the identification isomorphisms. □

We immediately deduce the following corollary:

**Corollary 3.2.16.** *For all $q < m/2$ and for all $p \leq n$,*

$$ {}_mE^2_{p,q} = \begin{cases} \ker\left(H_q(O_{n-p,n-p}) \xrightarrow{i_*} H_q(O_{n-p+1,n-p+1}))\right), & p \text{ odd} \\ \operatorname{coker}\left(H_q(O_{n-p-1,n-p-1}) \xrightarrow{i_*} H_q(O_{n-p,n-p}))\right), & p \text{ even.} \end{cases} $$

□

To prove homological stability, we will need to prove the following.

**Proposition 3.2.17.** *The differentials $d^r_{p,q}$ in spectral sequence (3.16) are zero for $r \geq 2$ and $q < m/2$, $p \leq n$. Hence, for all $q < m/2$ and $p \leq n$, ${}_mE^2_{p,q} \cong {}_mE^\infty_{p,q}$.*

*Proof.* Similar to [NS89] and [Sch17], we argue by induction on $n$. For $n = 0, 1$, the spectral sequence under consideration is located in columns 0 and 1. Therefore, the differentials $d^r$ for $r \geq 2$ are zero by dimension arguments.

Assume $n \geq 2$. We seek to define a homomorphism of complexes of $O_{n-2,n-2}$-modules

$$ \tau : C_*(n-2)[-2] \to C_*(n). $$

For $(v_1, \dots, v_{p-2}) \in C_p(n-2)[-2]$, define

$$ \begin{aligned} \tau_0(v_1, \dots, v_{p-2}) &:= (e_1, e_2, \bar{v}_1, \dots, \bar{v}_{p-2}) \\ \tau_1(v_1, \dots, v_{p-2}) &:= (e_1, e_2 - e_1, \bar{v}_1, \dots, \bar{v}_{p-2}) \\ \tau_2(v_1, \dots, v_{p-2}) &:= (e_2, e_2 - e_1, \bar{v}_1, \dots, \bar{v}_{p-2}), \end{aligned} $$

where

$$ \bar{v}_i := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_i \end{pmatrix} \in R^{2n}. $$

Define $\tau := \tau_0 - \tau_1 + \tau_2$. Note that $\tau$ commutes with differentials and commutes with $O_{n-2,n-2}$ multiplication from the left, so that it indeed defines a homomorphism of chain complexes of $O_{n-2,n-2}$-modules. We need to check that $\tau$ is equivariant for the global $R^*$-actions on the spectral sequences so that $\tau$ induces a map on the localised spectral sequences. By Proposition 3.2.11, the global action is induced by the map $(C_{B_a}, B_a)$ on the spectral sequence (3.12). Therefore, $R^*$-equivariance follows from the fact that for every $a \in R^*$ and for all $j = 0, 1, 2$, the diagrams

$$\begin{array}{ccc} (O_{n-2,n-2}, C_{p-2}(n-2)) & \xrightarrow{(i,\tau_j)} & (O_{n,n}, C_p(n)) \\ {\scriptstyle (C_{B_a},B_a)}\uparrow & & \uparrow{\scriptstyle (C_{B_a},B_a)} \\ (O_{n-2,n-2}, C_{p-2}(n-2)) & \xrightarrow[(i,\tau_j)]{} & (O_{n,n}, C_p(n)), \end{array}$$

commute, where $i : O_{n-2,n-2} \hookrightarrow O_{n,n}$ denotes the inclusion. The point is that $B_a(e_i) = e_i$. Therefore, $\tau$ induces a map of spectral sequences of $R^*$-modules

$$\tau_* : \tilde{E} \to E$$

where $\tilde{E} := E(n-2)[-2, 0]$ and $E := E(n)$.

Recall from Propositions 3.2.12 and 3.2.13 that the local actions are globally induced by multiplication with $a^{-2}$ for $a \in R^*$. Localising with respect to the last action, we obtain a map on the localised spectral sequences

$${}_m\tau_* : {}_m\tilde{E} \to {}_mE.$$

Note that for all $q < m/2$ and $2 \le p \le n$,

$${}_m\tilde{E}^1_{p,q} = {}_mE^1_{p,q}(n-2)[-2,0] = {}_mE^1_{p-2,q}(n-2) \cong H_q(O_{n-p,n-p})$$

The claim will then follow by induction on $r$ using the following lemma.

**Lemma 3.2.18.** *The map ${}_m\tau_* : {}_m\tilde{E}^1_{p,q} \to {}_mE^1_{p,q}$ is the identity for all $q < m/2$ and $2 \le p \le n$.*

*Proof.* If we can show that for $j = 0, 1, 2$, the diagrams

$$\begin{array}{ccc} H_q(O_{n-p,n-p}) & \xrightarrow{(\varepsilon,(e_1,\ldots,e_{p-2}))_*} & H_q(O_{n-2,n-2}, C_{p-2}(n-2)) \\ = \downarrow & & \downarrow {\scriptstyle (\varepsilon,\tau_j)_*} \\ H_q(O_{n-p,n-p}) & \xrightarrow[(\varepsilon,(e_1,\ldots,e_p))_*]{} & H_q(O_{n,n}, C_p(n)), \end{array} \tag{3.18}$$

commute, where the $\varepsilon$'s denote inclusions, we will be done, as $\tau = \tau_0 - \tau_1 + \tau_2$.

Again, we will prove these diagrams commute using lemma 3.2.9. Specifically, consider diagram

$$(O_{n-p,n-p}, \mathbb{Z}) \underset{(\varepsilon,(e_1,\dots,e_p))}{\overset{(\varepsilon,\tau_j(e_1,\dots,e_{p-2}))}{\rightrightarrows}} (O_{n,n}, C_p(n))$$

Note that $\tau_0(e_1, \dots, e_{p-2}) = (e_1, e_2, e_3, \dots, e_p)$, so that diagram (3.18) commutes in the case for $j = 0$ by functoriality of group homology. For $j = 1, 2$, we have $\tau_1(e_1, \dots, e_{p-2}) = (e_1, e_2 - e_1, e_3, \dots, e_p)$ and $\tau_2(e_1, \dots, e_{p-2}) = (e_2, e_2 - e_1, e_3, \dots, e_p)$. Define a matrix $A \in O_{n,n}(R)$ by

$$\begin{aligned}
&e_1 \mapsto e_1\\
&e_2 \mapsto e_2 - e_1\\
&f_1 \mapsto f_1 + f_2\\
&f_2 \mapsto f_2\\
&e_j \mapsto e_j \text{ for all } 3 \leq j \leq n\\
&f_j \mapsto f_j \text{ for all } 3 \leq j \leq n.
\end{aligned}$$

Similarly, define $B \in O_{n,n}$ by

$$\begin{aligned}
&e_1 \mapsto e_2\\
&e_2 \mapsto e_2 - e_1\\
&f_1 \mapsto f_1 + f_2\\
&f_2 \mapsto -f_1\\
&e_j \mapsto e_j \text{ for all } 3 \leq j \leq n\\
&f_j \mapsto f_j \text{ for all } 3 \leq j \leq n.
\end{aligned}$$

Then, $A(e_1, \dots e_p) = \tau_1(e_1, \dots, e_{p-2})$, $B(e_1, \dots e_p) = \tau_2(e_1, \dots, e_{p-2})$ and for every $M \in O_{n-p,n-p}$, $\varepsilon(M) = A\varepsilon(M)A^{-1} = B\varepsilon(M)B^{-1}$. Thus, by lemma 3.2.9, diagram (3.18) commutes for every $j = 0, 1, 2$. These diagrams still commute after localisation, but now the horizontal maps become the identification isomorphisms. □

This proves the lemma, and thus proposition 3.2.17. □

**Theorem 3.2.19.** *Let $R$ be a commutative local ring with infinite residue field such that $2 \in R^*$. Then, the natural homomorphism*

$$H_k(O_{n,n}(R)) \longrightarrow H_k(O_{n+1,n+1}(R))$$

*is an isomorphism for $k \leq n-1$ and surjective for $k \leq n$.*

**Remark 24.** This improves Mirzaii's result [Mir04] by 1 and matches the analogous result for fields obtained by Sprehn-Wahl [SW20].

*Proof.* Choose $m > 0$ sufficiently large so that we may apply corollary 3.2.16 when $q \leq n-1$.

Recall from theorem 3.1.9 that $H_q(C_*(n)) = 0$ for all $q \leq n-1$. Thus, from the spectral sequences (3.2) and (3.16), corollary 3.2.16 and proposition 3.2.17, we deduce

$$\begin{aligned} \operatorname{coker}\left(H_q(O_{n-1,n-1}) \xrightarrow{i_*} H_q(O_{n,n})\right) &= {}_mE^2_{0,q} \\ &\cong {}_mE^\infty_{0,q} \\ &= 0 \end{aligned}$$

for all $q \leq n-1$, and

$$\begin{aligned} \ker\left(H_q(O_{n-1,n-1}) \xrightarrow{i_*} H_q(O_{n,n})\right) &= {}_mE^2_{1,q} \\ &\cong {}_mE^\infty_{1,q} \\ &= 0 \end{aligned}$$

for all $q \leq n-2$.

The theorem follows. □

# Chapter 4

# Homological Stability for $SO_{n,n}$

Recall we have two hyperhomology spectral sequences

$$E^2_{p,q}(n) = H_p(SO_{n,n}, H_q(C_*(n))) \Rightarrow H_{p+q}(SO_{n,n}, C_*(n)) \tag{4.1}$$

$$E^1_{p,q}(n) = H_q(SO_{n,n}, C_p(n)) \Rightarrow H_{p+q}(SO_{n,n}, C_*(n)). \tag{4.2}$$

Moreover, recall that in Theorem 3.1.9, we proved $H_q(C_*(n)) = 0$ for every $q \leq n-1$. As we expect the homological stability range for $SO_{n,n}$ to be the same as for $O_{n,n}$, a reasonable proof strategy is to localise spectral sequence (4.2) in the same manner as we did for the $O_{n,n}$ and analyse the localised spectral sequences. The analysis will turn out to be very similar to the $O_{n,n}$ case, except for the situation when $p = n$, corresponding to the fact that the action of $SO_{n,n}$ on $\mathcal{IU}_p(R^{2n})$ is *transitive* only for $p < n$, see lemma 4.1.2. But in the end, this will not prove to be too significant.

Note that for all $n > 0$, we have short exact sequences

$$1 \to SO_{n,n} \to O_{n,n} \to \mathbb{Z}_2 \to 1,$$

where the right arrow given is by the determinant map. Moreover, if we define $ST_k \leq T_k$ to be the subgroup of matrices in $T_k$ having determinant 1, the projection map $\rho : T_k \to O_{n-k,n-k}$ restricts to a map $\rho : ST_k \to SO_{n-k,n-k}$. Note that

$$\ker(\rho : T_k \to O_{n-k,n-k}) = \ker(\rho : ST_k \to SO_{n-k,n-k}),$$

since, by inspection on the matrices in $T_k$, we deduce that $\det A = \det \rho(A)$ for every $A \in T_k$, and that both kernels consist precisely of those matrices that map to the identity matrix. This observation will turn out to be significant in the forthcoming analysis. Furthermore, note that $ST_n = T_n$. We use the conventions that $SO_{0,0} = 1$

and $ST_0 = SO_{n,n}$.

We obtain short exact sequences for every $0 \leq k \leq n$.

$$1 \to L_k \to ST_k \to SO_{n-k,n-k} \to 1. \tag{4.3}$$

## 4.1 Local $R^*$-actions and transitivity

Define a local $R^*$-action on short exact sequence (4.3) in exactly the same was as we did in section 3.2.2.2, namely we conjugate matrices in $ST_k$ with the matrix $D_{a,k} \in SO_{n,n}$. As $\ker(\rho : T_k \to O_{n-k,n-k}) = \ker(\rho : ST_k \to SO_{n-k,n-k})$, the *exact* same reasoning as in section 3.2.2.2 can be used to conclude that, *after localistaion*, the homology of $ST_k$ and $SO_{n-k,n-k}$ coincide:

**Corollary 4.1.1.** *The inclusion $SO_{n-k,n-k} \hookrightarrow T_k$ induces isomorphisms*

$$H_t(SO_{n-k,n-k}) \xrightarrow{\cong} s_m^{-1} H_t(ST_k)$$

*for all $t < m/2$.*

Next, we study the transitivity of the $SO_{n,n}$ action on $\mathcal{IU}_p(R^{2n})$.

**Lemma 4.1.2.** *The action of $SO_{n,n}$ on $\mathcal{IU}_p(R^{2n})$ is transitive for all $p < n$.*

*Proof.* Let $(u_1, \ldots, u_p), (v_1, \ldots, v_p) \in \mathcal{IU}_p(R^{2n})$. By lemma 3.2.1, we deduce there exists an $u_1^\#, \ldots, u_p^\#$ such that $(u_1, u_1^\#, \ldots, u_p, u_p^\#)$ has Gram matrix $\psi_{2p}$, and which may be extended to hyperbolic basis $(u_1, u_1^\#, \ldots, u_p, u_p^\#, x_1, x_1^\#, \ldots, x_{n-p}, x_{n-p}^\#)$. Similarly, there exists an $v_1^\#, \ldots, v_p^\#$ such that $(v_1, v_1^\#, \ldots, v_p, v_p^\#)$ has Gram matrix $\psi_{2p}$, and which may be extended to hyperbolic basis $(v_1, v_1^\#, \ldots, v_p, v_p^\#, y_1, y_1^\#, \ldots, y_{n-p}, y_{n-p}^\#)$.

Let $B \in O_{n,n}$ be the matrix

$$B := (u_1 \quad u_1^\# \quad \cdots \quad u_p \quad u_p^\# \quad x_1 \quad x_1^\# \quad \cdots \quad x_{n-p} \quad x_{n-p}^\#),$$

and let $C \in O_{n,n}$ be the matrix

$$C := (v_1 \quad v_1^\# \quad \cdots \quad v_p \quad v_p^\# \quad y_1 \quad y_1^\# \quad \cdots \quad y_{n-p} \quad y_{n-p}^\#).$$

If $\det B = \det C$, then $CB^{-1} \in SO_{n,n}$ and maps $(u_1, \ldots, u_p)$ to $(v_1, \ldots, v_p)$. If $\det B \neq \det C$, then define $\hat{C} := CT$, where $T$ is the matrix that swaps $y_{n-p}$ and $y_{n-p}^\#$ in the columns of $C$. Then, as $\det T = -1$, it follows $\det B = \det \hat{C}$, and we are in the previous case. □

Recall that Shapiro's lemma gives an isomorphism

$$\bigoplus_{[x]\in S/G} (i_x, x)_* : \bigoplus_{[x]\in S/G} H_*(G_x, \mathbb{Z}) \xrightarrow{\cong} H_*(G, \mathbb{Z}[S])$$

of homology groups, where the direct sum is over a set of representatives $x \in S$ of equivalence classes $[x] \in S/G$; the group $G_x$ is the *stabiliser* of $G$ at $x \in S$; the homomorphism $i_x : G_x \subseteq G$ is the inclusion; and $x$ also denotes the homomorphism of abelian groups $\mathbb{Z} \to \mathbb{Z}[S] : 1 \mapsto x$.

We apply Shapiro's lemma in the case $G = SO_{n,n}(R)$ and $S = \mathcal{IU}_p(R^{2n})$.

Thus, by lemma 4.1.2 we have identification isomorphisms given by Shapiro's lemma for every, $0 \leq p < n$ and $q \geq 0$

$$H_q(ST_p) \xrightarrow{\cong} H_q(SO_{n,n}, C_p(n)). \tag{4.4}$$

For $p = n$, we claim that the action of $SO_{n,n}$ on $\mathcal{IU}_n(R^{2n})$ has two orbits:

**Proposition 4.1.3.** *For $n \geq 1$, the action of $SO_{n,n}$ on $\mathcal{IU}_n(R^{2n})$ has orbits corresponding to $\mathbb{Z}_2$.*

*Proof.* We know by lemma 3.2.1 that the action of $O_{n,n}$ on $\mathcal{IU}_n(R^{2n})$ is transitive, so that we have an isomorphism of $O_{n,n}-$sets

$$O_{n,n}/T_n \cong \mathcal{IU}_n(R^{2n}).$$

Furthermore, note that $T_n = ST_n \leq SO_{n,n} \leq O_{n,n}$, so that we have a canonical surjection $O_{n,n}/T_n \to O_{n,n}/SO_{n,n}$ with fibre $SO_{n,n}/T_n$. Therefore, we have an isomorphism of $O_{n,n}$-sets $\mathcal{IU}_n(R^{2n})/SO_{n,n} \cong O_{n,n}/SO_{n,n}$. This gives us

$$\left|\mathcal{IU}_n(R^{2n})/SO_{n,n}\right| = |O_{n,n}/SO_{n,n}| = |\mathbb{Z}_2|,$$

where the last equality follows from the short exact sequence

$$1 \to SO_{n,n} \to O_{n,n} \to \mathbb{Z}_2 \to 1.$$

□

Therefore, Shapiro's lemma gives us an isomorphism

$$H_q(St(e_1, \ldots, e_n)) \oplus H_q(St(e_1, \ldots, e_{n-1}, f_n)) \xrightarrow{\cong} H_q(SO_{n,n}, C_n, (n)). \tag{4.5}$$

where $St(e_1, \ldots, e_{n-1}, f_n)$ denotes the stabiliser of $(e_1, \ldots, e_{n-1}, f_n)$ in $SO_{n,n}$, and the identification map is given by Shapiro's lemma. To ease notation, we define $\overline{T}_n := St(e_1, \ldots, e_{n-1}, f_n)$.

We will compute $\overline{T}_n$ and show that, *after localisation*, all non-zero homology groups of $\overline{T}_n$ vanish.

### 4.1.1 Computation of $\overline{T}_n$ and a local $R^*$ action

We first compute $\overline{T}_n$.

**Proposition 4.1.4.** *Matrices $A \in \overline{T}_n$ are of the form*

$$A = \begin{pmatrix} 1 & c_1^1 & 0 & c_2^1 & \cdots & 0 & c_{n-1}^1 & c_n^1 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c_1^2 & 1 & c_2^2 & \cdots & 0 & c_{n-1}^2 & c_n^2 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_1^{n-1} & 0 & c_2^{n-1} & \cdots & 1 & c_{n-1}^{n-1} & c_n^{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & c_1^n & 0 & c_2^n & \cdots & 0 & c_{n-1}^n & 0 & 1 \end{pmatrix}$$

*where $c_j^i \in R$, subject to the conditions*

$$c_j^i + c_i^j = 0. \tag{4.6}$$

*Proof.* Let $A \in \overline{T}_n$. Then, $Ae_1 = e_1, \ldots, Ae_{n-1} = e_{n-1}$ and $Af_n = f_n$. This gives the 1st, 3rd, ...., $(2n-3)$rd and $2n$-th column of $A$. Moreover, for a fixed $1 \leq i < n$ and for any $1 \leq j < n$, we have

$$\langle e_i, Af_j \rangle = \langle Ae_i, Af_j \rangle = \langle e_i, f_j \rangle = \delta_{ij}$$

and

$$\langle e_i, Ae_n \rangle = \langle Ae_i, Ae_n \rangle = \langle e_i, e_n \rangle = 0.$$

This gives the 2nd, 4th, $\ldots$, $(2n-2)$nd rows of $A$.

Furthermore, note that for $1 \leq j < n$,

$$\langle f_n, Af_j \rangle = \langle Af_n, Af_j \rangle = \langle f_n, f_j \rangle = 0,$$

$$\langle f_n, Ae_j \rangle = \langle Af_n, Ae_j \rangle = \langle f_n, e_j \rangle = 0$$

and

$$\langle f_n, Ae_n \rangle = \langle Af_n, Ae_n \rangle = \langle f_n, e_n \rangle = 1.$$

This gives the $(2n-1)$th row of $A$. Filling in the remaining entries by constants to be determined, we have that

$$A = \begin{pmatrix} 1 & c_1^1 & 0 & c_2^1 & \cdots & 0 & c_{n-1}^1 & c_n^1 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c_1^2 & 1 & c_2^2 & \cdots & 0 & c_{n-1}^2 & c_n^2 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_1^{n-1} & 0 & c_2^{n-1} & \cdots & 1 & c_{n-1}^{n-1} & c_n^{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & c_1^n & 0 & c_2^n & \cdots & 0 & c_{n-1}^n & \Delta & 1 \end{pmatrix}$$

where $c_j^i, \Delta \in R$.

We use the equation ${}^tA\psi_{2n}A = \psi_{2n}$ to determine the conditions on these variables. Specifically, we have that

$$ {}^tA\psi A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c_1^1 & 1 & c_1^2 & 0 & \cdots & c_1^{n-1} & 0 & 0 & c_1^n \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c_2^1 & 0 & c_2^2 & 1 & \cdots & c_2^{n-1} & 0 & 0 & c_2^n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ c_{n-1}^1 & 0 & c_{n-1}^2 & 0 & \cdots & c_{n-1}^{n-1} & 1 & 0 & c_{n-1}^n \\ c_n^1 & 0 & c_n^2 & 0 & \cdots & c_n^{n-1} & 0 & 1 & \Delta \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix} \psi \begin{pmatrix} 1 & c_1^1 & 0 & c_2^1 & \cdots & 0 & c_{n-1}^1 & c_n^1 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c_1^2 & 1 & c_2^2 & \cdots & 0 & c_{n-1}^2 & c_n^2 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_1^{n-1} & 0 & c_2^{n-1} & \cdots & 1 & c_{n-1}^{n-1} & c_n^{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & c_1^n & 0 & c_2^n & \cdots & 0 & c_{n-1}^n & \Delta & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & c_1^1 + c_1^1 & 0 & c_2^1 + c_1^2 & \cdots & 0 & c_{n-1}^1 + c_1^{n-1} & c_n^1 + c_1^n & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c_2^1 + c_1^2 & 1 & c_2^2 + c_2^2 & \cdots & 0 & c_{n-1}^2 + c_2^{n-1} & c_n^2 + c_2^n & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & c_{n-1}^1 + c_1^{n-1} & 0 & c_{n-1}^2 + c_2^{n-1} & \cdots & 1 & c_{n-1}^{n-1} + c_{n-1}^{n-1} & c_n^{n-1} + c_{n-1}^n & 0 \\ 0 & c_n^1 + c_1^n & 0 & c_n^2 + c_2^n & \cdots & 0 & c_n^{n-1} + c_{n-1}^n & 2\Delta & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \psi.$$

Thus, we conclude $\Delta = 0$ necessarily and we derive the required equations. □

We now define a local $R^*$ action on $\overline{T}_n$. This will be slightly different from the local actions on $T_k$. We will show that, *after localisation*, the non-zero homology groups of $\overline{T}_n$ *vanish*. Eventually, we will show the global action considered in subsection 3.2.2.3 induces this local action on $\overline{T}_n$.

**Definition 4.1.5** (Local action)**.** Let $a \in R^*$. Define a $2n \times 2n$ matrix $\overline{D}_{a,n}$ by

$$\overline{D}_{a,n} := \begin{pmatrix} a & & & & & & \\ & a^{-1} & & & & & \\ & & \ddots & & & & \\ & & & a & & & \\ & & & & a^{-1} & & \\ & & & & & a^{-1} & \\ & & & & & & a \end{pmatrix} = \left(\bigoplus_{1}^{n-1} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) \bigoplus \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}.$$

Note that $\overline{D}_{a,n} \in SO_{n,n}(R)$. The *local action* of $R^*$ on $\overline{T}_n$ is the conjugation action of $\overline{D}_{a,n}$ on $\overline{T}_n$.

The local action preserves $\overline{T}_n$ because

$$\overline{D}_{a,n} \begin{pmatrix} 1 & c_1^1 & 0 & c_2^1 & \cdots & 0 & c_{n-1}^1 & c_n^1 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c_1^2 & 1 & c_2^2 & \cdots & 0 & c_{n-1}^2 & c_n^2 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_1^{n-1} & 0 & c_2^{n-1} & \cdots & 1 & c_{n-1}^{n-1} & c_n^{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & c_1^n & 0 & c_2^n & \cdots & 0 & c_{n-1}^n & 0 & 1 \end{pmatrix} \overline{D}_{a,n}^{-1}$$

$$= \begin{pmatrix} 1 & a^2c_1^1 & 0 & a^2c_2^1 & \cdots & 0 & a^2c_{n-1}^1 & a^2c_n^1 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a^2c_1^2 & 1 & a^2c_2^2 & \cdots & 0 & a^2c_{n-1}^2 & a^2c_n^2 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & a^2c_1^{n-1} & 0 & a^2c_2^{n-1} & \cdots & 1 & a^2c_{n-1}^{n-1} & a^2c_n^{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & a^2c_1^n & 0 & a^2c_2^n & \cdots & 0 & a^2c_{n-1}^n & 0 & 1 \end{pmatrix} \in \overline{T}_n.$$

We show that localising with respect to the elements $s_m$ *kills* the non-zero homology groups of $\overline{T}_n$ when $m$ is taken to infinity. This is used to make the identifications in corollary 4.2.3.

**Lemma 4.1.6.** *We have* $s_m^{-1}H_0(\overline{T}_n) = \mathbb{Z}$ *and for all* $1 \leq 2q < m$, $s_m^{-1}H_q(\overline{T}_n) = 0$.

*Proof.* We claim there is a short exact sequence of groups

$$1 \to (R^{\binom{n}{2}}, +) \to \overline{T}_n \to 1 \to 1. \tag{4.7}$$

The first arrow maps

$$(c_1, \dots) \mapsto A_{(c_1,\dots)}$$

where $A_{(c_1,\dots)} \in \overline{T}_n$ is defined by conditions (4.6) (with some ordering specified beforehand). The second maps $A \in \overline{T}_n$ to its bottom right identity matrix. One may check that $\overline{T}_n$ is abelian, and these arrows define a short exact sequence of abelian groups.

Furthermore, this short exact sequence of abelian groups is $R^*$-equivariant where $b \in R^*$ acts on $(R^{\binom{n}{2}}, +)$ via pointwise multiplication by $b^2$, the element $b \in R^*$ acts on $\overline{T}_n$ via conjugation by $\overline{D}_{b,n}$ and the action of $b$ on 1 is taken to be trivial.

By Proposition 3.2.4, $s_m^{-1}H_q(\overline{T}_n) = 0$ for all $1 \leq 2q < m$. The equality $s_m^{-1}H_0(\overline{T}_n) = \mathbb{Z}$ follows from fact that $R^*$ acts trivially on $H_0(\overline{T}_n)$. □

## 4.2 A global action on the $SO_{n,n}$ spectral sequence

As before, we want to realise these 'local actions' as a 'global action' on the spectral sequence

$$E^1_{p,q}(n) = H_q(SO_{n,n}, C_p(n)) \Rightarrow H_{p+q}(SO_{n,n}, C_*(n)). \tag{4.8}$$

Again, we do this by defining an action on the associated exact couple with abutment. Specifically, the spectral sequence

$$E^1_{p,q} = H_q(SO_{n,n}, C_p(n)) \Rightarrow H_{p+q}(SO_{n,n}, C_*(n))$$

may be obtained from the exact couple with abutment

$$\begin{array}{ccccc} \bigoplus_{p,q} E^1_{p,q} & \xrightarrow{k} & \bigoplus_{p,q} D^1_{p,q} & \xrightarrow{\sigma} & \bigoplus_{p+q} A_{p+q} \\ & \nwarrow^{j} & \downarrow i & \nearrow^{\sigma} & \\ & & \bigoplus_{p,q} D^1_{p,q} & & \end{array} \tag{4.9}$$

with $E^1_{p,q} = H_{p+q}(SO_{n,n}, C_{\leq p}(n)/C_{\leq p-1}(n))$; $D^1_{p,q} = H_{p+q}(SO_{n,n}, C_{\leq p}(n))$; $A_{p+q} = H_{p+q}(SO_{n,n}, C_*(n))$; the maps $i, j, k$ being the maps of the long exact sequence of homology groups associated to the short exact sequence of complexes

$$0 \to C_{\leq p-1}(n) \to C_{\leq p}(n) \to C_{\leq p}(n)/C_{\leq p-1}(n) \to 0,$$

and $\sigma$ is induced by the inclusion.

For $a \in R^*$, we define the global action on spectral sequence (4.8) to be the action induced by the map $(C_{B_{a^{-2}}}, B_{a^{-2}})$ on exact couple (4.9), where $C_{B_{a^{-2}}}$ denotes conjugation by the matrix $B_{a^{-2}}$ of section 3.2.2.3, and $B_{a^{-2}}$ also refers to multiplication by this matrix.

As $D_{a,k} \in SO_{n,n}$ for every $0 \leq k \leq n$, the proof of proposition 3.2.12 may be used to prove following proposition continues to hold for $SO_{n,n}$.

**Proposition 4.2.1.** *Let $k, q \geq 0$ and $n \geq 1$. Then, for all $a \in R^*$, the following diagram commutes:*

$$\begin{array}{ccc} H_q(SO_{n,n}, C_k(n)) & \xrightarrow{(C_{B_{a^{-2}}}, B_{a^{-2}})_*} & H_q(SO_{n,n}, C_k(n)) \\ {\scriptstyle id}\uparrow & & \uparrow{\scriptstyle id} \\ H_q(SO_{n,n}, C_k(n)) & \xrightarrow[(id,\phi_a)_*]{} & H_q(SO_{n,n}, C_k(n)), \end{array}$$

*where for $a \in R^*$, the map*

$$(id, \phi_a) : (SO_{n,n}, C_k(n)) \to (SO_{n,n}, C_k(n))$$

*is defined to be the identity on $SO_{n,n}$ and on basis elements of $C_k(n)$ as*

$$\phi_a : (v_1, \ldots, v_k) \mapsto (a^{-1}v_1, \ldots, a^{-1}v_k).$$

For the next proposition, we need to treat the case $k = n$ separately:

**Proposition 4.2.2.** *Let $q \geq 0$ and $0 \leq k < n$. Then, for all $a \in R^*$, the following diagram commutes:*

$$\begin{array}{ccc} H_q(SO_{n,n}, C_k(n)) & \xrightarrow{(id,\phi_a)_*} & H_q(SO_{n,n}, C_k(n)) \\ {\scriptstyle (i,(e_1,\ldots,e_k))_*}\uparrow\cong & & \cong\uparrow{\scriptstyle (i,(e_1,\ldots,e_k))_*} \\ H_q(ST_k, \mathbb{Z}) & \xrightarrow[C_{D_{a,k}}]{} & H_q(ST_k, \mathbb{Z}), \end{array}$$

*where the vertical arrows are the isomorphisms given by Shapiro's lemma and the* *map $C_{D_{a,k}}$ denotes the map induced by conjugation with the element $D_{a,k}$ on the stabiliser $ST_k$.*

*Moreover, for $q \geq 0$ and $k = n$, the following diagram commutes for all $a \in R^*$:*

$$\begin{array}{ccc} H_q(SO_{n,n}, C_n(n)) & \xrightarrow{(id,\phi_a)_*} & H_q(SO_{n,n}, C_n(n)) \\ {\scriptstyle (i,(e_1,\dots,e_n))_* \oplus (i,(e_1,\dots,f_n))_*} \uparrow \cong & & \cong \uparrow {\scriptstyle (i,(e_1,\dots,e_n))_* \oplus (i,(e_1,\dots,f_n))_*} \\ H_q(T_n, \mathbb{Z}) \oplus H_q(\overline{T}_n, \mathbb{Z}) & \xrightarrow[C_{D_{a,n}} \oplus C_{\overline{D}_{a,n}}]{} & H_q(T_n, \mathbb{Z}) \oplus H_q(\overline{T}_n, \mathbb{Z}), \end{array}$$

*where the vertical arrows are the isomorphisms given by Shapiro's lemma and the map $C_{D_{a,n}} \oplus C_{\overline{D}_{a,n}}$ denotes the map induced by conjugation with the element $D_{a,n}$ on the stabiliser $T_n$ sum with the map induced by conjugation with the element $\overline{D}_{a,n}$ on the stabiliser $\overline{T}_n$ .*

*Proof.* The proof of the first half of the proposition is exactly the same as the $O_{n,n}$ case, since $D_{a,k} \in SO_{n,n}$. See proposition 3.2.13. The proof that the first component commutes is exactly the same as the $O_{n,n}$ case, since $D_{a,n} \in SO_{n,n}$. For the second component, consider diagram

$$(\overline{T}_n, \mathbb{Z}) \underset{(f_2,\varphi_2)}{\overset{(f_1,\varphi_1)}{\rightrightarrows}} (SO_{n,n}, C_n(n))$$

where $(f_1, \varphi_1) := (i, (a^{-1}e_1, \dots, a^{-1}f_n))$ and $(f_2, \varphi_2) := (iC_{\overline{D}_{a,n}}, (e_1, \dots, f_n))$, and $i : \overline{T}_n \to SO_{n,n}$ is the natural inclusion of groups. Let $\kappa = \overline{D}_{a,n} \in SO_{n,n}$. Then, for every $A \in \overline{T}_n$,

$$f_2(A) = iC_{\overline{D}_{a,n}}(A) = \overline{D}_{a,n} A \overline{D}_{a,n}^{-1} = \overline{D}_{a,n} i(A) \overline{D}_{a,n}^{-1} = \kappa f_1(A) \kappa^{-1}$$

and

$$(e_1 \dots, f_n) = \overline{D}_{a,n}(a^{-1}e_1, \dots, a^{-1}f_n) = \kappa(a^{-1}e_1, \dots, a^{-1}f_n).$$

By lemma 3.2.9, the diagram commutes. □

Thus, we have shown that there exists an $R^*$-action on the spectral sequence

$$E^1_{p,q}(n) = H_q(SO_{n,n}, C_p(n)) \Rightarrow H_{p+q}(SO_{n,n}, C_*(n))$$

which induces the desired local actions considered previously. Putting everything together, we obtain the following.

**Corollary 4.2.3.** *For every $m \geq 1$ and every $q < m/2$, the localised spectral sequence*

$$ {}_mE^1_{p,q}(n) = s_m^{-1}E^1_{p,q}(n) \Rightarrow s_m^{-1}H_{p+q}(SO_{n,n}, C_*(n)) \tag{4.10}$$

*has ${}_mE^1_{p,q}$ terms*

$$ {}_mE^1_{p,q}(n) = s_m^{-1}H_q(SO_{n,n}, C_p(n)) \cong \begin{cases} H_q(SO_{n-p,n-p}) & 0 \leq p < n \\ \mathbb{Z}[\mathbb{Z}_2] & p = n, q = 0 \\ 0 & p = n, q > 0. \end{cases}$$

□

Our next task is to compute the localised $d^1$ differentials $d^1 : {}_mE^1_{p,q} \to {}_mE^1_{p-1,q}$.

## 4.3 Computation of the localised $d^1$ differentials, and proof of homological stability

**Proposition 4.3.1.** *For all $q < m/2$ and $0 \leq p < n$, the homomorphism $d^1_{p,q}$ is*

$$ d^1_{p,q} = \begin{cases} 0, & p \text{ even} \\ i_*, & p \text{ odd}, \end{cases}$$

*where $i : SO_{n-p,n-p} \hookrightarrow SO_{n-p+1,n-p+1}$ denotes the inclusion. For $p = n$, the homomorphism $d^1_{n,q}$ is 0 if $q > 0$ or if $n$ is even; and for $q = 0$ and $n$ odd, $d^1_{n,0}$ is the augmentation map $\varepsilon : \mathbb{Z}[\mathbb{Z}_2] \to \mathbb{Z}$.*

*Proof.* For all $p < n$, we want to show that the following diagram commutes:

$$\begin{array}{ccc} H_q(SO_{n-p,n-p}) & \xrightarrow{(i,(e_1,\ldots,e_p))_*} & H_q(SO_{n,n}, C_p(n)) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle (d_i)_*} \\ H_q(SO_{n-p+1,n-p+1}) & \xrightarrow[(i,(e_1,\ldots,e_{p-1}))_*]{} & H_q(SO_{n,n}, C_{p-1}(n)), \end{array} \tag{4.11}$$

where $i : SO_{n-p,n-p} \hookrightarrow SO_{n,n}$ denotes the inclusion map; $(e_1, \ldots, e_p) : 1 \mapsto (e_1, \ldots, e_p)$ and recall that $d_i(v_1, \ldots, v_p) = (v_1, \ldots, \hat{v}_i \ldots, v_p)$.

The same proof as in proposition 3.2.15 will work, so long as we can show $\sigma$

has determinant 1. Note that permutation matrix $\sigma$ will be of the form

$$\sigma = (e_1 \quad f_1 \quad \cdots \quad e_{i-1} \quad f_{i-1} \quad e_p \quad f_p \quad e_i \quad f_i \cdots \quad e_{p-1} \quad f_{p-1}).$$

From this, we may write $id = \sigma \cdot T_1 \cdot T_2 \cdots T_{2(p-i)}$, where the matrices $T_i$ are the elementary matrices needed to swap columns in $\sigma$ to transform it into the identity matrix. Note that each of these matrices has determinant $-1$, and there are an even number of such matrices.

Thus, we deduce $1 = \det(id) = \det(\sigma \cdot T_1 \cdot T_2 \cdots T_{2(p-i)}) = (-1)^{2(p-i)} \det(\sigma) = \det(\sigma)$.

For $p = n$, it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}_2] & \xrightarrow{1 \mapsto (e_1,\dots,e_n), \sigma \mapsto (e_1,\dots,f_n)} & C_n(n)_{SO_{n,n}} \\ \varepsilon \downarrow & & \downarrow d_i \\ \mathbb{Z} & \xrightarrow[1 \mapsto (e_1,\dots,e_{n-1})]{} & C_{n-1}(n)_{SO_{n,n}}. \end{array}$$

For $i = n$, it is easy to see by inspection that the diagram commutes.

For $1 \leq i < n$, commutativity follows from the fact that $SO_{n,n}$ acts transitively on $\mathcal{IU}_{n-1}(R^{2n})$.

These diagrams still commutes after localisation, but now the horizontal arrows become the identification isomorphisms. □

We need to prove the following:

**Proposition 4.3.2.** *The differentials $d^r_{p,q}$ in spectral sequence (4.10) are zero for $r \geq 2$ and $q < m/2$, $p \leq n$. Hence, for all $q < m/2$ and $p \leq n$, ${}_mE^2_{p,q} \cong {}_mE^\infty_{p,q}$.*

*Proof.* For $n = 0, 1$, the spectral sequence under consideration is located in columns 0 and 1. Therefore, the differentials $d^r$ for $r \geq 2$ are zero by dimension arguments.

For $n \geq 2$, consider the homomorphism of complexes of $SO_{n-2,n-2}$-modules

$$\tau : C_*(n-2)[-2] \to C_*(n).$$

as defined in proposition 3.2.17. Note that the diagram

$$\begin{array}{ccc} (SO_{n-2,n-2}, C_{p-2}(n-2)) & \xrightarrow{(i,\tau_j)} & (SO_{n,n}, C_p(n)) \\ {\scriptstyle (C_{B_a},B_a)}\uparrow & & \uparrow{\scriptstyle (C_{B_a},B_a)} \\ (SO_{n-2,n-2}, C_{p-2}(n-2)) & \xrightarrow[(i,\tau_j)]{} & (SO_{n,n}, C_p(n)), \end{array}$$

still commutes, so that we have an induced map on localised spectral sequences

$$ {}_m\tau_* : {}_m\tilde{E} \to {}_mE. $$

The claim would then follow by induction on $r$ using the following lemma:

**Lemma 4.3.3.** *The map ${}_m\tau_* : {}_m\tilde{E}^1_{p,q} \to {}_mE^1_{p,q}$ is the identity for all $q < m/2$ and $2 \leq p \leq n$.*

*Proof.* For $2 \leq p < n$, the same proof as in lemma 3.2.18 works, as the matrices $A$ and $B$ in lemma 3.2.18 have determinant 1. Thus, we only need to consider the case $p = n$.

It suffices to show that for $j = 0, 1, 2$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{Z}[\mathbb{Z}_2] & \xrightarrow{1 \mapsto (e_1,\ldots,e_{n-2}),\sigma \mapsto (e_1,\ldots,f_{n-2})} & C_{n-2}(n-2)_{SO_{n-2,n-2}} \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \tau_j} \\
\mathbb{Z}[\mathbb{Z}_2] & \xrightarrow[1 \mapsto (e_1,\ldots,e_n),\sigma \mapsto (e_1,\ldots,f_n)]{} & C_n(n)_{SO_{n,n}}.
\end{array}
$$

For $j = 0$, the diagram is easily seen to commute by inspection.

For $j = 1$, we have that

$$
\begin{aligned}
A(e_1, \ldots, e_n) &= (e_1, e_2 - e_1, e_3, \ldots, e_n) \\
A(e_1, \ldots, f_n) &= (e_1, e_2 - e_1, e_3, \ldots, f_n),
\end{aligned}
$$

where $A \in SO_{n,n}$ is the matrix $A$ in the proof of lemma 3.2.18.

Similarly, for $j = 2$, we have that

$$
\begin{aligned}
B(e_1, \ldots, e_n) &= (e_2, e_2 - e_1, e_3, \ldots, e_n) \\
B(e_1, \ldots, f_n) &= (e_2, e_2 - e_1, e_3, \ldots, f_n),
\end{aligned}
$$

where $B \in SO_{n,n}$ is the matrix $B$ in the proof of lemma 3.2.18.

Thus, the diagrams commute. These diagrams still commute after localisation, but now the horizontal maps become the identification isomorphisms. □

This proves the lemma, and thus proposition 4.3.2. □

**Theorem 4.3.4.** *Let $R$ be a commutative local ring with infinite residue field such that $2 \in R^*$. Then, the natural homomorphism*

$$ H_k(SO_{n,n}(R)) \longrightarrow H_k(SO_{n+1,n+1}(R)) $$

*is an isomorphism for $k \leq n-1$ and surjective for $k \leq n$.*

**Remark 25.** This is the first known homological stability result for $SO_{n,n}$ over a local ring and generalises the result obtained by Essert [Ess13] for infinite fields.

*Proof.* Choose $m > 0$ sufficiently large. We have a spectral sequence (4.10) with $E^1$-terms given by corollary 4.2.3 and $d^1_{p,q}$ was computed for all $q < m/2$ in proposition 4.3.1. From theorem 3.1.9, spectral sequences (4.1) and (4.10) and proposition 4.3.2, we deduce ${}_mE^2_{p,q} = {}_mE^\infty_{p,q}$ for all $p+q \leq n-1$ and $q < m/2$. The theorem follows. □

# Chapter 5

# Homological Stability for $EO_{n,n}$ and $\mathrm{Spin}_{n,n}$

We define $EO_{n,n}$ as follows.

**Definition 5.0.1.** Define $EO_{n,n}$ to be the image of the map

$$EO_{n,n} := \mathrm{Im}(\pi : \mathrm{Spin}_{n,n} \longrightarrow SO_{n,n}),$$

where $\pi : \mathrm{Spin}_{n,n} \longrightarrow SO_{n,n}$ is the canonical map from the Spin group into the special orthogonal group.

From this definition, we see that $EO_{n,n}$ sits inside short exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_{n,n} \xrightarrow{\pi} EO_{n,n} \to 1.$$

We will study the homological stability of $EO_{n,n}$ and will later apply the relative Hochschild-Serre spectral sequence to deduce a homological stability result for $\mathrm{Spin}_{n,n}$.

**Remark 26.** Our definition agrees with the usual definition of the elementary orthogonal group $EO_{n,n}$ by [HO89, Theorem 9.2.8].

**Remark 27.** Note that [HO89, Theorem 9.2.8] as stated is true for $n \geq 2$. For $n = 1$, we use the convention that $EO_{1,1}(R) = R^{*2}$, so that the above short exact sequence is still true.

To prove homological stability of $EO_{n,n}$, we will study the hyperhomology spectral sequences

$$E^2_{p,q}(n) = H_p(EO_{n,n}, H_q(C_*(n))) \Rightarrow H_{p+q}(EO_{n,n}, C_*(n)) \tag{5.1}$$

$$E^1_{p,q}(n) = H_q(EO_{n,n}, C_p(n)) \Rightarrow H_{p+q}(EO_{n,n}, C_*(n)). \tag{5.2}$$

As the action of $EO_{n,n}$ on $\mathcal{IU}_p(R^{2n})$ is *transitive* only for $p < n$, see lemma 5.1.1, it is reasonable to expect that the analysis for the $EO_{n,n}$ case should be similar to the $SO_{n,n}$ case. This is indeed what happens.

By Theorem 2.4.21, $EO_{n,n}$ also sits inside the short exact sequence

$$1 \to EO_{n,n} \to SO_{n,n} \xrightarrow{\theta} R^*/R^{*2} \to 1,$$

where the first arrow is the inclusion and the second arrow $\theta$ is the *spinor norm* (this short exact sequence is also true for $n = 1$, given our convention $EO_{1,1}(R) = R^{*2}$). We refer the reader to the preliminaries for more details about the spinor norm. See also [Sch12] and [HO89] as additional references. From this short exact sequence, note that we have inclusion $[SO_{n,n}, SO_{n,n}] \subseteq EO_{n,n}$, where $[SO_{n,n}, SO_{n,n}]$ denotes the commutator subgroup of $SO_{n,n}$. Therefore, to prove a matrix is in $EO_{n,n}$, it will be sufficient to prove it is in $[SO_{n,n}, SO_{n,n}]$. This will be very convenient for us.

## 5.1 Transitivity and local $R^*$-actions

We want to prove that the canonical action of $EO_{n,n}$ on $\mathcal{IU}_p(R^{2n})$ is transitive for all $p < n$.

**Lemma 5.1.1.** *The action of $EO_{n,n}$ on $\mathcal{IU}_p(R^{2n})$ is transitive for all $p < n$.*

*Proof.* Let $(v_1, \ldots, v_p) \in \mathcal{IU}_p(R^{2n})$. It suffices to show that there exists an $A \in EO_{n,n}$ such that $A(e_1, \ldots, e_p) = (v_1, \ldots, v_p)$.

We know by lemma 4.1.2 that the action of $SO_{n,n}$ is transitive for all $p < n$. Therefore, there exists a $B \in SO_{n,n}$ such that $B(e_1, \ldots, e_p) = (v_1, \ldots, v_p)$.

Furthermore, note that we have surjections $ST_p = \mathrm{Stab}_{SO_{n,n}}(e_1, \ldots, e_p) \twoheadrightarrow SO_{n-p,n-p}$ and $SO_{n-p,n-p} \twoheadrightarrow R^*/R^{*2}$.

Therefore, we deduce that there exists a $C \in ST_p \leq SO_{n,n}$ such that $\theta(BC) = \theta(B)\theta(C) = 1$ and $BC(e_1, \ldots, e_p) = B(e_1, \ldots, e_p) = (v_1, \ldots, v_p)$.

As $\theta(BC) = 1$, we have that $BC \in \ker(\theta) = EO_{n,n}$. We may thus set $A := BC$. □

Define $ET_k := \mathrm{Stab}_{EO_{n,n}}(e_1, \ldots, e_k)$. Note that $ET_k$ is precisely $\ker(ST_k \xrightarrow{\theta}$

$R^*/R^{*2}$). This gives us the following diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & ET_k & \longrightarrow & ST_k & \xrightarrow{\theta} & R^*/R^{*2} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \rho & & \downarrow = & & \\ 1 & \longrightarrow & EO_{n-k,n-k} & \longrightarrow & SO_{n-k,n-k} & \xrightarrow{\theta} & R^*/R^{*2} & \longrightarrow & 1 \end{array},$$

where the existence of the dashed arrow for all $n \geq 2$ will follow if we can show that the right square commutes (the map trivially exists for $n = 1$). Let us prove this.

**Proposition 5.1.2.** *For every $n \geq 2$, the square*

$$\begin{array}{ccc} ST_k & \xrightarrow{\theta} & R^*/R^{*2} \\ \downarrow & & \downarrow = \\ SO_{n-k,n-k} & \xrightarrow{\theta} & R^*/R^{*2} \end{array}$$

*commutes.*

*Proof.* Let $n \geq 2$. Recall that $[SO_{n,n}, SO_{n,n}] \subseteq EO_{n,n}$. Let $A \in ST_k$ such that $\rho(A) = B$. Want to show $\theta(A) = \theta\left(\begin{pmatrix} 1_{2k} & \\ & B \end{pmatrix}\right)$. Equivalently, want to show $\theta\left(\begin{pmatrix} 1_{2k} & \\ & B^{-1} \end{pmatrix} A\right) = 1$. Therefore, we want to show that $\begin{pmatrix} 1_{2k} & \\ & B^{-1} \end{pmatrix} A \in [SO_{n,n}, SO_{n,n}] \subseteq EO_{n,n}$.

Note that $\begin{pmatrix} 1_{2k} & \\ & B^{-1} \end{pmatrix} A \in L_k$, therefore it suffices to prove that $L_k \subseteq [SO_{n,n}, SO_{n,n}]$.

The inclusion $L_k \hookrightarrow SO_{n,n}$ induces a map on homology $H_q(L_k) \to H_q(SO_{n,n})$. We claim this is the zero map for every $q \geq 1$.

Recall from lemma 3.2.5 that $s_m^{-1} H_q(L_k) = 0$ for every $1 \leq 2q < m$ and note that $s_m^{-1} H_q(SO_{n,n}) = H_q(SO_{n,n})$, as the $R^*$-action defining this localization is trivial on $H_q(SO_{n,n})$.

Taking $m$ sufficiently large, we obtain for every $q \geq 1$ commutative diagrams

$$\begin{array}{ccc} H_q(L_k) & \longrightarrow & H_q(SO_{n,n}) \\ \downarrow & & \downarrow = \\ s_m^{-1} H_q(L_k) = 0 & \longrightarrow & s_m^{-1} H_q(SO_{n,n}) = H_q(SO_{n,n}) \end{array}.$$

We therefore deduce that $H_q(L_k) \to H_q(SO_{n,n})$ is the zero map for every $q \geq 1$. In particular, as $H_1$ corresponds to taking abelianization, we have that the diagram

$$\begin{array}{ccc} L_k & \hookrightarrow & SO_{n,n} \\ \downarrow & & \downarrow \\ L_k/[L_k, L_k] & \xrightarrow{0} & SO_{n,n}/[SO_{n,n}, SO_{n,n}] \end{array}$$

commutes. Therefore, we conclude that the inclusion $L_k \hookrightarrow SO_{n,n}$ factors through $[SO_{n,n}, SO_{n,n}]$. ☐

Thus, the projection map $\rho : ST_k \twoheadrightarrow SO_{n-k,n-k}$ induces a map $\rho : ET_k \twoheadrightarrow EO_{n-k,n-k}$. Moreover, the above proof shows that $ET_n = ST_n = T_n$ and $L_k = \ker(\rho : ET_k \twoheadrightarrow EO_{n-k,n-k})$, so that we have short exact sequence

$$1 \to L_k \to ET_k \to EO_{n-k,n-k} \to 1.$$

The associated Hochschild-Serre spectral sequence is

$$E^2_{p,q} = H_p(EO_{n-k,n-k}; H_q(L_k)) \Rightarrow H_{p+q}(ET_k).$$

Knowing that $ET_n = ST_n = T_n$ allows us to prove the following proposition:

**Proposition 5.1.3.** *For $n \geq 1$, the action of $EO_{n,n}$ on $\mathcal{IU}_n(R^{2n})$ has orbits corresponding to $R^*/R^{*2} \times \mathbb{Z}_2$.*

*Proof.* We know by lemma 3.2.1 that the action of $O_{n,n}$ on $\mathcal{IU}_n(R^{2n})$ is transitive, so that we have an isomorphism of $O_{n,n}-$sets

$$O_{n,n}/T_n \cong \mathcal{IU}_n(R^{2n}).$$

Furthermore, note that $T_n = ET_n \leq EO_{n,n} \leq O_{n,n}$, so that we have a canonical surjection $O_{n,n}/T_n \to O_{n,n}/EO_{n,n}$ with fibre $EO_{n,n}/T_n$. Therefore, we have an isomorphism of $O_{n,n}$-sets $\mathcal{IU}_n(R^{2n})/EO_{n,n} \cong O_{n,n}/EO_{n,n}$. This gives us

$$\left|\mathcal{IU}_n(R^{2n})/EO_{n,n}\right| = |O_{n,n}/EO_{n,n}| = \left|R^*/R^{*2} \times \mathbb{Z}_2\right|,$$

where the last equality follows from the short exact sequence

$$1 \to EO_{n,n} \to O_{n,n} \to R^*/R^{*2} \times \mathbb{Z}_2 \to 1, \tag{5.3}$$

see theorem 2.4.21. ☐

### 5.1.1 The local $R^*$-action

Note that for every $a \in R^*$,

$$\theta(D_{a^2}) = \theta\left(\begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix}\right) = 1,$$

so that

$$D_{a^2,k} := \begin{pmatrix} D_{a^2} & & & \\ & \ddots & & \\ & & D_{a^2} & \\ & & & 1_{2n-2k} \end{pmatrix} \in EO_{n,n}.$$

We will define the local action of $R^*$ on $ET_k$ to be the conjugation action of $D_{a^2,k}$ on $ET_k$.

Replacing every unit by its square where necessary in the proof of lemma 3.2.5 shows that:

**Corollary 5.1.4.** *The inclusion $EO_{n-k,n-k} \hookrightarrow ET_k$ induces isomorphisms*

$$H_t(EO_{n-k,n-k}) \xrightarrow{\cong} s_m^{-1} H_t(ET_k)$$

*for all $t < m/2$.*

## 5.2 A global action on the $EO_{n,n}$ spectral sequence

As before, we want to realise these local actions as a global action on the spectral sequence

$$E^1_{p,q}(n) = H_q(EO_{n,n}, C_p(n)) \Rightarrow H_{p+q}(EO_{n,n}, C_*(n)). \tag{5.4}$$

Again, we do this by defining an action on the associated exact couple with abutment. Specifically, the spectral sequence

$$E^1_{p,q} = H_q(EO_{n,n}, C_p(n)) \Rightarrow H_{p+q}(EO_{n,n}, C_*(n))$$

may be obtained from the exact couple with abutment

$$\begin{array}{ccccc} \bigoplus_{p,q} E^1_{p,q} & \xrightarrow{k} & \bigoplus_{p,q} D^1_{p,q} & \xrightarrow{\sigma} & \bigoplus_{p+q} A_{p+q} \\ & \nwarrow^{j} & \downarrow i & \nearrow^{\sigma} & \\ & & \bigoplus_{p,q} D^1_{p,q} & & \end{array} \tag{5.5}$$

with $E^1_{p,q} = H_{p+q}(EO_{n,n}, C_{\leq p}(n)/C_{\leq p-1}(n))$; $D^1_{p,q} = H_{p+q}(EO_{n,n}, C_{\leq p}(n))$; $A_{p+q} = H_{p+q}(EO_{n,n}, C_*(n))$; the maps $i, j, k$ being the maps of the long exact sequence of homology groups associated to the short exact sequence of complexes

$$0 \to C_{\leq p-1}(n) \to C_{\leq p}(n) \to C_{\leq p}(n)/C_{\leq p-1}(n) \to 0,$$

and $\sigma$ is induced by the inclusion.

For $a \in R^*$, we define the global action on spectral sequence (5.4) to be the action induced by the map $(C_{B_{a^{-4}}}, B_{a^{-4}})$ on exact couple (5.5), where $C_{B_{a^{-4}}}$ denotes conjugation by the matrix $B_{a^{-4}}$ of section 3.2.2.3, and $B_{a^{-4}}$ also refers to multiplication by this matrix.

**Proposition 5.2.1.** *Let $k, q \geq 0$ and $n \geq 1$. Then, for all $a \in R^*$, the following diagram commutes:*

$$\begin{array}{ccc} H_q(EO_{n,n}, C_k(n)) & \xrightarrow{(C_{B_{a^{-4}}}, B_{a^{-4}})_*} & H_q(EO_{n,n}, C_k(n)) \\ {\scriptstyle id}\uparrow & & \uparrow{\scriptstyle id} \\ H_q(EO_{n,n}, C_k(n)) & \xrightarrow[(id, \phi_{a^2})_*]{} & H_q(EO_{n,n}, C_k(n)), \end{array}$$

*where for $a \in R^*$, the map*

$$(id, \phi_{a^2}) : (EO_{n,n}, C_k(n)) \to (EO_{n,n}, C_k(n))$$

*is defined to be the identity on $EO_{n,n}$ and on basis elements of $C_k(n)$ as*

$$\phi_{a^2} : (v_1, \ldots, v_k) \mapsto (a^{-2}v_1, \ldots, a^{-2}v_k).$$

*Proof.* We use lemma 3.2.9. Specifically, consider the diagram

$$(EO_{n,n}, C_k(n)) \underset{(f_2,\varphi_2)}{\overset{(f_1,\varphi_1)}{\rightrightarrows}} (EO_{n,n}, C_k(n))$$

where $(f_1, \varphi_1) := (id, \phi_{a^2})$ and $(f_2, \varphi_2) := (C_{B_{a^{-4}}}, B_{a^{-4}})$. Define

$$\kappa := D_{a^2,n} = \begin{pmatrix} a^2 & & & & \\ & a^{-2} & & & \\ & & \ddots & & \\ & & & a^2 & \\ & & & & a^{-2} \end{pmatrix}.$$

Denoting for $a \in R^*$,

$$\underline{a} := \begin{pmatrix} a & & & & \\ & a & & & \\ & & \ddots & & \\ & & & a & \\ & & & & a \end{pmatrix},$$

note that $B_{a^{-4}} = \kappa \underline{a}^{-2}$, so that $C_{B_{a^{-4}}} = C_\kappa C_{\underline{a}^{-2}}$. But, $C_{\underline{a}^{-2}} = id$, so that $C_{B_{a^{-4}}} = C_\kappa$. Furthermore, note that for every $(v_1, \ldots, v_k) \in \mathcal{IU}_k(R^{2n})$, $B_{a^{-4}}(v_1, \ldots, v_k) = \kappa\phi_{a^2}(v_1, \ldots, v_k)$, since $B_{a^{-4}} = \kappa\underline{a}^{-2}$. Thus, by lemma 3.2.9, the diagram commutes. □

**Proposition 5.2.2.** *Let $q \geq 0$ and $0 \leq k < n$. Then, for all $a \in R^*$, the following diagram commutes:*

$$\begin{array}{ccc} H_q(EO_{n,n}, C_k(n)) & \xrightarrow{(id,\phi_{a^2})_*} & H_q(EO_{n,n}, C_k(n)) \\ {\scriptstyle (i,(e_1,\ldots,e_k))_*}\uparrow\cong & & \cong\uparrow{\scriptstyle (i,(e_1,\ldots,e_k))_*} \\ H_q(ET_k, \mathbb{Z}) & \xrightarrow[C_{D_{a^2,k}}]{} & H_q(ET_k, \mathbb{Z}), \end{array}$$

*where the vertical arrows are the isomorphisms given by Shapiro's lemma and the map $C_{D_{a^2,k}}$ denotes the map induced by conjugation with the element $D_{a^2,k}$ on the stabiliser $ET_k$.*

*Proof.* We use lemma 3.2.9. Specifically, we have to consider the diagram

$$(ET_k, \mathbb{Z}) \underset{(f_2,\varphi_2)}{\overset{(f_1,\varphi_1)}{\rightrightarrows}} (EO_{n,n}, C_k(n))$$

where $(f_1, \varphi_1) := (i, (a^{-2}e_1, \ldots, a^{-2}e_k))$ and $(f_2, \varphi_2) := (iC_{D_{a^2,k}}, (e_1, \ldots, e_k))$, and $i : ET_k \to EO_{n,n}$ is the natural inclusion of groups. Let $\kappa = D_{a^2,k} \in EO_{n,n}$. Then, for every $A \in T_k$,

$$f_2(A) = iC_{D_{a^2,k}}(A) = D_{a^2,k}AD_{a^2,k}^{-1} = D_{a^2,k}i(A)D_{a^2,k}^{-1} = \kappa f_1(A)\kappa^{-1}$$

and

$$(e_1 \ldots, e_k) = D_{a^2,k}(a^{-2}e_1, \ldots, a^{-2}e_k) = \kappa(a^{-2}e_1, \ldots, a^{-2}e_k).$$

By lemma 3.2.9, the diagram commutes. □

Furthermore, we need to compute $H_q(EO_{n,n}, C_n(n))$ and show that, *after localisation*, they vanish for all $q > 0$.

**Proposition 5.2.3.** *For every $m \geq 1$ and $q < m/2$, we have*

$$s_m^{-1}H_q(EO_{n,n}, C_n(n)) \cong \begin{cases} \mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2] & q = 0 \\ 0 & q > 0. \end{cases}$$

*Proof.* We have isomorphisms

$$\begin{aligned} H_q(EO_{n,n}, \mathbb{Z}[\mathcal{IU}_n]) &\cong \mathrm{Tor}_q^{EO_{n,n}}(\mathbb{Z}, \mathbb{Z}[\mathcal{IU}_n]) \\ &\cong \mathrm{Tor}_q^{O_{n,n}}(\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2], \mathbb{Z}[\mathcal{IU}_n]) \\ &\cong H_q(\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2] \otimes^{\mathbb{L}}_{O_{n,n}} \mathbb{Z}[\mathcal{IU}_n]) \\ &\cong H_q(\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2] \otimes^{\mathbb{L}}_{O_{n,n}} \mathbb{Z}[O_{n,n}/T_n]) \\ &\cong H_q(\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2] \otimes^{\mathbb{L}}_{T_n} \mathbb{Z}) \\ &\cong H_q(T_n, \mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2]) \\ &\cong \mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2] \otimes_{\mathbb{Z}} H_q(T_n, \mathbb{Z}), \end{aligned}$$

where we have used short exact sequence (5.3), transitivity of the $O_{n,n}$-action and the Universal Coefficient Theorem. We therefore want to show that the action kills the $H_q(T_n)$ terms for all $q > 0$, whilst leaving the $\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2]$ term invariant. This will follow from the commutativity of the following two diagrams, which we state as lemmas.

**Lemma 5.2.4.** *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Tor}_q^{EO_{n,n}}(\mathbb{Z}, \mathbb{Z}[\mathcal{IU}_n]) & \xrightarrow{(id,id,\phi_{a^2})_*} & \mathrm{Tor}_q^{EO_{n,n}}(\mathbb{Z}, \mathbb{Z}[\mathcal{IU}_n]) \\ {\scriptstyle (i,i,id)_*}\downarrow\cong & & \cong\downarrow{\scriptstyle (i,i,id)_*} \\ \mathrm{Tor}_q^{O_{n,n}}(\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2], \mathbb{Z}[\mathcal{IU}_n]) & \xrightarrow[(id,id,\phi_{a^2})_*]{} & \mathrm{Tor}_q^{O_{n,n}}(\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2], \mathbb{Z}[\mathcal{IU}_n]), \end{array}$$

*where the vertical maps are the isomorphisms given by short exact sequence (5.3); $i : \mathbb{Z} \hookrightarrow \mathbb{Z}[R^*/R^{*2}]$ and $i : EO_{n,n} \hookrightarrow O_{n,n}$ denote the canonical inclusions and recall that $\phi_{a^2} : \mathbb{Z}[\mathcal{IU}_n] \to \mathbb{Z}[\mathcal{IU}_n]$ is the map defined on basis elements by $(v_1, \ldots, v_n) \mapsto (a^{-2}v_1, \ldots, a^{-2}v_n)$.*

*Proof.* Easily seen by inspection. $\square$

**Lemma 5.2.5.** *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Tor}_q^{O_{n,n}}(\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2], \mathbb{Z}[\mathcal{IU}_n]) & \xrightarrow{(id,id,\phi_{a^2})_*} & \mathrm{Tor}_q^{O_{n,n}}(\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2], \mathbb{Z}[\mathcal{IU}_n]) \\ {\scriptstyle (id,i,(e_1,\ldots,e_n))_*}\uparrow\cong & & \cong\uparrow{\scriptstyle (id,i,(e_1,\ldots,e_n))_*} \\ \mathrm{Tor}_q^{T_n}(\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2], \mathbb{Z}) & \xrightarrow[(D^{-1}_{a^2,n}, C_{D_{a^2,n}}, id)_*]{} & \mathrm{Tor}_q^{T_n}(\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2], \mathbb{Z}), \end{array}$$

*where the vertical maps are the isomorphisms given by the transitivity of the $O_{n,n}$-action; $D^{-1}_{a^2,n}$ denotes the map induced right multiplication by $D^{-1}_{a^2,n} \in O_{n,n}$ and $C_{D_{a^2,n}}$ denotes the map induced by conjugation with $D_{a^2,n}$.*

*Proof.* We use lemma 3.2.10. Specifically, consider the diagram

$$(\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2], T_n, \mathbb{Z}) \underset{(f_2,\varphi_2,g_2)}{\overset{(f_1,\varphi_1,g_1)}{\rightrightarrows}} (\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2], O_{n,n}, \mathbb{Z}[\mathcal{IU}_n])$$

where $(f_1, \varphi_1, g_1) := (id, i, (a^{-2}e_1, \ldots, a^{-2}e_n))$ and $(f_2, \varphi_2, g_2) := (D^{-1}_{a^2,n}, iC_{D_{a^2,n}}, (e_1, \ldots, e_n))$. Let $\kappa := D_{a^2,n} \in O_{n,n}$. We have that $\varphi_2 = \kappa\varphi_1\kappa^{-1}$; $g_2 = \kappa g_1$ and $f_2 = f_1\kappa^{-1}$, so that by lemma 3.2.10, the diagram commutes. $\square$

Note that in the previous lemma, $D_{a^2,n} \in EO_{n,n}$, so that the action on $\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2]$ is *trivial.* It follows therefore that the action on $\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2] \otimes_{\mathbb{Z}} H_q(T_n, \mathbb{Z})$ is trivial on $\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2]$ and is the action induced by conjugation by $D_{a^2,n}$ on $H_q(T_n)$. By lemma 3.2.5 (using the fact that $T_n = L_n$), the proposition follows. $\square$

Thus, we have shown that there exists an $R^*$-action on the spectral sequence

$$E^1_{p,q}(n) = H_q(EO_{n,n}, C_p(n)) \Rightarrow H_{p+q}(EO_{n,n}, C_*(n))$$

which induces the desired local actions considered previously. Using corollary 5.1.4 and proposition 5.2.3, we obtain the following.

**Corollary 5.2.6.** *For every $m \geq 1$ and every $q < m/2$, the localised spectral sequence*

$$ {}_mE^1_{p,q}(n) = s_m^{-1}E^1_{p,q}(n) \Rightarrow s_m^{-1}H_{p+q}(EO_{n,n}, C_*(n)) \tag{5.6} $$

*has ${}_mE^1_{p,q}$ terms*

$$ {}_mE^1_{p,q}(n) = s_m^{-1}H_q(EO_{n,n}, C_p(n)) \cong \begin{cases} H_q(EO_{n-p,n-p}), & 0 \leq p < n \\ \mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2], & p = n, q = 0 \\ 0 & p = n, q > 0. \end{cases} $$

□

Our next task is to compute the localised $d^1$ differentials $d^1 : {}_mE^1_{p,q} \to {}_mE^1_{p-1,q}$.

## 5.3 Computation of the localised $d^1$ differentials, and proof of homological stability

**Proposition 5.3.1.** *For all $q < m/2$ and $0 \leq p < n$, the homomorphism $d^1_{p,q}$ is*

$$ d^1_{p,q} = \begin{cases} 0, & p \text{ even} \\ i_*, & p \text{ odd}, \end{cases} $$

*where $i : EO_{n-p,n-p} \hookrightarrow EO_{n-p+1,n-p+1}$ denotes the inclusion. For $p = n$, the homomorphism $d^1_{n,q}$ is 0 if $q > 0$ or if $n$ is even; and for $q = 0$ and $n$ odd, $d^1_{n,0}$ is the augmentation map $\varepsilon : \mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2] \to \mathbb{Z}$.*

*Proof.* For all $p < n$, we want to show that the following diagram commutes:

$$ \begin{array}{ccc} H_q(EO_{n-p,n-p}) & \xrightarrow{(i,(e_1,\ldots,e_p))_*} & H_q(EO_{n,n}, C_p(n)) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle (d_i)_*} \\ H_q(EO_{n-p+1,n-p+1}) & \xrightarrow[(i,(e_1,\ldots,e_{p-1}))_*]{} & H_q(EO_{n,n}, C_{p-1}(n)), \end{array} \tag{5.7} $$

where $i : EO_{n-p,n-p} \hookrightarrow EO_{n,n}$ denotes the inclusion map; $(e_1, \ldots, e_p) : 1 \mapsto (e_1, \ldots, e_p)$ and recall that $d_i(v_1, \ldots, v_p) = (v_1, \ldots, \hat{v}_i \ldots, v_p)$. Suppose the matrix $A$ in the proof of proposition 3.2.15 has spinor norm $\theta(A) = a$. Note that, as $A = \begin{pmatrix} \sigma & \\ & 1_{2(n-p)} \end{pmatrix}$, it follows that $\theta(A) = \theta(\sigma)$. If $a = 1$, we are done. Otherwise, define $\hat{A} \in O_{n,n}$ by sending a hyperbolic basis to a hyperbolic basis as follows:

$$\begin{aligned}
(e_1, \dots, \hat{e}_i, \dots, e_p) &\mapsto (e_1, \dots, e_{p-1}) \\
(f_1, \dots, \hat{f}_i, \dots, f_p) &\mapsto (f_1, \dots, f_{p-1}) \\
e_i &\mapsto a e_p \\
f_i &\mapsto a^{-1} f_p \\
e_j \mapsto e_j \text{ and } f_j &\mapsto f_j \text{ for all } p+1 \leq j \leq n.
\end{aligned}$$

Write $\hat{A} = \begin{pmatrix} \hat{\sigma} & \\ & 1_{2(n-p)} \end{pmatrix}$, so that $\theta(\hat{A}) = \theta(\hat{\sigma})$. We prove that $\hat{\sigma} \in EO_{p,p}$. Indeed, this follows from the matrix equation

$$\hat{\sigma} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & a & \\ & & & & a^{-1} \end{pmatrix} \sigma.$$

Clearly, we still have $(e_1, \dots, e_{p-1}) = \hat{A}(e_1, \dots, \hat{e}_i, \dots e_p)$ and for every $B \in EO_{n-p,n-p}$,

$$\varepsilon \circ i(B) = \hat{A} \varepsilon(B) \hat{A}^{-1},$$

so that by lemma 3.2.9, the diagram commutes.

For $p = n$, it suffices to show that the diagram commutes:

$$\begin{array}{ccc}
\mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2] & \longrightarrow & C_n(n)_{EO_{n,n}} \\
\varepsilon \big\downarrow & & \big\downarrow d_i \\
\mathbb{Z} & \xrightarrow[1 \mapsto (e_1, \dots, e_{n-1})]{} & C_{n-1}(n)_{EO_{n,n}},
\end{array}$$

where the top horizontal arrow maps a given basis element $x \in R^*/R^{*2} \times \mathbb{Z}_2$ to the element given by the isomorphism of $O_{n,n}$-sets $R^*/R^{*2} \times \mathbb{Z}_2 \cong O_{n,n}/EO_{n,n} \cong \mathcal{IU}_n(R^{2n})/EO_{n,n}$, see proposition 5.1.3. But this follows from the fact that $EO_{n,n}$ acts transitively on $\mathcal{IU}_{n-1}(R^{2n})$. $\square$

In the proof of the next proposition, we will use the so called *hyperbolic map.*

**Definition 5.3.2.** Define the hyperbolic map as a group homomorphism $H : GL_n(R) \to$

$O_{n,n}(R)$ given by

$$H : GL_n(R) \longrightarrow O_{n,n}(R)$$
$$A \longmapsto \begin{pmatrix} A & \\ & {}^t(A^{-1}) \end{pmatrix}.$$

**Remark 28.** In the above definition, we have used the convention that $R^{2n}$ is equipped with symmetric bilinear form given by $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, and has ordered basis $e_1, \ldots, e_n, f_1, \ldots, f_n$, so that $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$ and $\langle e_i, f_j \rangle = \delta_{ij}$. We have done this for the sake of notation. It is clear that this convention differs from our usual convention up to matrix conjugation (by a suitable permutation matrix). We tacitly assume this whenever using the hyperbolic map.

We need to prove the following proposition:

**Proposition 5.3.3.** *The differentials $d^r_{p,q}$ in spectral sequence (5.6) are zero for $r \geq 2$ and $q < m/2$, $p \leq n$. Hence, for all $q < m/2$ and $p \leq n$, ${}_mE^2_{p,q} \cong {}_mE^\infty_{p,q}$.*

*Proof.* For $n = 0, 1$, the spectral sequence under consideration is located in columns 0 and 1. Therefore, the differentials $d^r$ for $r \geq 2$ are zero by dimension arguments.

For $n \geq 2$, consider the homomorphism of complexes of $EO_{n-2,n-2}$-modules

$$\tau : C_*(n-2)[-2] \to C_*(n).$$

as defined in proposition 3.2.17. Note that the diagram

$$\begin{array}{ccc} (EO_{n-2,n-2}, C_{p-2}(n-2)) & \xrightarrow{(i,\tau_j)} & (EO_{n,n}, C_p(n)) \\ {\scriptstyle (C_{B_a},B_a)}\uparrow & & \uparrow{\scriptstyle (C_{B_a},B_a)} \\ (EO_{n-2,n-2}, C_{p-2}(n-2)) & \xrightarrow[(i,\tau_j)]{} & (EO_{n,n}, C_p(n)), \end{array}$$

still commutes, so that we have an induced map on localised spectral sequences

$${}_m\tau_* : {}_m\tilde{E} \to {}_mE.$$

The claim would then follow by induction on $r$ using the following lemma:

**Lemma 5.3.4.** *The map ${}_m\tau_* : {}_m\tilde{E}^1_{p,q} \to {}_mE^1_{p,q}$ is the identity for all $q < m/2$ and $2 \leq p \leq n$.*

*Proof.* For $2 \leq p < n$, the same proof as in lemma 3.2.18 will work, as long as the matrices $A$ and $B$ of lemma 3.2.18 are in $EO_{n,n}$. Recall that $A \in O_{n,n}$ was defined by

$$\begin{aligned}
&e_1 \mapsto e_1 \\
&e_2 \mapsto e_2 - e_1 \\
&f_1 \mapsto f_1 + f_2 \\
&f_2 \mapsto f_2 \\
&e_j \mapsto e_j \text{ for all } 3 \leq j \leq n \\
&f_j \mapsto f_j \text{ for all } 3 \leq j \leq n,
\end{aligned}$$

and $B \in O_{n,n}$ was defined by

$$\begin{aligned}
&e_1 \mapsto e_2 \\
&e_2 \mapsto e_2 - e_1 \\
&f_1 \mapsto f_1 + f_2 \\
&f_2 \mapsto -f_1 \\
&e_j \mapsto e_j \text{ for all } 3 \leq j \leq n \\
&f_j \mapsto f_j \text{ for all } 3 \leq j \leq n.
\end{aligned}$$

It suffices to prove that

$$M := \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

and

$$N := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

are in $EO_{2,2}$. Note that the hyperbolic map $H : GL_2(R) \to O_{2,2}(R)$ is a *group homomorphism*. In addition, note that $SL_2(R)$ is *perfect*. For example, this follows from [HO89, Theorem 4.3.9.] and [Sch17, Lemma 3.8.]. Therefore, we deduce $H(SL_2(R)) \subseteq [SO_{2,2}(R), SO_{2,2}(R)] \subseteq EO_{2,2}(R)$. We then note that $M = H\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)$ and $N = H\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right)$.

Finally, we need to consider the case $p = n$. It suffices to show that for $j = 0, 1, 2$, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2] & \longrightarrow & C_{n-2}(n-2)_{EO_{n-2,n-2}} \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \tau_j} \\ \mathbb{Z}[R^*/R^{*2} \times \mathbb{Z}_2] & \longrightarrow & C_n(n)_{EO_{n,n}}, \end{array}$$

where the top and bottom horizontal arrows map a given basis element $x \in R^*/R^{*2} \times \mathbb{Z}_2$ to the element given by the isomorphism of $O_{n-2,n-2}$-sets $R^*/R^{*2} \times \mathbb{Z}_2 \cong O_{n-2,n-2}/EO_{n-2,n-2} \cong \mathcal{IU}_{n-2}(R^{2(n-2)})/EO_{n-2,n-2}$ and isomorphism of $O_{n,n}$-sets $R^*/R^{*2} \times \mathbb{Z}_2 \cong O_{n,n}/EO_{n,n} \cong \mathcal{IU}_n(R^{2n})/EO_{n,n}$ respectively, see proposition 5.1.3.

Under the isomorphism $R^*/R^{*2} \times \mathbb{Z}_2 \cong \mathcal{IU}_{n-2}(R^{2(n-2)})/EO_{n-2,n-2}$, an element $x \in R^*/R^{*2} \times \mathbb{Z}_2$ is sent to the element $P(e_1, \ldots, e_{n-2})$ for some $P \in O_{n-2,n-2}$, and under the isomorphism $R^*/R^{*2} \times \mathbb{Z}_2 \cong \mathcal{IU}_n(R^{2n})/EO_{n,n}$, the same element $x \in R^*/R^{*2} \times \mathbb{Z}_2$ is sent to the element $\tilde{P}(e_1, \ldots, e_n)$, where $\tilde{P} := \begin{pmatrix} 1_4 & \\ & P \end{pmatrix}$. Recalling that each $\tau_j$ is a map of $O_{n-2,n-2}$-modules, we have that

$$\tau_j(P(e_1, \ldots, e_{n-2})) = \tilde{P}\tau_j(e_1, \ldots, e_{n-2}) = \begin{cases} \tilde{P}(e_1, \ldots, e_n), & j = 0 \\ \tilde{P}(e_1, e_2 - e_1, e_3, \ldots, e_n), & j = 1 \\ \tilde{P}(e_2, e_2 - e_1, e_3, \ldots, e_n), & j = 2. \end{cases}$$

Thus, for $j = 0$, the diagram commutes by inspection. For $j = 1$, we note that

$$A\tilde{P}(e_1, e_2, e_3, \ldots, e_n) = \tilde{P}A(e_1, e_2, e_3, \ldots, e_n) = \tilde{P}(e_1, e_2 - e_1, e_3, \ldots, e_n),$$

since

$$A\tilde{P} = \begin{pmatrix} M & \\ & 1 \end{pmatrix} \begin{pmatrix} 1_4 & \\ & P \end{pmatrix} = \begin{pmatrix} 1_4 & \\ & P \end{pmatrix} \begin{pmatrix} M & \\ & 1 \end{pmatrix} = \tilde{P}A.$$

Similarly, for $j = 2$, we note that

$$B\tilde{P}(e_1, e_2, e_3, \ldots, e_n) = \tilde{P}B(e_1, e_2, e_3, \ldots, e_n) = \tilde{P}(e_2, e_2 - e_1, e_3, \ldots, e_n),$$

where $B$ and $\tilde{P}$ commute for similar reasons. Thus, the diagrams commute. These diagrams still commute after localisation, but now the horizontal maps become the identification isomorphisms. □

This proves the lemma, and thus proposition 5.3.3. □

**Theorem 5.3.5.** *Let $R$ be a commutative local ring with infinite field such that $2 \in R^*$. Then, the natural homomorphism*

$$H_k(EO_{n,n}(R)) \longrightarrow H_k(EO_{n+1,n+1}(R))$$

*is an isomorphism for $k \leq n-1$ and surjective for $k \leq n$.*

**Remark 29.** To our best knowledge, this is the first known homological stability result for $EO_{n,n}$.

*Proof.* Choose $m > 0$ sufficiently large. We have a spectral sequence (5.6) with $E^1$-terms given by corollary 5.2.6 and $d^1_{p,q}$ was computed for all $q < m/2$ in proposition 5.3.1. From theorem 3.1.9, spectral sequences (5.1) and (5.6) and proposition 5.3.3, we deduce ${}_mE^2_{p,q} = {}_mE^\infty_{p,q}$ for all $p+q \leq n-1$ and $q < m/2$. The theorem follows. □

## 5.4 Homological Stability for $\mathrm{Spin}_{n,n}$

Homological stability for $EO_{n,n}$ immediately gives homological stability for $\mathrm{Spin}_{n,n}$:

**Theorem 5.4.1.** *Let $R$ be commutative local ring with infinite field such that $2 \in R^*$. Then, the natural homomorphism*

$$H_k(\mathrm{Spin}_{n,n}(R)) \longrightarrow H_k(\mathrm{Spin}_{n+1,n+1}(R))$$

*is an isomorphism for $k \leq n-1$ and surjective for $k \leq n$.*

**Remark 30.** This coincides with the $H_1$-stability result for $\mathrm{Spin}_{n,n}$ in [HO89, Theorem 9.1.15.] and the $H_2$-stability result for $\mathrm{Spin}_{n,n}$ in [HO89, Theorem 9.1.17, Theorem 9.1.19 and discussion thereafter]. To our best knowledge, this is the first known homological stability result for $\mathrm{Spin}_{n,n}$ that accounts for all homology groups.

*Proof.* Immediate from theorem 5.3.5 and the relative Hochschild-Serre spectral sequence

$$E^2_{p,q} = H_p(EO_{n,n}, EO_{n-1,n-1}; H_q(\mathbb{Z}_2)) \Rightarrow H_{p+q}(\mathrm{Spin}_{n,n}, \mathrm{Spin}_{n-1,n-1}).$$

□

# Chapter 6

# Summary and future work

To summarise, the main achievement of this thesis has been proving the following homological stability results:

**Theorem 6.0.1.** *Let $R$ be a commutative local ring with infinite residue field such that $2 \in R^*$. Then, the natural homomorphisms*

$$H_k(O_{n,n}(R)) \longrightarrow H_k(O_{n+1,n+1}(R))$$
$$H_k(SO_{n,n}(R)) \longrightarrow H_k(SO_{n+1,n+1}(R))$$
$$H_k(EO_{n,n}(R)) \longrightarrow H_k(EO_{n+1,n+1}(R))$$
$$H_k(\mathrm{Spin}_{n,n}(R)) \longrightarrow H_k(\mathrm{Spin}_{n+1,n+1}(R))$$

*are isomorphisms for $k \leq n-1$ and surjective for $k \leq n$.*

The following remain open problems and are suitable for future investigation:

- Prove our homological stability range is optimal.
- Compute the obstructions to further homological stability

$$H_n(O_{n,n}, O_{n-1,n-1})$$
$$H_n(SO_{n,n}, SO_{n-1,n-1})$$
$$H_n(EO_{n,n}, EO_{n-1,n-1})$$
$$H_n(\mathrm{Spin}_{n,n}, \mathrm{Spin}_{n-1,n-1}).$$

- Prove homological stability **without** the assumption that $2 \in R^*$.

We will end by giving a conjecture for what the obstructions to further homological stability should be, and give some rough calculations as to why we think our conjecture is true.

**Conjecture 6.0.2.** Let $R$ be a commutative local ring with infinite residue field such that $2 \in R^*$. Then, for all $n \geq 2$, we have

$$H_n(O_{n,n}, O_{n-1,n-1}) \cong K_n^M/2 \oplus K_{n-1}^M/2$$
$$H_n(SO_{n,n}, SO_{n-1,n-1}) \cong K_n^M \oplus K_{n-1}^M$$
$$H_n(EO_{n,n}, EO_{n-1,n-1}) \cong H_n(\mathrm{Spin}_{n,n}, \mathrm{Spin}_{n-1,n-1}) \cong K_n^{MW} \oplus K_{n-1}^{MW}.$$

For the case $n = 2$, we give a rough calculation that supports the validity of this conjecture.

We have the exceptional isomorphisms

$$\mathrm{Spin}_2 \cong GL_1$$
$$\mathrm{Spin}_3 \cong SL_2$$
$$\mathrm{Spin}_4 \cong SL_2 \times SL_2$$
$$\mathrm{Spin}_5 \cong Sp_4$$
$$\mathrm{Spin}_6 \cong SL_4,$$

where $\mathrm{Spin}_{2n}$ is understood as $\mathrm{Spin}_{n,n}$ and $\mathrm{Spin}_{2n+1}$ is understood as $\mathrm{Spin}_{n+1,n}$. These exceptional isomorphisms are proven in [HO89, §7.3]. We want to use these exceptional isomorphisms to compute $H_2(\mathrm{Spin}_{2,2}, \mathrm{Spin}_{1,1})$.

**Proposition 6.0.3.** $H_2(\mathrm{Spin}_{2,2}, \mathrm{Spin}_{1,1}) \cong H_2(\mathrm{Spin}_4, \mathrm{Spin}_3) \oplus H_2(\mathrm{Spin}_3, \mathrm{Spin}_2)$.

*Proof.* We want to show that the long exact sequence of the triple $(SL_2 \times SL_2, SL_2, GL_1)$ has the following sections as indicated:

$$\cdots \to H_3(SL_2 \times SL_2, GL_1) \overset{\curvearrowleft}{\longrightarrow} H_3(SL_2 \times SL_2, SL_2) \to H_2(SL_2, GL_1)$$
$$\to H_2(SL_2 \times SL_2, GL_1) \overset{\curvearrowleft}{\longrightarrow} H_2(SL_2 \times SL_2, SL_2) \to \cdots.$$

This will then prove the existence of a split short exact sequence

$$0 \to H_2(SL_2, GL_1) \to H_2(SL_2 \times SL_2, GL_1) \overset{\curvearrowleft}{\longrightarrow} H_2(SL_2 \times SL_2, SL_2) \to 0.$$

Note that $SL_2$ is *perfect*, so that by Künneth's theorem, $H_k(SL_2 \times SL_2) \cong H_k(SL_2) \oplus H_k(SL_2)$ for $k = 2, 3$.

Therefore, for $k = 2, 3$, we obtain the following diagram:

$$\begin{array}{ccccc} & & & & H_k(SL_2) \\ & & & & \big\downarrow \Delta \\ & & & & H_k(SL_2) \oplus H_k(SL_2) \\ & & & & \big\downarrow \beta\alpha \\ H_k(SL_2) \oplus H_k(SL_2) & \xrightarrow{\alpha} & H_k(SL_2 \times SL_2, GL_1) & \xrightarrow{\beta} & H_k(SL_2 \times SL_2, SL_2). \end{array}$$

Here, $\Delta$ is the diagonal map; $\alpha$ comes from the long exact sequence of the pair $(SL_2 \times SL_2, GL_1)$ and $\beta$ comes from the long exact sequence of the triple $(SL_2 \times SL_2, SL_2, GL_1)$. Furthermore, by functoriality of relative group homology, the second vertical arrow is the composition $\beta\alpha$.

We argue that

$$\operatorname{coker} \Delta \cong H_k(SL_2 \times SL_2, SL_2).$$

For $k = 2$, this follows from the fact that $H_1(SL_2) = 0$. For $k = 3$, note that

$$\operatorname{coker} \Delta \cong \ker(H_3(SL_2 \times SL_2, SL_2) \to H_2(SL_2))$$

and

$$\operatorname{Im}(H_3(SL_2 \times SL_2, SL_2) \to H_2(SL_2)) = \ker(H_2 SL_2 \xrightarrow{\Delta} H_2(SL_2) \oplus H_2(SL_2)) = 0.$$

The claim follows.

In general, note that for an abelian group $A$ and the diagonal map $\Delta : A \to A \oplus A$, we have an isomorphism $\operatorname{coker} \Delta \cong A$. This isomorphism is realised by the map $a \mapsto (a, 0)$, with inverse $(a, b) = (a - b, 0) \mapsto a - b$. Thus, we obtain an isomorphism $H_k(SL_2 \times SL_2, SL_2) \cong H_k(SL_2)$. We then use this isomorphism to define a map

$$H_k(SL_2 \times SL_2, SL_2) \cong H_k(SL_2) \xrightarrow{(1,0)} H_k(SL_2) \oplus H_k(SL_2) \xrightarrow{\alpha} H_k(SL_2 \times SL_2, GL_1),$$

which one can check defines a section for $\beta$.

Thus, we compute

$$\begin{aligned}H_2(\mathrm{Spin}_{2,2}, \mathrm{Spin}_{1,1}) &\cong H_2(SL_2 \times SL_2, GL_1)\\ &\cong H_2(SL_2 \times SL_2, SL_2) \oplus H_2(SL_2, GL_1)\\ &\cong H_2(\mathrm{Spin}_4, \mathrm{Spin}_3) \oplus H_2(\mathrm{Spin}_3, \mathrm{Spin}_2).\end{aligned}$$

□

By personal communications with Marco Schlichting, we have $H_2(\mathrm{Spin}_4, \mathrm{Spin}_3) \cong K_2^{MW}$ and $H_2(\mathrm{Spin}_3, \mathrm{Spin}_2) \cong K_1^{MW}$, so that

$$H_2(\mathrm{Spin}_{2,2}, \mathrm{Spin}_{1,1}) \cong K_2^{MW} \oplus K_1^{MW}.$$

Furthermore, this suggests that we have the following isomorphisms:

$$H_2(SO_{2,2}, SO_{1,1}) \cong K_2^M \oplus K_1^M \text{ and } H_2(O_{2,2}, O_{1,1}) \cong K_2^M/2 \oplus K_1^M/2.$$

In more detail, consider the spectral sequences associated to the short exact sequences

$$\begin{aligned}&1 \to \mathbb{Z}_2 \to \mathrm{Spin}_{n,n}(R) \xrightarrow{\pi} EO_{n,n}(R) \to 1\\ &1 \to EO_{n,n}(R) \to SO_{n,n}(R) \xrightarrow{\theta} R^*/R^{*2} \to 1\\ &1 \to SO_{n,n}(R) \to O_{n,n}(R) \xrightarrow{\det} \mathbb{Z}_2 \to 1.\end{aligned}$$

The first short exact sequence gives us spectral sequence

$$E^2_{p,q} = H_p(EO_{2,2}, EO_{1,1}; H_q(\mathbb{Z}_2)) \Rightarrow H_{p+q}(\mathrm{Spin}_{2,2}, \mathrm{Spin}_{1,1}).$$

By homological stability for $EO_{n,n}$ and the universal coefficient theorem, we compute

$$H_2(EO_{2,2}, EO_{1,1}) \cong H_2(\mathrm{Spin}_{2,2}, \mathrm{Spin}_{1,1}) \cong K_2^{MW} \oplus K_1^{MW}.$$

From the second short exact sequence, we obtain spectral sequence

$$E^2_{p,q} = H_p(R^*/R^{*2}; H_q(EO_{2,2}, EO_{1,1})) \Rightarrow H_{p+q}(SO_{2,2}, SO_{1,1}).$$

One computes

$$H_2(SO_{2,2}, SO_{1,1}) \cong E^\infty_{0,2} = E^2_{0,2} = H_0(R^*/R^{*2}; H_2(EO_{2,2}, EO_{1,1}))$$
$$\cong (K_2^{MW} \oplus K_1^{MW})_{R^*/R^{*2}}.$$

If we can show

$$(K_2^{MW} \oplus K_1^{MW})_{R^*/R^{*2}} = (K_2^{MW})_{R^*/R^{*2}} \oplus (K_1^{MW})_{R^*/R^{*2}}$$

and the action is given by multiplication by the map

$$Q : R^*/R^{*2} \to K_0^{MW}$$
$$r \mapsto 1 + \eta[r] =: \langle r \rangle,$$

we would have proven $H_2(SO_{2,2}, SO_{1,1}) \cong K_2^M \oplus K_1^M$. Indeed, it suffices to see that for $n \geq 1$, $(K_n^{MW})_{R^*/R^{*2}} \cong K_n^M$.

Recall that for $n \geq 1$, $K_n^{MW}$ is generated by the symbols $[u_1] \cdots [u_n]$ subject to the relation $[u][1-u] = 0$ for all $u, 1-u \in R^*$. We also have the relation $[ru] = [r] + \langle r \rangle [u]$, see for example [Sch17]. Therefore, in $(K_n^{MW})_{R^*/R^{*2}}$, we obtain the addition relation $[ru_1] \cdots [u_n] = [r][u_2] \cdots [u_n] + [u_1] \cdots [u_n]$, and similarly for the other components. Thus, we deduce for all $n \geq 1$, $(K_n^{MW})_{R^*/R^{*2}} \cong K_n^M$.

Finally, the third short exact sequence gives us spectral sequence

$$E^2_{p,q} = H_p(\mathbb{Z}_2, H_q(SO_{2,2}, SO_{1,1})) \Rightarrow H_{p+q}(O_{2,2}, O_{1,1}).$$

One computes

$$H_2(O_{2,2}, O_{1,1}) \cong E^\infty_{0,2} = E^2_{0,2} = H_0(\mathbb{Z}_2, H_2(SO_{2,2}, SO_{1,1}))$$
$$\cong (K_2^M \oplus K_1^M)_{\mathbb{Z}_2}.$$

If one can show

$$(K_2^M \oplus K_1^M)_{\mathbb{Z}_2} \cong (K_2^M)_{\mathbb{Z}_2} \oplus (K_1^M)_{\mathbb{Z}_2}$$

and the action of $\mathbb{Z}_2$ on $K_n^M$ is given by $g \cdot u_1 \otimes \cdots \otimes u_n = u_1^{-1} \otimes u_2 \cdots \otimes u_n$, then we would have $(K_n^M)_{\mathbb{Z}_2} \cong K_n^M/2$ so that $H_2(O_{2,2}, O_{1,1}) \cong K_2^M/2 \oplus K_1^M/2$.

These above arguments would turn into a proof if one could prove the actions trace through the isomorphisms as one expects. The author believes this seems reasonable, although a proof will require some work!

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