Finite range decomposition for families of gradient Gaussian measures

Stefan Adams[a], Roman Kotecký[a,b], Stefan Müller[c]

[a] *Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom*
[b] *Center for Theoretical Study, Charles University, Jilská 1, Prague, Czech Republic*
[c] *Institute for Applied Mathematics and Hausdorff Center for Mathematics, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany*

Abstract

Let a family of gradient Gaussian vector fields on $\mathbb{Z}^d$ be given. We show the existence of a uniform finite range decomposition of the corresponding covariance operators, that is, the covariance operator can be written as a sum of covariance operators whose kernels are supported within cubes of diameters $\sim L^k$. In addition we prove natural regularity for the subcovariance operators and we obtain regularity bounds as we vary within the given family of gradient Gaussian measures.

Keywords: Gradient Gaussian field, covariance, renormalisation, Fourier multiplier

1. Introduction

In this paper we construct a finite range decomposition for a family of translation invariant gradient Gaussian fields on $\mathbb{Z}^d$ ($d \geq 2$) which depends real-analytically on the quadratic from that defines the Gaussian field. More precisely, we consider a large torus $(\mathbb{Z}/L^N\mathbb{Z})^d$ and obtain a finite range decomposition with estimates that do not depend on N. Equivalently, we show that the discrete Greens function C_A of the (elliptic) translation invariant difference operator $\mathcal{A} = \nabla^* A \nabla$ can be written as a sum $C_A = \sum_k C_{A,k}$ of positive kernels $C_{A,k}$ which are supported in cubes of size $\sim L^k$ with natural estimates for their discrete derivatives $\nabla^\alpha C_{A,k}$ (see Theorem 2.1) as well as for their derivatives with respect to A (see Theorem 2.2).

To put this into perspective recall that an $\mathbb{R}^m$-valued Gaussian field ξ on $\mathbb{Z}^d$ (with vanishing expectation, $\mathbb{E}(\xi(x)) = 0$) is said to have range M if the correlation matrices $\mathbb{E}[\xi^r(x)\xi^s(y)]$, $r, s = 1, \ldots, m$, vanish whenever $|x - y| > M$. In the following we consider only translation invariant Gaussian fields. We say that Gaussian fields ξ_k form a finite range decomposition of ξ if $\xi = \sum_k \xi_k$ and ξ_k has range $\sim L^k$, where $L \geq 2$ is an integer. The existence of such a decomposition is equivalent to a decomposition of the correlation matrices $C_A(x, y)^{r,s} := \mathbb{E}[\xi^r(x)\xi^s(y)]$ as a sum of positive (semi-) definite matrix valued kernels $C_{A,k}$ with range $\sim L^k$, i.e., $\sum_{x,y} \sum_{r,s} C^{r,s}_{A,k}(x, y)\xi^r(y)\xi^s(x) \geq 0$ and $C_{A,k}(x, y) = 0$ if $|x - y| \gtrsim L^k$.

Email addresses: S.Adams@warwick.ac.uk (Stefan Adams), R.Kotecky@warwick.ac.uk (Roman Kotecký), stefan.mueller@hcm.uni-bonn.de (Stefan Müller)

We are interested in gradient Gaussian fields, i.e., Gaussian fields with σ-algebra determined by the gradients $\nabla\xi$. Such fields arise naturally e.g. in problems in elasticity where only the difference of values is relevant for the energy. In this case we seek a decomposition into gradient Gaussian fields such that the gradient-gradient correlation $\mathbb{E}[\nabla_i \xi^r(x)\nabla_j \xi^s(y)] = \nabla_i \nabla_j^* E[\xi^r(x)\xi^s(y)]$ vanishes for $|x-y| \gtrsim L^k$. Gradient Gaussian fields are more subtle to handle since they exhibit long-range correlations (the gradient-gradient correlation of the original field typically has only algebraic decay $|x-y|^{-d}$ with the critical exponent $-d$). In the language of quantum field theory gradient Gaussian fields are thus often referred to as massless fields.

Decomposition into a sum of positive definite operators has been discussed in [8] where a radial function is written as a weighted integral of tent functions. In [5], finite range decompositions of the resolvent of the Laplacian $(a-\Delta)^{-1}$, with $a \geq 0$, have been obtained both for the usual Laplacian and for finite difference Laplacian on the simple cubic lattice $\mathbb{Z}^d$. In [4] these results are extended and generalised by providing sufficient conditions for a positive definite function to admit decomposition into a sum of positive functions that are compactly supported within disks of increasing diameters $\frac{1}{2}L^k$. More precisely, the authors of [4] consider positive definite bilinear forms on C_0^∞ and prove that finite range decompositions do exist when the bilinear form is dual to a bilinear form $\varphi \mapsto \int |B\varphi(x)|^2 \, \mathrm{d}x$ where B is a vector valued partial differential operator satisfying some regularity conditions.

The main novelty of our paper is twofold. First we extend the finite range decomposition for the discrete Laplacian to a situation where no maximum principle is available (even in the scalar case there is no discrete maximum principle for general elliptic difference operator $\nabla^* A \nabla$ with constant coefficients). This can be seen as an adaptation of [4] to the discrete setting. Secondly, we show that the finite range decomposition can be chosen so that the kernels $C_{A,k}$ depend analytically on A as long as A is positive definite.

Our main motivation is the renormalization group (RG) approach to problems in statistical mechanics, following the longstanding research programme of Brydges and Yau [6], and in particular the recent work of Brydges [3], both inspired by the work of K.G. Wilson [11]. The goal is to get good control of the expectations $\mathbb{E}(K)$ of nonlinear functions that depends on a gradient Gaussian field ξ in a large region $\Lambda \subset \mathbb{Z}^d$ of the integer lattice. The size of Λ and the long range correlations in ξ make it difficult to obtain accurate estimates on the expectation $\mathbb{E}(K)$. In [2] we show that such control can nonetheless be obtained in many interesting cases using the RG approach. One key difference with the earlier work of Brydges and others is the necessity to drop the assumption of isotropy. Hence the relevant quadratic term is a general finite difference operator $\nabla^* A \nabla$ and reduces no longer to a multiple of the discrete Laplacian. For this reason we need a finite-range decomposition for general (elliptic) operators A and, in addition, we need to control derivatives of the finite range decomposition with respect to A.

In Section 2 we introduce the setting of gradient fields and the relevant Greens functions. Our two main results are given in Theorem 2.1 and Theorem 2.2. The existence of the finite range decomposition is proved in Section 3 where we adapt and extend the methods in [4] to our setting. The regularity estimate for a fixed A is established in Section 4. Real-analytic dependence on A is proved in Section 5.

Acknowledgments

The research of S.A. was supported by EPSRC grant EP/I003746/1, R.K. by the grants GAČR 201-09-1931, 201/12/2613, and MSM 0021620845, and S.M. by the DFG Research group 718 “Analysis and Stochastics in Complex Physical Systems”.

2. Notation and main results

We are interested in gradient Gaussian fields on bounded domains in $\mathbb{Z}^d$. For that let $L \geq 3$ be a fixed odd integer and consider for any integer N the space

$$\boldsymbol{\mathcal{V}}_N = \{\varphi : \mathbb{Z}^d \to \mathbb{R}^m;\ \varphi(x+z) = \varphi(x) \ \text{ for all } \ z \in (L^N\mathbb{Z})^d\} = (\mathbb{R}^m)^{\mathbb{T}_N}$$

of functions on the torus $\mathbb{T}_N := (\mathbb{Z}/L^N\mathbb{Z})^d$ equipped with with the scalar product

$$\langle \varphi, \psi \rangle = \sum_{x \in \mathbb{T}_N} \langle \varphi(x), \psi(x) \rangle_{\mathbb{R}^m}. \tag{2.1}$$

Notice that a function on $\mathbb{T}_N$ can be identified with an L^N-periodic function on $\mathbb{Z}^d$. We will later denote the corresponding space of $\mathbb{C}^m$-valued function, equipped with the usual scalar product in the same way.

We consider two distances on $\mathbb{Z}^d$: $\rho(x,y) = \inf\{|x-y+z| \colon z \in (L^N\mathbb{Z})^d\}$ and $\rho_\infty(x,y) = \inf\{|x-y+z|_\infty \colon z \in (L^N\mathbb{Z})^d\}$. Then the torus can be represented by the lattice cube $\mathbb{T}_N = \{x \in \mathbb{Z}^d \colon |x|_\infty \leq \frac{1}{2}(L^N-1)\}$ of side L^N, equipped with the metric ρ or ρ_∞. Gradient Gaussian fields can be easily defined as discrete gradients of Gaussian fields. However it turns out to be inconvenient to work directly with the space of discrete gradient fields, since the constraint of being curl free (in a discrete sense) leads to a complicated bookkeeping. Instead, we use that discrete gradient fields are in one-to-one relation to usual fields modulo a constant. To eliminate this constant we use the normalisation condition that the sum of the field over the torus vanishes. We thus denote by $\boldsymbol{\mathcal{X}}_N$ the subspace

$$\boldsymbol{\mathcal{X}}_N = \{\varphi \in \boldsymbol{\mathcal{V}}_N : \sum_{x \in \mathbb{T}_N} \varphi(x) = 0\}. \tag{2.2}$$

The forward and backward derivatives are defined as

$$\begin{aligned} (\nabla\varphi)^r_j(x) &= \varphi^r(x+\mathrm{e}_j) - \varphi^r(x), \\ (\nabla^*\varphi)^r_j(x) &= \varphi^r(x-\mathrm{e}_j) - \varphi^r(x), \quad r = 1,\dots,m;\ j = 1,\dots,d. \end{aligned} \tag{2.3}$$

Let $A \colon \mathbb{R}^{m\times d} \to \mathbb{R}^{m\times d}$ be a linear map that is symmetric with respect to the standard scalar product $(\cdot,\cdot)_{\mathbb{R}^{m\times d}}$ on $\mathbb{R}^{m\times d}$ and positive definite, that is, there exists a constant $c_0 > 0$ such that

$$(AF, F)_{\mathbb{R}^{m\times d}} \geq c_0 \|F\|^2_{\mathbb{R}^{m\times d}} \quad \text{ for all } F \in \mathbb{R}^{m\times d} \text{ with } \|F\|_{\mathbb{R}^{m\times d}} = (F,F)^{1/2}_{\mathbb{R}^{m\times d}}. \tag{2.4}$$

The corresponding Dirichlet form defines a scalar product on $\boldsymbol{\mathcal{X}}_N$,

$$(\varphi,\psi)_+ := \mathcal{E}(\varphi,\psi) = \sum_{x \in \mathbb{T}_N} \langle A(\nabla\varphi(x)), \nabla\psi(x) \rangle_{\mathbb{R}^{m\times d}}, \quad \varphi,\psi \in \boldsymbol{\mathcal{X}}_N. \tag{2.5}$$

Skipping the index N, we consider the triplet $\boldsymbol{\mathcal{H}}_- = \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}_+$ of (finite-dimensional) Hilbert spaces obtained by equipping the space $\boldsymbol{\mathcal{X}}_N$ with the norms $\|\cdot\|_-$, $\|\cdot\|_2$, and $\|\cdot\|_+$, respectively. Here, $\|\cdot\|_2$ denotes the ℓ_2-norm $\|\varphi\|_2 = \langle \varphi, \varphi \rangle^{1/2}$, $\|\varphi\|_+ = (\varphi,\varphi)^{1/2}_+$, and $\|\cdot\|_-$ is the dual norm

$$\|\varphi\|_- = \sup_{\psi : \|\psi\|_+ \leq 1} \langle \psi, \varphi \rangle. \tag{2.6}$$

One easily checks that $\|\cdot\|_-$ is again induced in a unique way by a scalar product $(\cdot,\cdot)_-$. The linear map A defines an isometry

$$\mathcal{A} \colon \boldsymbol{\mathcal{H}}_+ \to \boldsymbol{\mathcal{H}}_-, \quad \varphi \mapsto \mathcal{A}\varphi = \nabla^*(A\nabla\varphi). \tag{2.7}$$

Indeed, it follows from the Lax-Milgram theorem that, for each $f \in \mathcal{H}_-$, the equation

$$(\varphi, v)_+ = \langle f, v\rangle \text{ for all } v \in \mathcal{H}_+ \tag{2.8}$$

has a unique solution $\varphi \in \mathcal{H}_+$. Hence $\mathcal{A}$ is a bijection from $\mathcal{H}_+$ to $\mathcal{H}_-$. Moreover

$$\|\mathcal{A}\varphi\|_- = \sup\{\langle \mathcal{A}\varphi, v\rangle : \|v\|_+ \le 1\} = \sup\{(\varphi, v)_+ : \|v\|_+ \le 1\} = \|\varphi\|_+. \tag{2.9}$$

Thus $\mathcal{A}$ is an isometry from $\mathcal{H}_+$ to $\mathcal{H}_-$. In view of the symmetry of $\mathcal{A}$ it follows that

$$(\varphi, \psi)_- = (\mathcal{A}^{-1}\varphi, \mathcal{A}^{-1}\psi)_+ = \langle \mathcal{A}^{-1}\varphi, \mathcal{A}\mathcal{A}^{-1}\psi\rangle = \langle \mathcal{A}^{-1}\varphi, \psi\rangle. \tag{2.10}$$

In the more abstract construction of Brydges and Talarczyk [4] it is important that the operator $\mathcal{A}$ can be written as

$$\mathcal{A} = \mathcal{B}^*\mathcal{B}, \tag{2.11}$$

where $\mathcal{B}^*$ denotes the dual of $\mathcal{B}$. This is indeed possible in our case. Since the operator A is symmetric and positive definite it has a positive square root $A^{1/2}$ and we can define $\mathcal{B}$ by

$$(\mathcal{B}\varphi)(x) = (A^{1/2}\nabla\varphi)(x). \tag{2.12}$$

This yields

$$(\varphi, \psi)_+ = \langle \mathcal{B}\varphi, \mathcal{B}\psi\rangle, \quad \|\varphi\|_+ = \|\mathcal{B}\varphi\|_2. \tag{2.13}$$

In the following, however, we will not use the operator $\mathcal{B}$ explicitly and will, instead, directly use that $\langle \mathcal{A}\varphi, \psi\rangle = \langle A\nabla\varphi, \nabla\psi\rangle$ and exploit that the right hand side is sufficiently local in φ and ψ. We do, however, make crucial use of the assumption that A is positive definite in the proof of Lemma 3.4. For many of the other estimates it would be sufficient to assume that the operator $\mathcal{A}$ is positive which is implied by the weaker condition that A is positive definite on matrices of rank one, i.e., $\langle A(a \otimes b), a \otimes b\rangle \ge c_0|a|^2|b|^2$ for all $a \in \mathbb{R}^m$, $b \in \mathbb{R}^d$.

Consider now the inverse $\mathcal{C}_A = \mathcal{A}^{-1}$ of the operator $\mathcal{A}$ (or the Green function) and the corresponding bilinear form on $\boldsymbol{X}_N$ defined by

$$G_A(\varphi, \psi) = \langle \mathcal{C}_A\varphi, \psi\rangle = (\varphi, \psi)_-, \quad \varphi, \psi \in \boldsymbol{X}_N. \tag{2.14}$$

Given that the operator $\mathcal{A}$ and its inverse commutes with translations on $\mathbb{T}_N$, there exists a unique kernel C_A such that

$$(\mathcal{C}_A\varphi)(x) = \sum_{y \in \mathbb{T}_N} C_A(x-y)\varphi(y), \tag{2.15}$$

(see Lemma 3.5 below). We write $C_A \in \boldsymbol{\mathcal{M}}_N$, using $\boldsymbol{\mathcal{M}}_N$ (in analogy with $\boldsymbol{X}_N$) to denote the space of all matrix-valued maps on $\mathbb{T}_N$ with zero mean. Notice that if the kernel C_A is constant, $C_A(x) = C$ for any $x \in \mathbb{T}_N$, where C is a linear operator on $\mathbb{R}^m$, then $(C_A\varphi)(x) = \sum_{y\in\mathbb{T}_N} C_A(x - y)\varphi(y) = C\sum_{y\in\mathbb{T}_N}\varphi(y) = 0$ for any $\varphi \in \boldsymbol{X}_N$. It is easy to see that the function $G_{A,y}(\cdot) = C_A(\cdot - y)$ is the unique solution $G_{A,y} \in \boldsymbol{\mathcal{M}}_N$ of the equation

$$\mathcal{A}G_{A,y} = (\delta_y - \frac{1}{L^{Nd}})\mathbb{1}, \tag{2.16}$$

where $\mathbb{1}$ is the unit $m \times m$ matrix. Notice that for any $a \in \mathbb{R}^m$ one has:

$$(\mathcal{A}G_{A,y})a = (\delta_y - \frac{1}{L^{Nd}})a \in \boldsymbol{X}_N.$$

We now state our main results.

Theorem 2.1. *The operator $\mathcal{C}_A\colon \mathcal{H}_- \to \mathcal{H}_+$ admits a finite range decomposition, i.e., there exist translation invariant positive-definite operators*

$$\mathcal{C}_{A,k}\colon \mathcal{H}_- \to \mathcal{H}_+,\ (\mathcal{C}_{A,k}\varphi)(x) = \sum_{y\in\mathbb{T}_N} C_{A,k}(x-y)\varphi(y),\ k = 1,\dots,N+1, \tag{2.17}$$

such that

$$\mathcal{C}_A = \sum_{k=1}^{N+1} \mathcal{C}_{A,k}, \tag{2.18}$$

and for each associated kernel $C_{A,k} \in \boldsymbol{\mathcal{M}}_N$ there exists a constant matrix $C_{A,k}$ such that

$$C_{A,k}(x-y) = C_{A,k} \ \text{ whenever } \ \rho_\infty(x,y) \geq \frac{1}{2}L^k \quad \text{for } k = 1,\dots,N. \tag{2.19}$$

Moreover, for any multiindex α, there exist constants $C_\alpha(d) > 0$ and $\eta(\alpha,d)$, depending only on the dimension d, such that

$$\|\nabla^\alpha C_{A,k}(x)\| \leq C_\alpha(d) L^{-(k-1)(d-2+|\alpha|)} L^{\eta(\alpha,d)} \tag{2.20}$$

for all $x \in \mathbb{T}_N$ and $k = 1,\dots,N+1$. Here, $\nabla^\alpha = \prod_{i=1}^d \nabla_i^{\alpha_i}$ and $\nabla_i^0 = \mathrm{id}$, and $\|\cdot\|$ denotes the operator norm.

Note that since any function in $\boldsymbol{\mathcal{X}}_N$ has mean zero the kernel $\widetilde{C_{A,k}} = C_{A,k} - C_{A,k}$ generates the same operator $\mathcal{C}_{A,k}$. Thus (2.19) indeed guarantees that $\mathcal{C}_{A,k}$ has finite range. See Lemmas 3.6 and 3.5 for further details.

The operator $\mathcal{A}$, its inverse $\mathcal{C}_A$, and the finite range decomposition itself, depend on the linear map $A\colon \mathbb{R}^{m\times d} \to \mathbb{R}^{m\times d}$. Our major result is that the finite range decomposition can be defined in such a way that the maps $A \mapsto C_{A,k}$ are real-analytic, as long as A is positive definite.

Let $\mathcal{L}_{\mathrm{sym}}(\mathbb{R}^{m\times d})$ denote the space of linear maps $A\colon \mathbb{R}^{m\times d} \to \mathbb{R}^{m\times d}$ that are symmetric with respect to the standard scalar product on $\mathbb{R}^{m\times d}$ and let

$$U := \left\{A \in \mathcal{L}_{\mathrm{sym}}(\mathbb{R}^{m\times d})\colon (AF,F)_{\mathbb{R}^{m\times d}} > 0 \ \text{ for all } \ F \in \mathbb{R}^{m\times d}, F \neq 0\right\} \tag{2.21}$$

denote the open subset of positive definite symmetric maps.

Theorem 2.2. *Let $d \geq 2$ and let α be a multiindex. There exist constants $C_\alpha(d)$ and $\eta(\alpha,d)$ with the following properties. For each integer $N \geq 1$, each $k = 1,\dots,N+1$ and each odd integer $L \geq 16$ there exist real-analytic maps $A \mapsto C_{A,k}$ from U to $\boldsymbol{\mathcal{M}}_N$ such that the following three assertions hold.*

(i) If $\mathcal{C}_{A,k}$ denotes the translation invariant operator on induced by $C_{A,k}$ then

$$\mathcal{C}_A = \sum_{k=1}^{N+1} \mathcal{C}_{A,k}. \tag{2.22}$$

(ii) There exist constant $m \times m$ matrices $C_{A,k}$ such that

$$C_{A,k}(x) = C_{A,k} \quad \textit{if } \rho_\infty(x,0) \geq \frac{1}{2}L^k. \tag{2.23}$$

(iii) If $(A_0F, F)_{\mathbb{R}^{m\times d}} \geq c_0\|F\|^2_{\mathbb{R}^{m\times d}}$ for all $F \in \mathbb{R}^{m\times d}$ and $c_0 > 0$ then

$$\sup_{\|\dot{A}\|\leq 1} \left\|(\nabla^\alpha D^j_A C_{A_0,k}(x)(\dot{A},\ldots,\dot{A})\right\| \leq C_\alpha(d)\left(\frac{2}{c_0}\right)^j j!\, L^{-(k-1)(d-2+|\alpha|)} L^{\eta(\alpha,d)}. \tag{2.24}$$

for all $x \in \mathbb{T}_N$ and all $j \geq 0$. Here $\nabla^\alpha = \prod_{i=1}^d \nabla_i^{\alpha_i}$, we use $\|\dot{A}\|$ to denote the operator norm of a linear mapping $\dot{A}\colon \mathbb{R}^{m\times d} \to \mathbb{R}^{m\times d}$, and the jth derivative with respect to A in the direction $\dot{A}$ is taken at A_0.

The condition $L \geq 16$ can be dropped. However, in applications to renormalization group arguments, one needs to choose L large also for other reasons. Given $\delta \in (0, 1/2)$, the property (ii) can be strengthened to $C_{A,k}(x) = C_{A,k}$ if $\rho_\infty(x, 0) > \delta L^k$, provided that L is large enough. Then the constants $C_\alpha(d)$ and $\eta(\alpha, d)$ depend also on δ. We refer to Remark 3.10 for further details.

The proof of the existence of the finite range decomposition, i.e., (2.18) and (2.19) of Theorem 2.1, is given in Section 3. The remaining proof of the regularity bounds is given in Section 4 for a fixed A. Real-analytic dependence on A and the bounds (2.24) are established in Section 5.

The research of R.K. was partially supported by the grants GAČR 201-09-1931, 201/12/2613, and MSM 0021620845.

3. Construction of the finite range decomposition

In this section we prove the existence part of Theorem 2.1 via an extension and adaption of the methods in [4] to our case. The proof of the estimates is given in Sections 4 and 5.
The existence of a finite range decomposition is contained in Proposition 3.8 and Proposition 3.9 below and their proofs are built on the following auxiliary results.

First, for the construction of the decomposition we consider the discrete cube

$$Q = \{1, \ldots, l-1\}^d \tag{3.1}$$

for some $l \in \mathbb{N}, l \geq 3$. We can identify Q with a subset of $\mathbb{T}_N$ once $l - 1 < L^N$. Similarly, any shift $Q + x \subset \mathbb{T}_N$. For any $x \in \mathbb{T}_N$, consider the subspace

$$\mathcal{H}(Q + x) = \{\varphi \in \mathcal{H}\colon \varphi = 0 \text{ in } \mathbb{T}_N \setminus (Q + x)\}. \tag{3.2}$$

We write $\mathcal{H}_+(Q + x)$ and $\mathcal{H}_-(Q + x)$ for the same space equipped with the scalar products $(\cdot,\cdot)_+$ and $(\cdot,\cdot)_-$, respectively. We denote by Π_x the $(\cdot,\cdot)_+$-orthogonal projection $\mathcal{H}_+ \to \mathcal{H}_+(Q+x)$ and set $P_x = \mathrm{id} - \Pi_x$. Thus $\Pi_x\varphi \in \mathcal{H}_+(Q + x)$ and

$$(\Pi_x\varphi, \psi)_+ = (\varphi, \psi)_+ \text{ for all } \psi \in \mathcal{H}_+(Q + x). \tag{3.3}$$

For any set $M \subset \Lambda_N$, we define its closure by

$$\overline{M} = \{x \in \Lambda_N\colon \mathrm{dist}_\infty(x, M) \leq 1\}, \quad \mathrm{dist}_\infty(x, M) := \min\{\rho_\infty(x, y)\colon y \in M\}. \tag{3.4}$$

In particular,

$$\overline{Q} = \{0, \ldots, l\}^d. \tag{3.5}$$

We also define

$$Q_- := \{0, 1, \ldots, l-1\}^d. \tag{3.6}$$

Lemma 3.1. *For any $\varphi \in \mathcal{H}_+$ we have*

(i) $\mathcal{A}(P_x\varphi) = \text{const}$ *in* $Q + x$,

(ii) $P_x\varphi = \varphi$ *in* $\mathbb{T}_N \setminus (Q + x)$,

(iii) $\Pi_x\varphi = \varphi 1_{Q+x}$ *if* $\varphi = 0$ *on* $\overline{(Q + x)} \setminus (Q + x)$.

Remark 3.2. *This shows that $P_x\varphi$ is essentially the $\mathcal{A}$-harmonic extension in $Q + x$. Thus we would expect that P_x is (locally) smoothing and suppresses locally high frequency oscillations, while Π_x suppresses locally low frequencies. This will be made precise in Lemma 4.1 below where we show the corresponding estimates for the averaged operators $\mathcal{T} = l^{-d} \sum_{x\in\mathbb{T}_N} \Pi_x$ and $\mathcal{R} = \mathrm{id} - \mathcal{T}$.*

Proof. (i): By (3.3) we have for all $\psi \in \mathcal{H}_+(Q + x)$ the relation $(P_x\varphi, \psi)_+ = 0$ and hence $\langle \mathcal{A}(P_x\varphi), \psi\rangle = 0$. Taking $\psi = \delta_v - \delta_z$ for any pair of points $v, z \in Q + x$ we get $\mathcal{A}(P_x\varphi)(v) = \mathcal{A}(P_x\varphi)(z)$. This proves (i).
(ii): This follows from the fact that $\Pi_x\varphi$ belongs to $\mathcal{H}_+(Q + x)$ and hence vanishes outside $Q + x$.
(iii): It suffices to consider the case $x = 0$ and we write Π for Π_0. Let $\tilde\varphi = \varphi 1_Q$. Then $\tilde\varphi \in \mathcal{H}_+(Q)$ and hence $\Pi\tilde\varphi = \tilde\varphi$. Moreover $\varphi - \tilde\varphi$ vanishes in $\overline{Q}$. Thus $\nabla(\varphi - \tilde\varphi)$ vanishes in Q_-. Hence $(\varphi - \tilde\varphi, \psi)_+ = 0$ for all $\psi \in \mathcal{H}_+(Q)$ since $\nabla\psi$ is supported in Q_-. Therefore $\Pi(\varphi - \tilde\varphi) = 0$ which yields the assertion.

□

Lemma 3.3.

(i) $\Pi_x\Pi_y = 0$ *whenever* $(Q_- + x) \cap (Q_- + y) = \varnothing$,

(ii) $\Pi_x\varphi = 0$ *whenever* $\mathrm{supp}\varphi \cap (\overline{Q} + x) = \varnothing$.

Proof. (i): For any $\varphi, \psi \in \mathcal{H}_+$, the functions $\Pi_x\varphi$ and $\Pi_y\psi$ vanish on $\mathbb{T}_N \setminus (Q + x)$ and $\mathbb{T}_N \setminus (Q + y)$, respectively. Hence, $\nabla\Pi_x\psi$ and $\nabla\Pi_y\varphi$ vanish on $\mathbb{T}_N \setminus (Q_- + x)$ and on $\mathbb{T}_N \setminus (Q_- + y)$, respectively. Assuming now that $Q_- + x$ and $Q_- + y$ are disjoint and taking into account (2.5) we get

$$(\psi, \Pi_x\Pi_y\varphi)_+ = (\Pi_x\psi, \Pi_y\varphi)_+ = \sum_{z\in\mathbb{T}_N} \langle A(\nabla\Pi_x\psi)(z), (\nabla\Pi_y\varphi)(z)\rangle_{\mathbb{R}^{m\times d}} = 0. \tag{3.7}$$

(ii): For $\psi \in \mathcal{H}_+(Q + x)$ we have $\mathcal{A}\psi = 0$ in $\mathbb{T}_N \setminus (\overline{Q} + x)$. Thus for any $\varphi \in \mathcal{H}_+$ with $\mathrm{supp}\varphi \cap (\overline{Q} + x) = \varnothing$ we get $(\varphi, \psi)_+ = \langle \varphi, \mathcal{A}\psi\rangle = 0$. In view of (3.3) this yields $\Pi_x\varphi = 0$. □
Next, consider the symmetric operator

$$\mathcal{T} = \frac{1}{l^d} \sum_{x\in\mathbb{T}_N} \Pi_x \tag{3.8}$$

on $\mathcal{H}_+$. The following result is the key estimate for the finite range decomposition construction. Our proof is a slight modification of the argument in [4].

Lemma 3.4. *For any $\varphi \in \mathcal{H}_+$ we have*

(i) $0 \le (\Pi_x\varphi, \varphi)_+ \le \langle 1_{Q_-+x} A\nabla\varphi, \nabla\varphi\rangle$,

(ii) $0 \le (\mathcal{T}\varphi, \varphi)_+ \le (\varphi, \varphi)_+$ *and the inequalities are strict if* $\varphi \ne 0$,

(iii) $(\mathcal{T}\varphi, \mathcal{T}\varphi)_+ \le (\mathcal{T}\varphi, \varphi)_+$.

Proof. (i): We have $(\Pi_x\varphi, \varphi)_+ = (\varphi, \Pi_x\varphi)_+ = (\Pi_x\varphi, \Pi_x\varphi)_+ \ge 0$. For the other inequality we use that $\nabla \Pi_x \varphi$ is supported in $Q_- + x$. Thus

$$(\Pi_x\varphi, \varphi)_+ = \langle A\nabla\Pi_x\varphi, \nabla\varphi\rangle = \langle A\nabla\Pi_x\varphi, 1_{Q_-+x}\nabla\varphi\rangle. \tag{3.9}$$

Since A is symmetric and positive definite the expression $(F, G)_A := \langle AF, G\rangle$ is a scalar product on functions $\mathbb{Z}^d \to \mathbb{R}^{m\times d}$. Thus the Cauchy-Schwarz inequality yields

$$\begin{aligned} \langle A\nabla\Pi_x\varphi, 1_{Q_-+x}\nabla\varphi\rangle &\le \langle A\nabla\Pi_x\varphi, \nabla\Pi_x\varphi\rangle^{1/2}\langle A1_{Q_-+x}\nabla\varphi, 1_{Q_-+x}\nabla\varphi\rangle^{1/2} \\ &= (\Pi_x\varphi, \Pi_x\varphi)_+^{1/2}\langle 1_{Q_-+x}A\nabla\varphi, \nabla\varphi\rangle^{1/2}. \end{aligned} \tag{3.10}$$

Together with (3.9) this yields the assertion since $(\Pi_x\varphi, \varphi)_+ = (\Pi_x\varphi, \Pi_x\varphi)_+$.
(ii): Since $\sum_{x\in\mathbb{T}_N} 1_{Q_-+x}(y) = l^d$ for all $y \in \mathbb{T}_N$ the inequalities follow by summing (i) over $x \in \mathbb{T}_N$. If $(\mathcal{T}\varphi, \varphi)_+ = 0$ then $(\Pi_x\varphi, \varphi)_+ = 0$ for all $x \in \mathbb{T}_N$ and thus $\Pi_x\varphi = 0$ and $P_x\varphi = \varphi$. Lemma 3.1 implies that there exist constants c_x such that $(\mathcal{A}\varphi)(y) = c_x$ for all $y \in Q + x$. Since $l \ge 3$ the cubes $Q + x$ and $Q + (x + \mathrm{e}_i)$ overlap and this yields $c_x = c_{x+\mathrm{e}_i}$ for all $i = 1, \ldots, d$. Thus c_x is independent of x. Since $\mathcal{A}\varphi \in \boldsymbol{\mathcal{X}}_N$ this implies $c = 0$. Hence $\mathcal{A}\varphi = 0$ and therefore $\varphi = 0$.
Now suppose that $(\mathcal{T}\varphi, \varphi)_+ = (\varphi, \varphi)_+$. This implies that for all $x \in \mathbb{T}_N$ we have $(\Pi_x\varphi, \varphi)_+ = \langle 1_{Q_-+x}A\nabla\varphi, \nabla\varphi\rangle$. We claim that the last identity implies that $\nabla\varphi(x) = 0$. Indeed, if $1_{Q_-+x}\nabla\varphi = 0$ we are done. Otherwise the identity can only hold if the inequality in (3.10) is an identity. In particular we must have $\nabla\Pi_x\varphi = \lambda 1_{Q_-+x}\nabla\varphi$ and $\lambda = 1$. Now $\Pi_x\varphi$ vanishes outside $Q + x$ and in particular at the points x and $x + \mathrm{e}_i$. Thus $\nabla\Pi_x\varphi(x) = 0$ and hence $\nabla\varphi(x) = 0$. It follows that φ is constant on $\mathbb{T}_N$ and hence $\varphi = 0$ since φ has mean zero.
(iii): It follows from (ii) that $(\varphi, \psi)_* := (\mathcal{T}\varphi, \psi)_+$ defines a scalar product on $\boldsymbol{\mathcal{H}}_+$. Thus the Cauchy Schwarz inequality and (ii) yield

$$(\mathcal{T}\varphi, \psi)_+ \le (\mathcal{T}\varphi, \varphi)_+^{1/2}(\mathcal{T}\psi, \psi)_+^{1/2} \le (\mathcal{T}\varphi, \varphi)_+^{1/2}(\psi, \psi)_+^{1/2}. \tag{3.11}$$

Taking $\psi = \mathcal{T}\varphi$ we obtain the desired estimate. □
Consider the operator $\mathcal{T}' : \boldsymbol{\mathcal{H}}_- \to \boldsymbol{\mathcal{H}}_-$ dual with respect to $\mathcal{T}$ and defined by

$$\langle \mathcal{T}'\varphi, \psi\rangle = \langle \varphi, \mathcal{T}\psi\rangle, \quad \varphi \in \boldsymbol{\mathcal{H}}_-, \psi \in \boldsymbol{\mathcal{H}}_+. \tag{3.12}$$

Notice that

$$\mathcal{T}' = \mathcal{A}\mathcal{T}\mathcal{A}^{-1}, \quad (\mathcal{T}'\varphi, \psi)_- = (\varphi, \mathcal{T}'\psi)_-, \quad \text{and } (\mathcal{T}'\varphi, \varphi)_- = (\mathcal{T}\mathcal{A}^{-1}\varphi, \mathcal{A}^{-1}\varphi)_+. \tag{3.13}$$

Indeed, for any $\varphi \in \boldsymbol{\mathcal{H}}_+$, we have

$$\langle \mathcal{T}'\mathcal{A}\varphi, \psi\rangle = \langle \mathcal{A}\varphi, \mathcal{T}\psi\rangle = (\varphi, \mathcal{T}\psi)_+ = (\mathcal{T}\varphi, \psi)_+ = \langle \mathcal{A}\mathcal{T}\varphi, \psi\rangle, \tag{3.14}$$

and this yields the first identity in (3.13). Now

$$(\mathcal{T}'\varphi, \psi)_- = \langle \mathcal{A}^{-1}\mathcal{A}\mathcal{T}\mathcal{A}^{-1}\varphi, \psi\rangle = \langle \mathcal{T}\mathcal{A}^{-1}\varphi, \mathcal{A}\mathcal{A}^{-1}\psi\rangle = (\mathcal{T}\mathcal{A}^{-1}\varphi, \mathcal{A}^{-1}\psi)_+. \tag{3.15}$$

Since the last expression is symmetric in φ and ψ we get the second identity in (3.13) and taking $\psi = \varphi$ we obtain the third identity. Similarly, we have $\Pi'_x = \mathcal{A}\Pi_x\mathcal{A}^{-1}$ for the dual of Π_x. Notice that

$$\Pi'_x\varphi = 0 \quad \text{whenever } \operatorname{supp}\varphi \cap (Q + x) = \emptyset. \tag{3.16}$$

Indeed, considering any test function $\psi \in \boldsymbol{\mathcal{X}}_N$, we have $\langle \Pi_x' \varphi, \psi \rangle = \langle \varphi, \Pi_x \psi \rangle = 0$. We also consider the operator

$$\mathcal{R} := \mathrm{id} - \mathcal{T} \quad \text{and its dual } \mathcal{R}' = \mathrm{id} - \mathcal{T}'. \tag{3.17}$$

It follows from Lemma 3.4(ii) and (3.13) that

$$(\mathcal{T}'\varphi, \varphi)_- > 0, \quad (\mathcal{R}'\varphi, \varphi)_- > 0, \quad (\mathcal{T}'\varphi, \mathcal{T}'\varphi)_- \leq (\mathcal{T}'\varphi, \varphi)_- \ \text{ for all } \ \varphi \in \boldsymbol{\mathcal{H}}_- \setminus \{0\}. \tag{3.18}$$

We next discuss the locality properties of translation invariant bilinear forms, operators and the corresponding kernels. These properties would be obvious if we consider bilinear forms on $\boldsymbol{\mathcal{V}}_N$ since then we can use the Dirac masses δ_x as test functions. Dirac masses, however, do not belong to $\boldsymbol{\mathcal{X}}_N$ and hence we need to use test functions with broader support which makes the conclusion of Lemma 3.6 below nontrivial. We begin by recalling the relation between bilinear forms, operators and kernels. The translation operator $\tau_a, a \in \mathbb{T}_N$, is defined by $\tau_a \varphi(y) = \varphi(y - a)$, so that $\tau_a \delta_y = \delta_{a+y}$. Recall that $\boldsymbol{\mathcal{M}}_N$ denotes the space of all matrix-valued, L^N periodic maps with zero mean.

Lemma 3.5. *Let B be a translation invariant bilinear form on $\boldsymbol{\mathcal{X}}_N$, i.e.,*

$$B(\tau_a \varphi, \tau_a \psi) = B(\varphi, \psi) \ \text{for all} \ \varphi, \psi \in \boldsymbol{\mathcal{X}}_N, \ \text{for all} \ a \in \mathbb{T}_N. \tag{3.19}$$

Then the following assertions hold.

(i) There exists a unique linear operator $\mathcal{B} \colon \boldsymbol{\mathcal{X}}_N \to \boldsymbol{\mathcal{X}}_N$ such that

$$\langle \mathcal{B}\varphi, \psi \rangle = B(\varphi, \psi) \ \text{for all} \ \varphi, \psi \in \boldsymbol{\mathcal{X}}_N. \tag{3.20}$$

Moreover $\mathcal{B}$ is translation invariant, i.e., $\mathcal{B}\tau_a = \tau_a \mathcal{B}$.

(ii) There exists a unique matrix-valued kernel $\mathscr{B} \in \boldsymbol{\mathcal{M}}_N$ such that

$$(\mathcal{B}\varphi)(x) = \sum_{y \in \mathbb{T}_N} \mathscr{B}(x - y)\varphi(y) \ \text{for all} \ x \in \mathbb{T}_N, \ \text{for all} \ \varphi \in \boldsymbol{\mathcal{X}}_N. \tag{3.21}$$

Moreover for $\tilde{\mathscr{B}} \colon \mathbb{T}_N \to \mathbb{R}^{m \times m}$ we have

$$(\mathcal{B}\varphi)(x) = \sum_{y \in \mathbb{T}_N} \tilde{\mathscr{B}}(x - y)\varphi(y) \ \text{for all} \ x \in \mathbb{T}_N \ \text{for all} \ \varphi \in \boldsymbol{\mathcal{X}}_N. \tag{3.22}$$

if and only if

$$\tilde{\mathscr{B}} - \mathscr{B} = C \tag{3.23}$$

with a constant $m \times m$ matrix C.

(iii) If $\mathscr{B}' \in \boldsymbol{\mathcal{X}}_N$ denotes the kernel of the dual operator $\mathcal{B}'$ then

$$\mathscr{B}'(z) = \mathscr{B}(-z). \tag{3.24}$$

(iv) If $\mathcal{B}_1$ and $\mathcal{B}_2$ are translation invariant operators on $\boldsymbol{\mathcal{X}}_N$ and $\mathcal{B}_3 = \mathcal{B}_1 \mathcal{B}_2$ then $\mathcal{B}_3$ is translation invariant and the corresponding kernels $\mathscr{B}_i \in \boldsymbol{\mathcal{M}}_N, i = 1, 2, 3,$ are related by discrete convolution, i.e.,

$$\mathscr{B}_3(x) = (\mathscr{B}_1 * \mathscr{B}_2)(x) := \sum_{y \in \mathbb{T}_N} \mathscr{B}_1(x - y)\mathscr{B}_2(y) \quad \text{for all } x \in \mathbb{T}_N. \tag{3.25}$$

Proof. We include the elementary proof for the convenience of the reader.
(i): Existence and uniqueness of $\mathcal{B}$ follows from the Riesz representation theorem. To prove translation invariance let $\mathcal{D} := \mathcal{B}\tau_a - \tau_a\mathcal{B}$. We have

$$\langle \mathcal{D}\varphi, \tau_a\psi\rangle = \langle \mathcal{B}\tau_a\varphi, \tau_a\psi\rangle - \langle \tau_a\mathcal{B}\varphi, \tau_a\psi\rangle. \tag{3.26}$$

Since τ_a is an isometry with respect to the scalar product $\langle\cdot,\cdot\rangle$ we get

$$\langle \mathcal{D}\varphi, \tau_a\psi\rangle = B(\tau_a\varphi, \tau_a\psi) - \langle \mathcal{B}\varphi, \psi\rangle = B(\tau_a\varphi, \tau_a\psi) - B(\varphi, \psi) = 0. \tag{3.27}$$

This holds for all $\varphi, \psi \in \boldsymbol{X}_N$. Hence $\mathcal{D} = 0$.
(ii): To show the existence of $\mathscr{B}$ note that $(\delta_0 - \frac{1}{L^{Nd}})a \in \boldsymbol{X}_N$ with any $a \in \mathbb{R}^m$ and define

$$\mathscr{B}(x)a := \mathcal{B}[a(\delta_0 - \tfrac{1}{L^{Nd}})](x). \tag{3.28}$$

Using that $\tau_y(\delta_0) = \delta_y$, we get

$$\begin{aligned}\mathscr{B}(x-y)a = \mathcal{B}[a(\delta_0 - \tfrac{1}{L^{Nd}})](x-y) = \tau_y(\mathcal{B}[a(\delta_0 - \tfrac{1}{L^{Nd}})])(x) = \\ = \mathcal{B}\tau_y[a(\delta_0 - \tfrac{1}{L^{Nd}})](x) = \mathcal{B}[a(\delta_y - \tfrac{1}{L^{Nd}})](x).\end{aligned} \tag{3.29}$$

Observing, further, that for any $\varphi \in \boldsymbol{X}_N$, we have

$$\varphi = \sum_{y\in\mathbb{T}_N} \varphi(y)(\delta_y - \tfrac{1}{L^{Nd}}), \tag{3.30}$$

we get

$$(\mathcal{B}\varphi)(x) = \sum_{y\in\mathbb{T}_N} \mathcal{B}[\varphi(y)(\delta_y - \tfrac{1}{L^{Nd}})](x) = \sum_{y\in\mathbb{T}_N} \mathscr{B}(x-y)\varphi(y). \tag{3.31}$$

This shows the existence of $\mathscr{B}$. Suppose now that $\mathcal{B}\varphi(x) = \sum_{y\in\mathbb{T}_N} \tilde{\mathscr{B}}(x-y)\varphi(y)$ for all $\varphi \in \boldsymbol{X}_N$ and set $C(z) = \tilde{\mathscr{B}}(z) - \mathscr{B}(z)$. The choice $\varphi = (\delta_{\mathrm{e}_i} - \delta_0)a$ with an arbitrary $a \in \mathbb{R}^m$ yields

$$C(x - \mathrm{e}_i) = C(x) \text{ for all } x \in \mathbb{T}_N, \text{ for all } i = 1, \dots, d. \tag{3.32}$$

Thus $\tilde{\mathscr{B}} - \mathscr{B} = C$ with a constant $m\times m$ matrix C. If in addition $\tilde{\mathscr{B}} \in \boldsymbol{\mathcal{M}}_N$ then this implies $\tilde{\mathscr{B}} = \mathscr{B}$. Conversely if $\tilde{\mathscr{B}} = \mathscr{B} + C$ then $\tilde{\mathscr{B}}$ and $\mathscr{B}$ generate the same operator since $\sum_{x\in\mathbb{T}_N}\varphi(x) = 0$ for $\varphi \in \boldsymbol{X}_N$.
(iii): We have

$$\begin{aligned}\langle \mathcal{B}'\varphi, \psi\rangle &= \langle \varphi, \mathcal{B}\psi\rangle = \sum_{x,y\in\mathbb{T}_N} \langle \varphi(x), \mathscr{B}(x-y)\psi(y)\rangle_{\mathbb{R}^m} = \sum_{x,y\in\mathbb{T}_N} \langle \varphi(y), \mathscr{B}(y-x)\psi(x)\rangle_{\mathbb{R}^m} \\ &= \sum_{x,y\in\mathbb{T}_N} \langle \mathscr{B}(-[x-y])\varphi(y), \psi(x)\rangle_{\mathbb{R}^m}.\end{aligned} \tag{3.33}$$

Hence $(\mathcal{B}'\varphi)(x) = \sum_{y\in\mathbb{T}_N} \mathscr{B}(-[x-y])\varphi(y)$ and the uniqueness result in (ii) implies that $\mathscr{B}'(z) = \mathscr{B}(-z)$.
(iv): One easily verifies that $\mathscr{B}_1 * \mathscr{B}_2 \in \boldsymbol{\mathcal{M}}_N$ and that $(\mathcal{B}_1\mathcal{B}_2\varphi)(x) = \sum_{y\in\mathbb{T}_N} \mathscr{B}_1 * \mathscr{B}_2(x-y)\varphi(y)$. Thus the assertion follows from the uniqueness result in (ii). □

For two sets $M_1, M_2 \subset \mathbb{T}_N$ we define

$$\operatorname{dist}_\infty(M_1, M_2) := \min\{\rho_\infty(x,y) \colon x \in M_1, y \in M_2\}. \tag{3.34}$$

Lemma 3.6. *Let B be a translation invariant bilinear form on $\boldsymbol{\mathcal{X}}_N$ and let $\mathcal{B}$ and $\mathscr{B} \in \boldsymbol{\mathcal{M}}_N$ be the associated operator and the associated kernel, respectively. Let n be an integer and suppose that $L^N > 2n+3$. Then the following three statements are equivalent.*

(i) $B(\varphi, \psi) = 0$ *whenever* $\operatorname{dist}_\infty(\operatorname{supp}\varphi, \operatorname{supp}\psi) > n$.

(ii) *There exists an $m \times m$ matrix C such that $\mathscr{B}(z) = C$ whenever $\rho_\infty(z, 0) > n$.*

(iii) $\operatorname{supp}\mathcal{B}\varphi \subset \operatorname{supp}\varphi + \{-n, \dots, n\}^d$ *for all* $\varphi \in \boldsymbol{\mathcal{X}}_N$.

Proof. The implication (ii) $\Longrightarrow$ (iii) is easy. Set $\tilde{\mathscr{B}}(z) = \mathscr{B}(z) - C$. Then $\tilde{\mathscr{B}}(z) = 0$ if $\rho_\infty(z) > n$ with $\rho_\infty(z) = \rho_\infty(z, 0)$ and by Lemma 3.5(ii) we have

$$(\mathcal{B}\varphi)(x) = \sum_{y \in \mathbb{T}_N} \tilde{\mathscr{B}}(x-y)\varphi(y). \tag{3.35}$$

If $x \notin \operatorname{supp}\varphi + \{-n, \dots, n\}^d$ then either $y \notin \operatorname{supp}\varphi$ or $y \in \operatorname{supp}\varphi$ and $\rho_\infty(x-y, 0) > n$. In either case $\mathcal{B}\varphi(x) = 0$.

The implication (iii) $\Longrightarrow$ (i) is also easy. Suppose that $\operatorname{dist}_\infty(\operatorname{supp}\varphi, \operatorname{supp}\psi) > n$. Then (iii) implies that $\operatorname{dist}_\infty(\operatorname{supp}\mathcal{B}\varphi, \operatorname{supp}\psi) > 0$, i.e, $\mathcal{B}\varphi$ and ψ have disjoint support. Thus $B(\varphi, \psi) = \langle \mathcal{B}\varphi, \psi \rangle = 0$.

To prove the implication (i) $\Longrightarrow$ (ii), consider the torus $\mathbb{T}_N$ with the fundamental domain $\Lambda_N = \{-\frac{L^N-1}{2}, \dots, \frac{L^N-1}{2}\}^d$ and set

$$\begin{aligned} M &:= \{-n, \dots, n\}^d, & (3.36)\\ M_- &:= \{-n, \dots, n+1\}^d, \text{ and the closure of } M, & (3.37)\\ \overline{M} &= \{-(n+1), -n, \dots, n+1\}^d. & (3.38) \end{aligned}$$

Note that by the assumption $L^N > 2n+3$ the set $\mathbb{T}_N \setminus \overline{M}$ is nonempty.
We first show that for all $i, j \in \{1, \dots, d\}$ we have

$$\nabla_i \nabla_j^* \mathscr{B} = 0 \quad \text{in } \mathbb{T}_N \setminus \overline{M}. \tag{3.39}$$

To see this let $\xi \in \mathbb{T}_N \setminus \overline{M}$ and consider

$$\psi = (\delta_{\xi + \mathrm{e}_i} - \delta_\xi)a, \quad \varphi = (\delta_{\mathrm{e}_j} - \delta_0)b \text{ with } a, b \in \mathbb{R}^m. \tag{3.40}$$

Since $|\mathrm{e}_i - \mathrm{e}_j|_\infty \le 1$ we have

$$\operatorname{dist}_\infty(\operatorname{supp}\varphi, \operatorname{supp}\psi) \ge \rho_\infty(0, \xi) - 1 \ge n+1. \tag{3.41}$$

Hence

$$0 = B(\varphi, \psi) = \langle (\mathscr{B}(\xi + \mathrm{e}_i - \mathrm{e}_j) - \mathscr{B}(\xi + \mathrm{e}_i) - (\mathscr{B}(\xi - \mathrm{e}_j) - \mathscr{B}(\xi)))a, b \rangle = \langle \nabla_i \nabla_j^* \mathscr{B}(\xi)a, b \rangle \tag{3.42}$$

for every $a, b \in \mathbb{R}^m$ and thus

$$\nabla_i \nabla_j^* \mathscr{B}(\xi) = 0. \tag{3.43}$$

Next we show that for $j \in \{1, \dots, d\}$ there exist a matrix C_j such that

$$\nabla_j^* \mathscr{B} = C_j \quad \text{in } \mathbb{T}_N \setminus \overline{M}. \tag{3.44}$$

Fix j and set $f = \nabla_j^* \mathcal{B}$. Using the shorthand $I = \{-n-1, \dots, n+1\}$, let

$$x_1 \in \{-\tfrac{L^N-1}{2}, \dots, \tfrac{L^N-1}{2}\} \setminus I, \quad x' := (x_2, \dots, x_d) \in \{-\tfrac{L^N-1}{2}, \dots, \tfrac{L^N-1}{2}\}^{d-1}. \tag{3.45}$$

Then $x = (x_1, x') \in \mathbb{T}_N \setminus \overline{M}$ and hence for $i \neq 1$ we have $(\nabla_i f)(x_1, x') = 0$. Thus there exists a matrix-valued function g_1 on $\{-\frac{L^N-1}{2}, \dots, \frac{L^N-1}{2}\} \setminus I$ such that

$$f(x) = g_1(x_1) \quad \text{if } x_1 \notin I. \tag{3.46}$$

In the same manner, there exists a function g_2 such that $f(x) = g_2(x_2)$ if $x_2 \notin I$. This eventually implies (3.44).

Further,

$$\nabla_j^* \mathcal{B}(-n-1, x') - \nabla_j^* \mathcal{B}(-n-2, x') = (\nabla_1 \nabla_j^* \mathcal{B})(-n-2, x') = 0. \tag{3.47}$$

Hence we also have $\nabla_j^* \mathcal{B}(x) = C_j$ if $x_1 = -n-1$. Arguing similarly for the other components of x we get

$$\nabla_j^* \mathcal{B} = C_j \quad \text{in } \mathbb{T}_N \setminus M_-. \tag{3.48}$$

We now show that $C_j = 0$. Assume without loss of generality $j \neq 1$. For $x_1 \notin (I \setminus \{-n-1\})$ we have

$$\nabla_j^* \mathcal{B}(x_1, x') = C_j \ \text{ for all } \ x' \in \{-\tfrac{L^N-1}{2}, \dots, \tfrac{L^N-1}{2}\}^{d-1}. \tag{3.49}$$

On the other hand $\sum_{x' \in \{-\frac{L^N-1}{2}, \dots, \frac{L^N-1}{2}\}^{d-1}} \nabla_j^* \mathcal{B}(x_1, x') = 0$ since $\mathcal{B}$ is periodic and $j \neq 1$. Thus $C_j = 0$.

Arguing as in the derivation of (3.44) we conclude from (3.48) and the fact that $C_j = 0$ that there exists a matrix C such that

$$\mathcal{B} = C \quad \text{in } \mathbb{T}_N \setminus M_-. \tag{3.50}$$

In addition we have

$$\mathcal{B}(n+1, x') - \mathcal{B}(n+2, x') = (\nabla_1^* \mathcal{B})(n+2, x') = 0. \tag{3.51}$$

Hence $\mathcal{B}(x) = C$ for $x_1 = n+1$. Arguing similarly for the other components of x we get $\mathcal{B} = C$ in $\mathbb{T}_N \setminus M$. This finishes the proof of Lemma 3.6 (ii). □

Lemma 3.7. *Suppose that* $\operatorname{dist}_\infty(\operatorname{supp}\varphi, \operatorname{supp}\psi) > l - 1$. *Then*

$$\langle \mathcal{T}\varphi, \psi \rangle = 0, \quad \langle \mathcal{T}'\varphi, \psi \rangle = 0, \quad \langle \mathcal{R}\varphi, \psi \rangle = 0, \quad \langle \mathcal{R}'\varphi, \psi \rangle = 0. \tag{3.52}$$

Proof. It suffices to prove the first identity. The second follows by exchanging φ and ψ and the third and fourth follow since $\mathcal{R} = \mathrm{id} - \mathcal{T}$ and $\mathcal{R}' = \mathrm{id} - \mathcal{T}'$. By Lemma 3.3 we have

$$\Pi_x \varphi = 0 \quad \text{if } \operatorname{supp}\varphi \cap (\overline{Q} + x) = \varnothing \tag{3.53}$$

and it follows from the definition of Π_x that $\operatorname{supp}\Pi_x\varphi \subset Q + x$. Assume $\langle \mathcal{T}\varphi, \psi \rangle \neq 0$. Then there exist $x \in \mathbb{T}_N$ such that $\langle \Pi_x \varphi, \psi \rangle \neq 0$. Thus $\operatorname{supp}\psi \cap (Q + x) \neq \varnothing$ and $\operatorname{supp}\varphi \cap (\overline{Q} + x) \neq \varnothing$. Therefore there exist $\xi \in \overline{Q}$ and $\zeta \in Q$ such that $x + \xi \in \operatorname{supp}\varphi$, $x + \zeta \in \operatorname{supp}\psi$. Thus

$$x + \xi - (x + \zeta) = \xi - \zeta \in \{-(l-1), \dots, l-1\}^d. \tag{3.54}$$

Hence $\operatorname{dist}_\infty(\operatorname{supp}\varphi, \operatorname{supp}\psi) \leq l - 1$. □

Consider now the inverse $\mathcal{C} = \mathcal{A}^{-1}$. (For time being, we omit the reference to the matrix A in the notation for $\mathcal{C}$. We will reinstate it in Section 5, where we explicitly discuss the smoothness with respect to A.) The main step toward the decomposition, is to subtract a positive definite operator from $\mathcal{C}$ in such a way that the remnant is positive definite and of finite range. We define

$$\mathcal{C}_1 := \mathcal{C} - \mathcal{R}\mathcal{C}\mathcal{R}', \text{ which yields } C_1 = C - \mathcal{R} * C * \mathcal{R}'. \tag{3.55}$$

Proposition 3.8. *Both $\mathcal{C}_1$ and $\mathcal{R}\mathcal{C}\mathcal{R}'$ are positive definite and $\mathcal{C}_1$ has finite range, i.e.,*

$$\langle \mathcal{C}_1\varphi, \psi\rangle = 0 \quad \textit{if } \operatorname{dist}_\infty(\operatorname{supp}\varphi, \operatorname{supp}\psi) > 2l - 3. \tag{3.56}$$

In particular, there exists an $m \times m$ matrix C such that

$$C_1(z) = C \textit{ if } \rho_\infty(z, 0) > 2l - 3. \tag{3.57}$$

Proof. For any $\varphi, \psi \in \boldsymbol{X}_N$, we use (2.14) to get

$$\langle \mathcal{R}\mathcal{C}\mathcal{R}'\varphi, \varphi\rangle = (\mathcal{R}'\varphi, \mathcal{R}'\varphi)_- \geq 0. \tag{3.58}$$

If $\mathcal{R}'\varphi = 0$ then (3.18) implies that $\varphi = 0$. Thus $\mathcal{R}\mathcal{C}\mathcal{R}'$ is positive definite. Furthermore,

$$\begin{aligned} \langle \mathcal{C}_1\varphi, \psi\rangle &= \langle \mathcal{C}\varphi, \psi\rangle - \langle \mathcal{C}\mathcal{R}'\varphi, \mathcal{R}'\psi\rangle \\ &= (\varphi, \psi)_- - (\mathcal{R}'\varphi, \mathcal{R}'\psi)_- = (\mathcal{T}'\varphi, \psi)_- + (\varphi, \mathcal{T}'\psi)_- - (\mathcal{T}'\varphi, \mathcal{T}'\psi)_-. \end{aligned} \tag{3.59}$$

Thus (3.18) implies that $\mathcal{C}_1$ is positive definite.

To evaluate the range of the quadratic form $\langle \mathcal{C}_1\varphi, \psi\rangle$, we inspect the terms on the right hand side of (3.59). For the first (and similarly the second) term, we have

$$(\mathcal{T}'\varphi, \psi)_- = \frac{1}{l^d} \sum_{x \in \mathbb{T}_N} (\Pi_x'\varphi, \psi)_- = \frac{1}{l^d} \sum_{x \in \mathbb{T}_N} (\Pi_x'\varphi, \Pi_x'\psi)_-. \tag{3.60}$$

In view of (3.16), a term in the sum vanishes at x except when the supports of φ and ψ both intersect $Q + x$. Therefore, the scalar product is zero whenever the distance of the supports is strictly greater than $l-1$. The second term of the bilinear form $G_1(\varphi, \psi) := \langle \mathcal{C}_1\varphi, \psi\rangle$ is the double sum

$$(\mathcal{T}'\varphi, \mathcal{T}'\psi)_- = \frac{1}{l^d} \sum_{y \in \mathbb{T}_N} \frac{1}{l^d} \sum_{x \in \mathbb{T}_N} (\Pi_y'\varphi, \Pi_x'\psi)_-. \tag{3.61}$$

By Lemma 3.3 we have $\Pi_x'\Pi_y' = \mathcal{A}\Pi_x\Pi_y\mathcal{A}^{-1} = 0$ whenever $(Q_- + x) \cap (Q_- + y) = \emptyset$, i.e., if $\rho_\infty(x, y) > l - 1$. Hence the double sum only contains a non-zero contribution if there exist x and y such that $\rho_\infty(x, y) \leq l - 1$, $\operatorname{supp}\varphi \cap Q + x \neq \emptyset$, and $\operatorname{supp}\psi \cap Q + y \neq \emptyset$. Hence there must exist $\xi, \zeta \in Q$ such that $x + \xi \in \operatorname{supp}\varphi$ and $y + \zeta \in \operatorname{supp}\psi$. Hence

$$\operatorname{dist}_\infty(\operatorname{supp}\varphi, \operatorname{supp}\psi) \leq \rho_\infty(x + \xi - (y + \zeta), 0) \leq \rho_\infty(x - y, 0) + \rho_\infty(\xi - \zeta, 0) \leq l - 1 + l - 2 \leq 2l - 3. \tag{3.62}$$

This proves (3.56), and (3.57) follows from Lemma 3.6. □

We construct a finite range decomposition by an iterated application of Proposition 3.8. Let $L \geq 16$ and consider

$$Q_j = \{1, \ldots, l_j - 1\}^d \quad \text{with } l_j = \lfloor \tfrac{1}{8} L^j \rfloor + 1 \quad \text{for } j = 1, \ldots, N. \tag{3.63}$$

Here $\lfloor a \rfloor$ denotes the integer part of a, the largest integer not greater than a. In particular we have

$$\tfrac{1}{8}L^j < l_j \le \tfrac{1}{8}L^j + 1. \tag{3.64}$$

We define $\mathcal{T}_j$, $\mathcal{T}'_j$, and $\mathcal{R}'_j$ as before with Q replaced by Q_j and set

$$\mathcal{C}_k := (\mathcal{R}_1 \dots \mathcal{R}_{k-1})\mathcal{C}(\mathcal{R}'_{k-1} \dots \mathcal{R}'_1) - (\mathcal{R}_1 \dots \mathcal{R}_{k-1}\mathcal{R}_k)\mathcal{C}(\mathcal{R}'_k\mathcal{R}'_{k-1} \dots \mathcal{R}'_1),\ k = 1, \dots, N, \tag{3.65}$$

and

$$\mathcal{C}_{N+1} := (\mathcal{R}_1 \dots \mathcal{R}_{N-1} \dots \mathcal{R}_N)\mathcal{C}(\mathcal{R}'_N\mathcal{R}'_{N-1} \dots \mathcal{R}'_1). \tag{3.66}$$

With these definitions, we show that the sequence $\{\mathcal{C}_k\}_{k=1,\dots,N+1}$ yields a finite range decomposition.

Proposition 3.9. *Suppose that $L \ge 16$. Then the operators $\mathcal{C}_k$ satisfy*

(i) $\mathcal{C} = \sum_{k=1}^{N+1} \mathcal{C}_k$.

(ii) *$\mathcal{C}_k$ is positive definite for $k = 1, \dots, N + 1$.*

(iii) *For $k = 1, \dots, N$ the range of $\mathcal{C}_k$ is bounded by $\frac{1}{2}L^k$, i.e.,*

$$\langle \mathcal{C}_k\varphi, \psi\rangle = 0 \quad \textit{if}\ \mathrm{dist}_\infty(\mathrm{supp}\varphi, \mathrm{supp}\psi) > \tfrac{1}{2}L^k \tag{3.67}$$

and there exist $m \times m$ matrices C_k such that

$$C_k(z) = C_k \ \ \textit{if}\ \ \rho_\infty(z, 0) > \tfrac{1}{2}L^k. \tag{3.68}$$

Remark 3.10.
(i) Let $\delta \in (0, 1/2)$. Then we can obtain the sharper conclusion $C_k(z) = C_k$ if $\rho(z, 0) > \delta L^k$, provided that L is large enough and we choose the integers l_j sufficiently small, e.g. we may take $l_j = \lfloor \delta L^j/4 \rfloor + 1$ if $L \ge 8/\delta$. Of course the regularity estimates (2.20) *then depend on δ and degenerate for $\delta \to 0$.*
(ii) The restriction $L \ge 16$ can be removed. If $6 \le L \le 15$ we can take $l_1 = 3$ and for $j \ge 2$ define l_j as before. One can easily check that in this case we still have $-1 + 2\sum_{j=1}^{k}(l_j - 1) \le L^k/2$ and this yields the desired assertion (see the proof below). If $3 \le L \le 5$ one can skip the first few renormalization steps. Formally one can take $l_1 = l_2 = 2$ and define l_j as before for $j \ge 3$. Then $\mathcal{T}_1 = \mathcal{T}_2 = 0$, $\mathcal{R}_1 = \mathcal{R}_2 = \mathrm{id}$, $\mathcal{C}_1 = \mathcal{C}_2 = 0$ and $-1 + 2\sum_{j=3}^{k}(l_j - 1) \le L^k/2$.

Proof. Assertion (i) follows directly from the definition. To prove (ii), set

$$\varphi_k := \mathcal{R}'_{k-1} \dots \mathcal{R}'_1\varphi, \quad \psi_k := \mathcal{R}'_{k-1} \dots \mathcal{R}'_1\psi, \quad k = 1, \dots, N + 1. \tag{3.69}$$

Inductive application of (3.18) shows that $\varphi_k = 0$ implies $\varphi = 0$. Now, directly from definitions, $\langle \mathcal{C}_{N+1}\varphi, \varphi\rangle = (\varphi_{N+1}, \varphi_{N+1})_-$. Thus $\mathcal{C}_{N+1}$ is positive definite. For $k = 1, \dots, N$ we have

$$\langle \mathcal{C}_k\varphi, \varphi\rangle = \langle (\mathcal{C} - \mathcal{R}_k\mathcal{C}\mathcal{R}'_k)\varphi_k, \varphi_k\rangle. \tag{3.70}$$

Hence by Proposition 3.8 we get $\langle \mathcal{C}_k\varphi, \varphi\rangle \ge 0$ with equality only holding if $\varphi_k = 0$, which implies $\varphi = 0$. Thus $\mathcal{C}_k$ is positive definite.
(iii): In view of the equation $\langle \mathcal{C}_k\varphi, \psi\rangle = \langle (\mathcal{C} - \mathcal{R}_k\mathcal{C}\mathcal{R}'_k)\varphi_k, \psi_k\rangle$, Proposition 3.8 implies that

$$\langle \mathcal{C}_k\varphi, \psi\rangle = 0 \quad \text{if } \mathrm{dist}_\infty(\mathrm{supp}\varphi_k, \mathrm{supp}\psi_k) > 2l_k - 3. \tag{3.71}$$

Iterative application of Lemma 3.7 and Lemma 3.6 yields

$$\mathrm{supp}\varphi_k \subset \mathrm{supp}\varphi + \{-n_k, \dots, n_k\}^d, \quad \mathrm{supp}\psi_k \subset \mathrm{supp}\psi + \{-n_k, \dots, n_k\}^d, \quad n_k = \sum_{j=1}^{k-1}(l_j - 1). \tag{3.72}$$

Thus

$$\langle \mathcal{C}_k\varphi, \psi\rangle = 0 \quad \text{if } \mathrm{dist}_\infty(\mathrm{supp}\varphi_k, \mathrm{supp}\psi_k) > -1 + 2\sum_{j=1}^{k}(l_j - 1). \tag{3.73}$$

Now since $l_j - 1 \le \frac{1}{8}L^j$ and $\sum_{n=0}^{\infty} L^{-n} \le 2$ we get $2\sum_{j=1}^{k}(l_j - 1) \le \frac{1}{2}L^k$. This finishes the proof. □

4. Estimates for fixed *A*

To prove the regularity bounds of Theorem 2.1 and Theorem 2.2 we derive estimates for the Fourier multipliers of the relevant operators. To this end, we first extend the space $\boldsymbol{\mathcal{V}}_N$ to the set $\boldsymbol{\mathcal{V}}_N = (\mathbb{C}^m)^{L^{Nd}}$ of complex-valued vectors with the subspace $\boldsymbol{\mathcal{X}}_N$ defined, again, as the subset of functions $\varphi \in \boldsymbol{\mathcal{V}}_N$ with vanishing sum, $\sum_{x\in\mathbb{T}_N}\varphi(x) = 0$. Various scalar products and norms are extended to complex-valued functions in the usual way, $\langle\varphi,\psi\rangle := \sum_{x\in\mathbb{T}_N}\langle\varphi(x),\psi^*(y)\rangle_{\mathbb{C}^m}$, $\varphi,\psi \in \boldsymbol{\mathcal{V}}_N$, with $\psi^*(x)$ denoting the complex conjugate of $\psi(x)$.

To introduce the discrete Fourier transform, consider the set of (scalar) functions $f_p(x) = \mathrm{e}^{i\langle p,x\rangle}$, $p \in \widehat{\mathbb{T}}_N$, labelled by the dual torus

$$\widehat{\mathbb{T}}_N = \{p = (p_1, \dots, p_d)\colon\, p_j \in \{\tfrac{(-L^N+1)\pi}{L^N}, \tfrac{(-L^N+3)\pi}{L^N}, \dots, 0, \dots, \tfrac{(L^N-1)\pi}{L^N}\}, j = 1, \dots, d\}. \tag{4.1}$$

The family of functions $\{L^{-\frac{Nd}{2}} f_p\}_{p\in\widehat{\mathbb{T}}_N}$ forms an orthonormal basis of $\mathbb{C}^{L^{Nd}}$. For any $\psi \in \boldsymbol{\mathcal{V}}_N$, we can define the Fourier transform component-wise,

$$\widehat{\psi}(p) := \sum_{x\in\mathbb{T}_N} f_p(-x)\psi(x) \;\text{ for }\; p \in \widehat{\mathbb{T}}_N. \tag{4.2}$$

For $\psi \in \boldsymbol{\mathcal{X}}_N$, we get $\widehat{\psi}(0) = \sum_{x\in\mathbb{T}_n}\psi(x) = 0$ with the inverse

$$\psi(x) = L^{-Nd}\sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}} f_p(x)\widehat{\psi}(p) \tag{4.3}$$

and

$$\langle\varphi,\psi\rangle = L^{-Nd}\sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}}\langle\widehat{\varphi}(p),\widehat{\psi}(p)\rangle_{\mathbb{C}^m}. \tag{4.4}$$

In the same way (component-wise) we get the Fourier transform for a matrix-valued kernel $\mathcal{K} \in \boldsymbol{\mathcal{M}}_N$,

$$\widehat{\mathcal{K}}(p) := \sum_{x\in\mathbb{T}_N} f_p(-x)\mathcal{K}(x). \tag{4.5}$$

For a translation invariant operator $\mathcal{K}\colon \boldsymbol{\mathcal{X}}_N \to \boldsymbol{\mathcal{X}}_N$ with the kernel $\mathcal{K} \in \boldsymbol{\mathcal{M}}_N$, we get

$$\widehat{\mathcal{K}\psi}(p) = \widehat{\mathcal{K}}(p)\widehat{\psi}(p). \tag{4.6}$$

Indeed,

$$\begin{aligned}\widehat{\mathcal{K}\psi}(p) &= \sum_{x\in\mathbb{T}_N}\sum_{y\in\mathbb{T}_N}\mathcal{K}(x-y)\psi(y)f_p(-x) = \sum_{x\in\mathbb{T}_N}\sum_{y\in\mathbb{T}_N}\mathcal{K}(x-y)\psi(y)f_p(-(x-y))f_p(-y) = \\ &= \sum_{z\in\mathbb{T}_N}\mathcal{K}(z)f_p(-z)\sum_{y\in\mathbb{T}_N}\psi(y)f_p(-y) = \widehat{\mathcal{K}}(p)\widehat{\psi}(p). \end{aligned} \tag{4.7}$$

Henceforth, we call $\widehat{\mathcal{K}}$ the Fourier multiplier of $\mathcal{K}$. Applying the equality (4.6) with $\psi = af_p$, $p \neq 0$, we get

$$\mathcal{K}af_p = \widehat{\mathcal{K}}(p)af_p. \tag{4.8}$$

Indeed, taking into account that $\widehat{f_p}(p') = L^{Nd}\delta_{p,p'}$, we have $\widehat{\mathcal{K}af_p}(p') = \widehat{\mathcal{K}}(p')\widehat{af_p}(p') = L^{Nd}\delta_{p,p'}\widehat{\mathcal{K}}(p')a = L^{Nd}\delta_{p,p'}\widehat{\mathcal{K}}(p)a$ and thus

$$\mathcal{K}af_p(x) = L^{-Nd}\sum_{p'\in\widehat{\mathbb{T}}_N\setminus\{0\}} f_{p'}(x)\widehat{\mathcal{K}af_p}(p') = \sum_{p'\in\widehat{\mathbb{T}}_N\setminus\{0\}} f_{p'}(x)\delta_{p,p'}\widehat{\mathcal{K}}(p)a = \widehat{\mathcal{K}}(p)af_p. \tag{4.9}$$

Notice also that, by Lemma 3.5, the kernel of a product of two translation invariant operators is given by the discrete convolution of the kernels and thus

$$\widehat{\mathcal{K}_1 * \mathcal{K}_2}(p) = \widehat{\mathcal{K}}_1(p)\widehat{\mathcal{K}}_2(p). \tag{4.10}$$

Now, we will study the Fourier multipliers of the operators $\mathcal{A} = \nabla^* A\nabla$ as well as the operators $\mathcal{T}$ and $\mathcal{R}$ introduced in the previous section. Given that $\nabla_j f_p = q_j(p)f_p$ with $q_j(p) = \mathrm{e}^{ip_j} - 1$ and $\nabla_j^* f_p = q_j^*(p)f_p$, $j = 1,\ldots,d$, we have

$$(\mathcal{A}af_p)_r = \sum_{j,k,s} q_j^*(p)A_{r,j;s,k}a_s q_k(p)f_p \ \text{ for any } a\in\mathbb{R}^m, \tag{4.11}$$

where $A_{r,j;s,k}$ are matrix elements of A, $(A(a\otimes q))_{r,j} = \sum_{s=1}^m\sum_{k=1}^d A_{r,j;s,k}a_s q_k$. In view of (4.8), we get, for the Fourier multiplier $\widehat{\mathcal{A}}(p)$ of the operator $\mathcal{A}$, the expession

$$(\widehat{\mathcal{A}}(p)a)_r = \sum_{j,s,k} q_j^*(p)A_{r,j;s,k}a_s q_k(p) \ \text{ for any } a\in\mathbb{R}^m. \tag{4.12}$$

Alternatively, we can express this in terms of corresponding quadratic form,

$$\langle\widehat{\mathcal{A}}(p)a, b\rangle_{\mathbb{C}^m} = \langle A(a\otimes q(p)), b\otimes q(p)\rangle_{\mathbb{C}^{m\times d}} \ \text{ for any } a, b\in\mathbb{R}^m. \tag{4.13}$$

It follows that the multiplier $\widehat{\mathcal{A}}(p)$ is Hermitian and positive definite. More precisely, for any $p\in\widehat{\mathbb{T}}_N\setminus\{0\}$, we have

$$\|\widehat{\mathcal{A}}(p)\| \le \|A\|\,|p|^2 \ \text{ and } \ \|\widehat{\mathcal{A}}(p)^{-1}\| \le \frac{\pi^2}{4c_0|p|^2}. \tag{4.14}$$

Indeed, the first bound follows directly from (4.13) and the definition of the operator norm $\|A\|$ of the linear map A,

$$\|A\| := \max\{|AF| \colon F\in\mathbb{C}^{m\times d}, |F|\le 1\}. \tag{4.15}$$

We also took into account that $|q(p)|^2 \leq |p|^2$ as follows from the upper bound in the estimate $\frac{4}{\pi^2}t^2 \leq |e^{it}-1|^2 \leq t^2$ valid for all $t \in [-\pi, \pi]$. To get the second inequality, we use the lower bound (2.4) as well as the lower bound above, to get the estimate $\langle \widehat{\mathcal{A}}(p)a, a\rangle_{\mathbb{C}^m} \geq \frac{4}{\pi^2}c_0|p|^2|a|^2$ for any $p \in \hat{\mathbb{T}}_N$.

For the Fourier multiplier of $\mathcal{T}$ defined in (3.8), we first recall that the translation operator τ_x is defined by $(\tau_x\psi)(y) = \psi(y-x)$. Hence we have $\Pi_x\psi = \tau_x\Pi_0(\tau_{-x}\psi)$. Note also that $\tau_{-x}f_p = e^{i\langle x,p\rangle}f_p$. Writing Π as a shorthand for Π_0, we get

$$\widehat{\mathcal{T}}(p)a = l^{-d}\widehat{\Pi(af_p)}(p) \tag{4.16}$$

for any $a \in \mathbb{C}^m$. Indeed, applying $\mathcal{T}$ to af_p, we get

$$\begin{aligned} l^d\mathcal{T}(af_p)(y) &= \sum_{x\in\mathbb{T}_N}\Pi_x(af_p)(y) = \sum_{x\in\mathbb{T}_N}\Pi(\tau_{-x}af_p)(y-x) \\ &= \sum_{x\in\mathbb{T}_N}\Pi(e^{i\langle p,x\rangle}af_p)(y-x) = \sum_{x\in\mathbb{T}_N}\Pi(af_p)(y-x)\ e^{i\langle p,x-y\rangle}e^{i\langle p,y\rangle} \\ &= \sum_{z\in\mathbb{T}_N}\Pi(af_p)(z)e^{-i\langle p,z\rangle}\ e^{i\langle p,y\rangle} = \widehat{\Pi(af_p)}(p)f_p(y). \end{aligned} \tag{4.17}$$

Thus, $\mathcal{T}(af_p) = l^{-d}\widehat{\Pi(af_p)}(p)f_p$ implying the claim by comparing with (4.8).

We now use the symmetry and boundedness of $\mathcal{T}$, with respect to the scalar product $(\cdot,\cdot)_+$, to deduce the corresponding properties for $\widehat{\mathcal{T}}(p)$. According to (4.8), we have

$$(\mathcal{T}(af_p), bf_p)_+ = \langle \mathcal{A}\mathcal{T}(af_p), bf_p\rangle = L^{Nd}\langle \widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}(p)a, b\rangle_{\mathbb{C}^m} \tag{4.18}$$

for any $a, b \in \mathbb{C}^m$. Combining this with the fact that $\mathcal{T}$ is Hermitian with respect to $(\cdot,\cdot)_+$ and with Lemma 3.4(ii), we infer that

$$\langle \widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}(p)a, b\rangle_{\mathbb{C}^m} = \langle a, \widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}(p)b\rangle_{\mathbb{C}^m} \text{ and } 0 \leq \langle \widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}(p)a, a\rangle_{\mathbb{C}^m} \leq \langle \widehat{\mathcal{A}}(p)a, a\rangle_{\mathbb{C}^m} \tag{4.19}$$

for all$a, b \in \mathbb{C}^m$. Since $\widehat{\mathcal{A}}(p)$ is Hermitian and positive definite, it has a unique Hermitian positive definite square root $\widehat{\mathcal{A}}(p)^{1/2}$ with inverse $\widehat{\mathcal{A}}(p)^{-1/2}$. Applying (4.14), we get

$$\|\widehat{\mathcal{A}}(p)^{1/2}\| \leq \|A\|^{1/2}|p| \text{ and } \|\widehat{\mathcal{A}}(p)^{-1/2}\| \leq \frac{\pi}{2\sqrt{c_0}}\frac{1}{|p|}. \tag{4.20}$$

Setting, finally

$$\widetilde{\mathcal{T}}(p) := \widehat{\mathcal{A}}(p)^{1/2}\widehat{\mathcal{T}}(p)\widehat{\mathcal{A}}(p)^{-1/2} = \widehat{\mathcal{A}}(p)^{-1/2}(\widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}(p))\widehat{\mathcal{A}}(p)^{-1/2} \tag{4.21}$$

and

$$\widetilde{\mathcal{R}}(p) := \widehat{\mathcal{A}}(p)^{1/2}\widehat{\mathcal{R}}(p)\widehat{\mathcal{A}}(p)^{-1/2} = \mathbb{1} - \widetilde{\mathcal{T}}(p), \tag{4.22}$$

we get the following bounds.

Lemma 4.1. *The operators $\widetilde{\mathcal{T}}(p)$ and $\widetilde{\mathcal{R}}(p)$ are Hermitian (with respect to the standard scalar product on $\mathbb{C}^m$) and satisfy,for any $p \in \widehat{\mathbb{T}}_N \setminus \{0\}$, the following bounds:*

(i) there is a constant $c < \infty$ (which depends only on $\|A\|$ and c_0, and the dimension d) such that

$$\|\mathbb{1} - \widetilde{\mathcal{R}}(p)\| = \|\widetilde{\mathcal{T}}(p)\| \leq \min(1, c(|p|l)^4), \tag{4.23}$$

(ii) there is a constant $c < \infty$ (which depends only on $\|A\|$, c_0, and the dimension d) such that

$$\|\widetilde{\mathcal{R}}(p)\| \leq \min(1, \tfrac{c}{l}(\tfrac{1}{|p|} + 1)), \tag{4.24}$$

These estimates show that $\mathcal{T}$ suppresses low frequencies, while $\mathcal{R}$ suppresses high frequencies, reflecting the idea that $P_x = \mathrm{id} - \Pi_x$ is a (locally) smoothing operator (cf. Remark 3.2).

Proof. From (4.19), we get

$$\langle \widetilde{\mathcal{T}}(p)a, b\rangle_{\mathbb{C}^m} = \langle a, \widetilde{\mathcal{T}}(p)b\rangle_{\mathbb{C}^m} \text{ and } 0 \leq \langle \widetilde{\mathcal{T}}(p)a, a\rangle_{\mathbb{C}^m} \leq \langle a, a\rangle_{\mathbb{C}^m} \tag{4.25}$$

for any $a, b \in \mathbb{C}^m$. The operators $\widetilde{\mathcal{T}}(p)$ and $\widetilde{\mathcal{R}}(p)$ are thus Hermitian and $0 \leq \widetilde{\mathcal{T}}(p) \leq \mathbb{1}$ and, equivalently, $0 \leq \widetilde{\mathcal{R}}(p) \leq \mathbb{1}$. This implies that

$$\|\widetilde{\mathcal{T}}(p)\| \leq 1, \quad \|\widetilde{\mathcal{R}}(p)\| \leq 1. \tag{4.26}$$

(i): In view of (4.26) we may assume that $|p|l \leq 1$. We first estimate the norm of the Hermitian matrix $\widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}(p)$. First, we show that

$$l^d \langle \widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}(p)a, a\rangle_{\mathbb{C}^m} = \|\Pi(af_p)\|_+^2 \tag{4.27}$$

To see this, we start from the right hand side,

$$\begin{aligned} &\|\Pi(af_p)\|_+^2 = (\Pi(af_p), af_p)_+ = \langle \Pi(af_p), \mathcal{A}(af_p)\rangle = L^{-Nd}\langle \widehat{\Pi(af_p)}, \widehat{\mathcal{A}(af_p)}\rangle = \\ &= L^{-Nd} \sum_{p' \in \widehat{\mathbb{T}}_N \setminus \{0\}} \langle \widehat{\Pi(af_p)}(p'), \widehat{\mathcal{A}}(p)a\widehat{f_p}(p')\rangle_{\mathbb{C}^m} = \langle \widehat{\Pi(af_p)}(p), \widehat{\mathcal{A}}(p)a\rangle_{\mathbb{C}^m} = l^d \langle \widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}(p)a, a\rangle_{\mathbb{C}^m}. \end{aligned} \tag{4.28}$$

Here, we first used the fact that Π is an orthogonal projection with respect to $(\cdot,\cdot)_+$, passing to the second line we used the equation $\widehat{\mathcal{A}(af_p)}(p') = \widehat{\mathcal{A}}(p)a\widehat{f_p}(p')$ obtained as the Fourier transform of (4.8) for $\mathcal{A}$, then the fact that $f_p(p') = L^{Nd}\delta_{p,p'}$ and, finally, we applied the equation (4.16).

Applying (4.8) and using that $af_p \in \boldsymbol{X}_N$ (for $p \neq 0$) and thus also $\Pi(af_p) \in \boldsymbol{X}_N$, we get

$$\|\Pi(af_p)\|_+^2 = \langle \Pi(af_p), \mathcal{A}(af_p)\rangle = \langle \Pi(af_p), \widehat{\mathcal{A}}(p)af_p\rangle = \langle \Pi(af_p), \widehat{\mathcal{A}}(p)a(f_p - 1)\rangle. \tag{4.29}$$

Further, given that $\Pi(af_p)$ is supported in Q and $|f_p(z) - 1| \leq \sqrt{d}\,|p|l$ for $z \in Q$,we have $\|\Pi(af_p)\|_2 = \|\Pi(af_p)\|_{\ell_2(Q)}$ and $\|(f_p - 1)1_Q\|_2 = \|f_p - 1\|_{\ell_2(Q)} \leq \sqrt{d}\,|p|l^{d/2+1}$. With the help of Schwarz inequality and this observation, we get

$$\|\Pi(af_p)\|_+^2 = \langle \Pi(af_p), \widehat{\mathcal{A}}(p)a(f_p - 1)1_Q\rangle \leq \|\Pi(af_p)\|_{\ell_2(Q)}\|\widehat{\mathcal{A}}(p)\||a|\sqrt{d}\,|p|l^{d/2+1}. \tag{4.30}$$

The Poincaré inequality ([9, 10]) implies that

$$\|\Pi(af_p)\|_{\ell_2(Q)} \leq \bar{c}l\|\nabla\Pi(af_p)\|_{\ell_2(Q)} \leq \frac{\bar{c}}{c_0^{1/2}} l\|\Pi(af_p)\|_+ \tag{4.31}$$

with a suitable constant $\bar{c}$. Combining this with (4.30) and (4.14), we get

$$\|\Pi(af_p)\|_+ \leq \sqrt{d}\,\bar{c}\,\frac{\|A\|}{c_0^{1/2}}|p|^3 l^{d/2+2}|a|. \tag{4.32}$$

Given that $\widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}(p)$ is Hermitian, we get

$$\|\widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}(p)\| = \max_{|a|\leq 1}\langle\widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}(p)a, a\rangle_{\mathbb{C}^m} \leq d\,\bar{c}^2(\|A\|^2/c_0)|p|^6 l^4. \tag{4.33}$$

The assertion follows using the second inequality in (4.14).
(ii): In view of (4.26), it suffices to consider the case $|p|l \geq 1$. Again, we first estimate $\|\widehat{\mathcal{A}}(p)\widehat{\mathcal{R}}(p)\|$. Since $\widehat{\mathcal{R}}(p) = \mathbb{1} - \widehat{\mathcal{T}}(p)$ we get from (4.27)

$$l^d\langle\widehat{\mathcal{A}}(p)\widehat{\mathcal{R}}(p)a, a\rangle_{\mathbb{C}^m} = l^d\langle\widehat{\mathcal{A}}(p)a, a\rangle - (\Pi(af_p), af_p)_+. \tag{4.34}$$

Let ω be a cut-off function such that

$$\omega(z) = 1 \text{ if } z \in \overline{Q}\setminus Q, \quad \omega = 0 \text{ if } \operatorname{dist}(z, \overline{Q}\setminus Q) \geq 1+\frac{1}{|p|}, \quad 0 \leq \omega \leq 1 \text{ and } \quad |\nabla\omega| \leq \bar{c}|p| \tag{4.35}$$

with a suitable constant $\bar{c}$. By Lemma 3.1 (ii) we have $\Pi(1-\omega)(af_p) = (1-\omega)1_Q af_p$. Hence

$$(\Pi(af_p), af_p)_+ = (\Pi(a\omega f_p), af_p)_+ + (a(1-\omega)1_Q f_p, af_p)_+ \tag{4.36}$$

and

$$\begin{aligned}(a(1-\omega)1_Q f_p, af_p)_+ &= \langle a(1-\omega)1_Q f_p, \mathcal{A}(af_p)\rangle = \\ &= \langle a(1-\omega)1_Q f_p, \widehat{\mathcal{A}}(p)af_p\rangle = \sum_{z\in Q_-}(1-\omega)\,\langle\widehat{\mathcal{A}}(p)a, a\rangle_{\mathbb{C}^m}.\end{aligned} \tag{4.37}$$

Here, in the last step, we used that $\sum_{z\in Q}(1-\omega) = \sum_{z\in Q_-}(1-\omega)$ since $\omega = 1$ on $Q_- \setminus Q$. Using that $|Q_-| = l^d$, the equations (4.34) and (4.36) with (4.37) yield

$$l^d\langle\widehat{\mathcal{A}}(p)\widehat{\mathcal{R}}(p)a, a\rangle_{\mathbb{C}^m} = -(\Pi(a\omega f_p), af_p)_+ + \sum_{z\in Q_-}\omega\langle\widehat{\mathcal{A}}(p)a, a\rangle_{\mathbb{C}^m}. \tag{4.38}$$

Given that ω is supported in a neighbourhood of order $1 + 1/|p|$ around the boundary $\overline{Q}\setminus Q$ of Q, the last term is easily estimated

$$\Big|\sum_{z\in Q_-}\omega\langle\widehat{\mathcal{A}}(p)a, a\rangle_{\mathbb{C}^m}\Big| \leq 4dl^{d-1}(1+\frac{1}{|p|})\|\widehat{\mathcal{A}}(p)\||a|^2 \leq 4dl^{d-1}(1+\frac{1}{|p|})\|A\||p|^2|a|^2. \tag{4.39}$$

To bound the remaining term we introduce another cut-off function $\tilde{\omega}$ that satisfies the following conditions,

$$\tilde{\omega}(z) = 1 \text{ if } \operatorname{dist}(z, \overline{Q}\setminus Q) \leq 2+\frac{2}{|p|}, \quad \tilde{\omega} = 0 \text{ if } \operatorname{dist}(z, \overline{Q}\setminus Q) \geq 3+\frac{3}{|p|}, \quad 0 \leq \tilde{\omega} \leq 1 \text{ and } |\nabla\tilde{\omega}| \leq \bar{c}|p|. \tag{4.40}$$

Then

$$\begin{aligned}(\Pi(a\omega f_p), af_p)_+ &= (a\omega f_p, \Pi(af_p))_+ = (a\omega f_p, \Pi(a\tilde{\omega} f_p)_+ + (a\omega f_p, (1-\tilde{\omega})1_Q af_p)_+ \\ &= (a\omega f_p, \Pi(a\tilde{\omega} f_p)_+ + \langle \mathcal{A}(a\omega f_p), (1-\tilde{\omega})1_Q af_p\rangle \\ &= (a\omega f_p, \Pi(a\tilde{\omega} f_p))_+ \end{aligned} \tag{4.41}$$

since $1-\tilde{\omega}$ and $\mathcal{A}(a\omega f_p)$ have disjoint support. Thus

$$|(\Pi(a\omega f_p), af_p)_+| \le \|a\omega f_p\|_+ \, \|\Pi(a\tilde{\omega} f_p)\|_+ \le \|a\omega f_p\|_+ \, \|a\tilde{\omega} f_p\|_+ \le \tfrac{c}{4}\|A\| |p|^2 l^{d-1}(3+\frac{3}{|p|})|a|^2, \tag{4.42}$$

where we used that ω and $\tilde{\omega}$ are supported in a strip of size $3+3/|p|$ around $\overline{Q}\setminus Q$, that $1/|p|\le l$, and that the gradients of ω, $\tilde{\omega}$ and f_p are bounded by $\bar{c}|p|$ and the constant c in (4.42) is suitably chosen in dependence on $\bar{c}$ and d. The combination of (4.38), (4.39) and (4.42) now yields the estimate

$$\|\widehat{\mathcal{A}}(p)\widehat{\mathcal{R}}(p)\| \le (\tfrac{3}{4}c+4d)\|A\||p|^2\frac{1}{l}(1+\frac{1}{|p|}). \tag{4.43}$$

In view of (4.14) this finishes the proof of (ii). □

As in the previous section assume that $L\ge 16$ and consider

$$Q_j = \{1,\dots,l_j-1\}^d \quad \text{with } l_j = \lfloor \tfrac{1}{8}L^j \rfloor + 1 \text{ for } j=1,\dots,N. \tag{4.44}$$

Also operators $\mathcal{T}_j, \mathcal{T}'_j$, and $\mathcal{R}'_j$, as well as $\mathcal{C}_k$, $k=1,\dots,N+1$, are defined as before (cf. (3.65) and (3.66)).

We define $\mathcal{A}^{1/2}$ via the action of the corresponding Fourier symbol, $\widehat{\mathcal{A}^{1/2}\varphi}(p) = \widehat{\mathcal{A}}^{1/2}(p)\widehat{\varphi}(p)$. Similarly we define $\mathcal{A}^{-1/2}$. Then the operators $\widetilde{\mathcal{R}}_k := \mathcal{A}^{1/2}\mathcal{R}_k\mathcal{A}^{-1/2}$ and $\widetilde{\mathcal{T}}_k := \mathcal{A}^{1/2}\mathcal{T}_k\mathcal{A}^{-1/2}$ are Hermitian. For $k=1,\dots,N$ define

$$\widetilde{\mathcal{M}}_k := \widetilde{\mathcal{R}}_1\dots\widetilde{\mathcal{R}}_{k-1}\widetilde{\mathcal{R}}_k \text{ and } \widetilde{\mathcal{M}}_0 := \text{id}. \tag{4.45}$$

Since $\tilde{\mathcal{R}}_k = \text{id} - \widetilde{\mathcal{T}}_k$ and $\mathcal{C} = \mathcal{A}^{-1} = \mathcal{A}^{-1/2}\mathcal{A}^{-1/2}$ we have for $k=1,\dots N$

$$\begin{aligned}\mathcal{C}_k &= \mathcal{A}^{-1/2}[\widetilde{\mathcal{M}}_{k-1}\widetilde{\mathcal{M}}'_{k-1} - \widetilde{\mathcal{M}}_k\widetilde{\mathcal{M}}'_k]\mathcal{A}^{-1/2} \\ &= \mathcal{A}^{-1/2}[(\widetilde{\mathcal{T}}_k\widetilde{\mathcal{M}}_{k-1})\widetilde{\mathcal{M}}'_{k-1} + \widetilde{\mathcal{M}}_{k-1}(\widetilde{\mathcal{T}}_k\widetilde{\mathcal{M}}_{k-1})' - \widetilde{\mathcal{T}}_k\widetilde{\mathcal{M}}_{k-1}(\widetilde{\mathcal{T}}_k\widetilde{\mathcal{M}}_{k-1})']\mathcal{A}^{-1/2}\end{aligned} \tag{4.46}$$

and

$$\mathcal{C}_{N+1} = \mathcal{A}^{-1/2}\widetilde{\mathcal{M}}_N\widetilde{\mathcal{M}}'_N\mathcal{A}^{-1/2}. \tag{4.47}$$

To estimate the corresponding kernels C_k and their derivatives we use formula (4.3) for the inverse Fourier transform. This yields

$$\sup_{x\in\mathbb{T}_N}\|C_k(x)\| \le \frac{1}{L^{Nd}}\sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}}\|\widehat{C}_k(p)\| \tag{4.48}$$

and for any multiindex $\alpha=(\alpha_1,\dots,\alpha_d)$,

$$\sup_{x\in\mathbb{T}_N}\|\nabla^\alpha C_k(x)\| \le \frac{1}{L^{Nd}}\sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}}\|\widehat{C}_k(p)q^\alpha\| \le \frac{1}{L^{Nd}}\sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}}\|\widehat{C}_k(p)\||p|^{|\alpha|}, \tag{4.49}$$

where $q_j = e^{ip_j} - 1$ and $q^\alpha = \prod_{j=1}^d q_j^{\alpha_j}$.

Finally, a bound on $\|\widehat{C}_k(p)\|$ will be based on (4.46) (respectively, (4.47)) combined with bounds on $\|\widetilde{\mathcal{M}}_k(p)\|$ and $\|\widetilde{\mathcal{T}}_{k+1}(p)\widetilde{\mathcal{M}}_k(p)\|$.

The estimate of the latter will depend on $|p|$. Namely, slicing the dual torus into the annuli

$$A_j := \{p \in \widehat{\mathbb{T}}_N \setminus \{0\} : \pi L^{-j} \le |p| < \pi L^{-j+1}\},\ j = 1, ..., N, \tag{4.50}$$

with the complement

$$A_0 := \{p \in \widehat{\mathbb{T}}_N \setminus \{0\} : |p| \ge \pi\}, \tag{4.51}$$

and defining, for any $c \ge 1$ and $j, k = 0, 1, \dots, N$, the step functions

$$M_{k,c,L}(p) := \begin{cases} 1, & \text{if } p \in A_j, j \ge k, \\ \frac{c^{k-j}}{L^{(k-j)(k-j+1)/2}}, & \text{if } p \in A_j, j < k, \end{cases} \tag{4.52}$$

and

$$\widetilde{M}_{k,c,L}(p) := \begin{cases} cL^8 L^{4(k-j)}, & \text{if } p \in A_j, j \ge k, \\ \frac{c^{k-j}}{L^{(k-j)(k-j+1)/2}}, & \text{if } p \in A_j, j < k, \end{cases} \tag{4.53}$$

we have the following estimates.

Lemma 4.2. *There exists a constant c (depending only on c_0, $\|A\|$, and d) such that for any odd $L \ge 16$ and any $N \in \mathbb{N}, N \ge 1$,*

$$\|\widetilde{\mathcal{M}}_k(p)\| \le M_{k,c,L}(p) \text{ for } k = 0, \dots, N, \tag{4.54}$$

and

$$\|\widetilde{\mathcal{T}}_{k+1}(p)\widetilde{\mathcal{M}}_k(p)\| \le \widetilde{M}_{k,c,L}(p) \text{ for } k = 0, \dots, N-1. \tag{4.55}$$

Proof. Let $p \in A_j$. For $k \le j$ both bounds follow from the bounds $\|\widetilde{\mathcal{R}}_n(p)\| \le 1$ for $n = 0, 1, \dots, k$ and $\|\widetilde{\mathcal{T}}_{k+1}(p)\| \le c(L^{k+1}|p|)^4$. Now, assume that $k > j$ and recall that (after increasing the constant c from (4.24) by a constant factor)

$$\|\widetilde{\mathcal{R}}_n(p)\| \le \frac{c}{L^n}\Big(1 + \frac{1}{|p|}\Big) \le \frac{c}{L^{n-j}}, \quad \text{for } n = j+1, \dots N. \tag{4.56}$$

Thus

$$\prod_{n=1}^{k} \|\widetilde{\mathcal{R}}_n(p)\| \le \prod_{n=j+1}^{k} \|\widetilde{\mathcal{R}}_n(p)\| \le \prod_{n=j+1}^{k} \frac{c}{L^{n-j}} = \frac{c^{k-j}}{L^{(k-j)(k+1-j)/2}}. \tag{4.57}$$

The first estimate for $k > j$ follows, with the second estimate implied since $\|\widetilde{\mathcal{T}}_{k+1}(p)\| \le 1$. □

To combine these bounds for an estimate on the Fourier multpliers $\widehat{C}_k(p)$, we use the following Lemma.

Lemma 4.3. *Let n be a nonegative integer and $c \ge 1$. Then there exists a constant c' (depending on parameters c, n, and the dimension d) such that with*

$$\eta = \eta(n, d) = \max(\tfrac{1}{4}(d + n - 1)^2, d + n + 6) + 2 \tag{4.58}$$

and for all integers $L \geq 3$, $N \geq 1$, and all $k = 1, \dots, N+1$, we have

$$\frac{1}{L^{dN}} \sum_{p \in \widehat{\mathbb{T}}_N \setminus \{0\}} M_{k-1,c,L}(p) \widetilde{M}_{k-1,c,L}(p) |p|^{n-2} \leq c' L^{\eta} L^{-(k-1)(d+n-2)}, \tag{4.59}$$

$$\frac{1}{L^{dN}} \sum_{p \in \widehat{\mathbb{T}}_N \setminus \{0\}} \widetilde{M}_{k-1,c,L}(p) \widetilde{M}_{k-1,c,L}(p) |p|^{n-2} \leq c c' L^{\eta+8} L^{-(k-1)(d+n-2)}, \tag{4.60}$$

$$\frac{1}{L^{dN}} \sum_{p \in \widehat{\mathbb{T}}_N \setminus \{0\}} M_{N,c,L}(p) M_{N,c,L}(p) |p|^{n-2} \leq c' L^{\eta} L^{-N}(d+n-2). \tag{4.61}$$

Proof. It suffices to prove the first bound for $k = 1, \dots N+1$. The second and third bounds follow employing the inequalities $\widetilde{M}_{k-1,c,L} \leq cL^8 M_{k-1,c,L}$ and $M_{N,c,L} \leq \widetilde{M}_{N,c,L}$, respectively.

To prove the first estimate we split the sum into the sum of contributions over the annuli A_j. For $p \in A_j$, we have

$$|p|^{n-2} \leq \pi^{n-2} d^{n/2} L^2 L^{(-j+1)(n-2)}. \tag{4.62}$$

Indeed, for $j \neq 0$, we get

$$|p|^{n-2} \leq L^{\max((2-n),0)} L^{(-j+1)(n-2)} \pi^{n-2} \leq \pi^{n-2} L^2 L^{(-j+1)(n-2)}. \tag{4.63}$$

The expression $(2-n)$ in the term $\max((2-n),0)$ stems from the fact that for $n=0$ and $n=1$, we actually employ the lower bound on $|p|$ from (4.50). For $j=0$, we have $\pi \leq |p| \leq \sqrt{d}|p|_\infty \leq \sqrt{d}\pi$ and thus $|p|^{n-2} \leq \pi^{n-2} d^{n/2} \leq \pi^{n-2} d^{n/2} L^n$. The size of the annuli can be bounded as

$$\frac{|A_j|}{L^{dN}} \leq \pi^d L^{(-j+1)d}. \tag{4.64}$$

As a result, for $j \geq k-1$,

$$\frac{1}{L^{dN}} \sum_{p \in A_j} M_{k-1,c,L}(p) \widetilde{M}_{k-1,c,L}(p) |p|^{n-2} \leq c\pi^{n+d-2} d^{n/2} L^2 L^{(-j+1)(d+n-2)} L^8 L^{4(k-1-j)} \leq$$
$$\leq c\pi^{n+d-2} d^{n/2} L^{8+d+n} L^{-(k-1)(d+n-2)} L^{-(j-(k-1))(d+n+2)}. \tag{4.65}$$

and

$$\frac{1}{L^{dN}} \sum_{j=k-1}^{N} \sum_{p \in A_j} M_{k-1,c,L}(p) \widetilde{M}_{k-1,c,L}(p) |p|^{n-2} \leq \tilde{c} L^{8+d+n} L^{-(k-1)(d+n-2)} \tag{4.66}$$

with $\tilde{c} = 2c\pi^{n+d-2} d^{n/2}$ since

$$\sum_{j=k-1}^{N} L^{-(j-(k-1))(d+n+2)} = \sum_{j'=0}^{N-k+1} L^{-j'(d+n+2)} \leq \frac{1}{1 - L^{-(d+n+2)}}. \tag{4.67}$$

Now consider $j < k-1$. We get

$$\frac{1}{L^{dN}} \sum_{p \in A_j} M_{k-1,c,L}(p) \widetilde{M}_{k-1,c,L}(p) |p|^{n-2} \leq \pi^{n+d-2} d^{n/2} L^2 L^{(-j+1)(d+n-2)} \frac{c^{2(k-1-j)}}{L^{(k-1-j)(k-j)}} \leq$$
$$\leq \pi^{n+d-2} d^{n/2} L^2 L^{(-k+1)(d+n-2)} L^{(k-j)(d+n-2)} \frac{c^{2(k-1-j)}}{L^{(k-1-j)(k-j)}} \tag{4.68}$$

Setting $j' = k-1-j$ we get

$$\frac{1}{L^{dN}}\sum_{j=0}^{k-2}\sum_{p\in A_j} M_{k-1,c,L}(p)\widetilde{M}_{k-1,c,L}(p)|p|^{n-2} \le \tilde{c}L^2L^{-(k-1)(d+n-2)}\sum_{j'=1}^{k-1}\frac{c^{2j'}}{L^{j'(j'+1)-(d+n-2)(j'+1)}}. \tag{4.69}$$

Consider the integer $\bar{\ell} = \lfloor \frac{\log(2c^2)}{\log 3}\rfloor + 1$ and split the sum above into terms with $j' \le \bar{j}$ and the rest with $j' > \bar{j}$, where $\bar{j} = d+n-2+\bar{\ell}$. We get,

$$\sum_{j'=1}^{k-2}\frac{c^{2j'}}{L^{j'(j'+1)-(d+n-2)(j'+1)}} \le c^{2\bar{j}}\sum_{j'=1}^{\bar{j}} L^{(d+n-2-j')(j'+1)} + \sum_{j'=\bar{j}+1}^{\infty}\frac{c^{2j'}}{L^{(j'+1)\bar{\ell}}} \le$$

$$\le \bar{j}c^{2\bar{j}}L^{\frac{1}{4}(d+n-1)^2} + \frac{(\frac{1}{2})^{\bar{j}+1}}{L^{\bar{\ell}}(1-\frac{1}{2})} \le \bar{j}c^{2\bar{j}}L^{\frac{1}{4}(d+n-1)^2} + 1 \tag{4.70}$$

Here, in the first sum, we bounded $(d+n-2-j')(j'+1)$ (with maximum at $j' = \frac{d+n-3}{2}$) by $\frac{1}{4}(d+n-1)^2$ and, in the second sum, we took into account that $L \ge 3$ and thus $c^2L^{-\bar{\ell}} \le \frac{1}{2}$.

Combining (4.66) and (4.69) with (4.70), we get the sought bound for (4.59) and (4.61) with constants $c' = \tilde{c}(2+\bar{j}c^{2\bar{j}})$ and $\eta = \max(\frac{1}{4}(d+n-1)^2+2, d+n+8)$. For (4.60), the constants must be increased by adding 8 to η and multiplying c' by c. □

Proof of Theorem 2.1. By (4.46), Lemma 4.2, and the bound (4.20) we have $\|\widehat{C}_k(p)\| \le 2cM_{k-1,c,L}(p)\widetilde{M}_{k-1,c,L}(p)|p|^{-2} + c\widetilde{M}_{k-1,c,L}(p)\widetilde{M}_{k-1,c,L}(p)|p|^{-2}$. Now, for $k = 1,\dots,N$, the desired bounds follow from Lemma 4.3 and (4.49). For $k = N+1$ we use (4.47) to get $\|\widehat{C}_{N+1}(p)\| \le cM_{N,c,L}(p)^2|p|^{-2}$ and the assertion follows again from Lemma 4.3 and (4.49). □

5. Analytic dependence on A and proof of Theorem 2.2

We now study the dependence of the finite range decomposition on the map A which appears in the operator $\mathcal{A} = \nabla^*A\nabla$. To this end we will show that the operators Π_x, $\mathcal{T}$ and $\mathcal{R}$ can be locally extended to holomorphic functions of A and the bounds derived previously for fixed A can be extended to a small complex ball. Then the Cauchy integral formula immediately yields bounds on all derivatives with respect to A. We do not claim that the extensions of Π_x, $\mathcal{T}$, $\mathcal{R}$ to complex A yield a finite range decomposition for complex A (indeed positivity is meaningless if $\mathcal{A}$ is not Hermitian). The extension is merely a convenient tool to show that the relevant quantities are real-analytic as functions of real, symmetric, positive definite A.

Let A be a linear map from $\mathbb{C}^{m\times d}$ to $\mathbb{C}^{m\times d}$ such that

$$A = A_0 + A_1 \tag{5.1}$$

with A_0 and A_1 satisfying the following conditions

$$\langle A_0F, G\rangle_{\mathbb{C}^{m\times d}} = \langle F, A_0G\rangle_{\mathbb{C}^{m\times d}}, \quad \langle A_0F, F\rangle_{\mathbb{C}^{m\times d}} \ge c_0|F|^2, \text{ for all } F, G \in \mathbb{C}^{m\times d}, \text{ and} \tag{5.2}$$

$$\|A_1\| \le \frac{c_0}{2}. \tag{5.3}$$

Here, $c_0 > 0$ is a fixed constant and, as before, $\langle\cdot,\cdot\rangle_{\mathbb{C}^{m\times d}}$ and $|\cdot|$ denote the standard scalar product and norm on $\mathbb{C}^{m\times d}$ and $\|A_1\|$ is the corresponding operator norm of A_1.

Again, we consider the operator

$$\mathcal{A} := \nabla^* A \nabla, \tag{5.4}$$

on $\boldsymbol{X}_N$, i.e.,

$$(\mathcal{A}\varphi)_r := \sum_{j=1}^{d} \nabla_j^* (A\nabla\varphi)_{j,r}, \quad \text{where} \quad (\nabla\varphi)_{j,r} = \nabla_j \varphi_r. \tag{5.5}$$

With operator $\mathcal{A}$ we associate the sesquilinear form

$$(\varphi, \psi)_A := \langle A\nabla\varphi, \nabla\psi \rangle \tag{5.6}$$

where $\langle \cdot, \cdot \rangle$ is the ℓ_2-scalar product on $\boldsymbol{X}_N$, defining the adjoint $\mathcal{A}^*$ by

$$\langle \mathcal{A}\varphi, \psi \rangle = (\varphi, \psi)_A = \langle \varphi, \mathcal{A}^*\psi \rangle, \quad \text{with} \quad \mathcal{A}^* = \nabla^* A^* \nabla, \tag{5.7}$$

where A^* is the adjoint of A. Note that for real, symmetric A the form $(\cdot, \cdot)_A$ is a scalar product and agrees with $(\cdot, \cdot)_+$. In the following, we use the previous notation $\boldsymbol{\mathcal{H}}_+$ for the Hilbert space with the scalar product $(\cdot, \cdot)_{A_0}$ and define $\|\varphi\|_{A_0} := (\varphi, \varphi)_{A_0}^{1/2}$.

Using $\Re z$ and z^* to denote the real part and the complex conjugate of a complex number z, we summarize the main properties of the sesquilinear form $(\cdot, \cdot)_A$.

Lemma 5.1. *Assume that an operator A satisfies the conditions* (5.1)*,* (5.2)*, and* (5.3)*.*
Then the sesquilinear form $(\cdot, \cdot)_A$ *on* $\boldsymbol{X}_N$ *satisifes*

$$\Re(\varphi, \varphi)_A \geq \tfrac{1}{2}\|\varphi\|_{A_0}^2, \tag{5.8}$$

$$|(\varphi, \psi)_A| \leq \tfrac{3}{2}\|\varphi\|_{A_0}\|\psi\|_{A_0}, \tag{5.9}$$

$$(\psi, \varphi)_A = (\varphi, \psi)_{A^*}^*. \tag{5.10}$$

Proof. The first claim follows using the definition of the form $(\cdot, \cdot)_A$ and the lower bound

$$\Re\langle AF, F \rangle_{\mathbb{C}^{m\times d}} \geq \langle A_0 F, F \rangle_{\mathbb{C}^{m\times d}} - \frac{c_0}{2}|F|^2 \geq \frac{1}{2}\langle A_0 F, F \rangle_{\mathbb{C}^{m\times d}} \tag{5.11}$$

implied by (5.2) and (5.3).

Using (5.3), the Cauchy-Schwarz inequality for the scalar product $\langle A_0 F, G \rangle_{\mathbb{C}^{m\times d}}$, and the bound from (5.2), we also get

$$\begin{aligned} |\langle AF, G \rangle_{\mathbb{C}^{m\times d}}| &\leq \langle A_0 F, G \rangle_{\mathbb{C}^{m\times d}} + \frac{c_0}{2}|F||G| \leq \langle A_0 F, F \rangle_{\mathbb{C}^{m\times d}}^{1/2} \langle A_0 G, G \rangle_{\mathbb{C}^{m\times d}}^{1/2} + \\ &+ \tfrac{1}{2}\langle A_0 F, F \rangle_{\mathbb{C}^{m\times d}}^{1/2} \langle A_0 G, G \rangle_{\mathbb{C}^{m\times d}}^{1/2} \leq \tfrac{3}{2}\langle A_0 F, F \rangle_{\mathbb{C}^{m\times d}}^{1/2} \langle A_0 G, G \rangle_{\mathbb{C}^{m\times d}}^{1/2} \end{aligned} \tag{5.12}$$

implying the second claim.

The last identity follows from the relation $\langle AG, F \rangle_{\mathbb{C}^{m\times d}} = \langle G, A^* F \rangle_{\mathbb{C}^{m\times d}} = \langle A^* F, G \rangle_{\mathbb{C}^{m\times d}}^*$. □

In view of the above Lemma, the complex version of the Lax-Milgram theorem can be used to ensure the existence of of the bounded inverse operator $\mathcal{C}_A = \mathcal{A}^{-1}$.

In the following, similarly as in the case of the Hilbert space $\boldsymbol{\mathcal{H}}_+$, we use $\boldsymbol{\mathcal{H}}_+(Q + x)$ to denote the corresponding Hilbert space (of functions from $\boldsymbol{X}_N$ with support in $Q + x$) with the scalar product $(\cdot, \cdot)_{A_0}$.

Next, we define an extension of the operators Π_x for a general complex A.

Lemma 5.2. *Assume that A satisfies* (5.1), (5.2), *and* (5.3). *Then, for each $\varphi \in \boldsymbol{X}_N$, there exists a unique $v \in \mathcal{H}_+(Q+x)$ such that*

$$(v, \psi)_A = (\varphi, \psi)_A \text{ for all } \psi \in \mathcal{H}_+(Q+x). \tag{5.13}$$

Proof. The assertion follows from Lemma 5.1 and the Lax-Milgram theorem. □

Lemma 5.3. *Assume that A satisfies* (5.1), (5.2), *and* (5.3). *For any $\varphi \in \boldsymbol{X}_N$, we set*

$$\Pi_{A,x}\varphi := v, \quad \Pi_A := \Pi_{A,0}, \tag{5.14}$$

with $v \in \mathcal{H}_+(Q+x)$ defined by (5.13). *Using, as before, τ_x to denote the translation by x, 1_Q for the characteristic function of a set Q, and D for the open unit disc $D = \{w \in \mathbb{C} \colon |w| < 1\}$, we have*

- (i) $\Pi_{A,x}\tau_x\varphi = \tau_x\Pi_A\varphi$,
- (ii) $\|\Pi_A\varphi\|_{A_0} \leq 3\|\varphi\|_{A_0}$,
- (iii) $\Pi_A\varphi = \varphi$ *for all* $\varphi \in \mathcal{H}_+(Q)$, $\quad \Pi_A\Pi_A = \Pi_A$,
- (iv) $\Pi_A\varphi = \varphi 1_Q$ *if* $\varphi = 0$ *on* $\overline{Q} \setminus Q$,
- (v) $(\Pi_A\varphi, \psi)_A = (\varphi, \Pi_{A^*}\psi)_A$,
- (vi) *The map $z \mapsto \Pi_{A_0+zA_1}\varphi$ is holomorphic for z in the open unit disc D.*

Proof.
(i): Given that the shift τ_{-x} is an isometry with respect to $(\cdot,\cdot)_{A_0}$ and maps $\mathcal{H}_+(Q+x)$ onto $\mathcal{H}_+(Q)$, we have the identities $(\tau_x\Pi_A\varphi, \psi)_A = (\Pi_A\varphi, \tau_{-x}\psi)_A = (\varphi, \tau_{-x}\psi)_A = (\tau_x\varphi, \psi)_A$ for all $\psi \in \mathcal{H}_+(Q+x)$. As $\tau_x\Pi_A\varphi \in \mathcal{H}_+(Q+x)$, this yields the assertion.
(ii): Taking $\psi = varPi_A\varphi$ in the definition (5.13) of $v = \Pi_A\varphi$ and using Lemma 5.1, we get

$$\tfrac{1}{2}\|\Pi_A\varphi\|^2_{A_0} \leq \Re(\Pi_A\varphi, \Pi_A\varphi)_A = \Re(\varphi, \Pi_A\varphi)_A \leq \tfrac{3}{2}\|\varphi\|_{A_0}\|\Pi_A\varphi\|_{A_0}. \tag{5.15}$$

(iii): The second assertion follows from the first. By definition, we have $(\Pi_A\varphi - \varphi, \psi)_A = 0$ for all $\psi \in \mathcal{H}_+(Q)$. In particular, by the assumption $\varphi \in \mathcal{H}_+(Q)$ we may take $\psi = \Pi_A\varphi - \varphi$ inferring that $\Pi_A\varphi = \varphi$.
(iv): Let $\tilde{\varphi} = \varphi 1_Q$. By (iii) we have $\Pi_A\tilde{\varphi} = \tilde{\varphi}$. Moreover $\varphi - \tilde{\varphi}$ vanishes in $\bar{Q}$. Thus $\nabla(\varphi - \tilde{\varphi})$ vanishes in Q_-. Hence $(\varphi - \tilde{\varphi}, \psi)_A = 0$ for all $\psi \in \mathcal{H}_+(Q)$ since $\nabla\psi$ is supported in Q_-. Therefore $\Pi_A(\varphi - \tilde{\varphi}) = 0$ which yields the assertion.
(v) Since $\Pi_{A^*}\psi \in \mathcal{H}^+(Q)$ we have

$$(\varphi, \Pi_{A^*}\psi)_A = (\Pi_A\varphi, \Pi_{A^*}\psi)_A = (\Pi_{A*}\psi, \Pi_A\varphi)^*_{A^*} = (\psi, \Pi_A\varphi)^*_{A^*} = (\Pi_A\varphi, \psi)_A, \tag{5.16}$$

where we used the relation $(\varphi, \psi)_A = (\psi, \varphi)^*_{A^*}$ and the definition of Π_{A^*}.
(vi): This follows from the complex inverse function theorem. Fix φ and consider the map R from $D \times \mathcal{H}^+(Q)$ into the dual of $\mathcal{H}^+(Q)$ given by

$$R(z, v)(\psi) = (v - \varphi, \psi)_{A_0+zA_1} = \langle (A_0 + zA_1)(\nabla v - \nabla\varphi), \nabla\psi \rangle. \tag{5.17}$$

Then R is complex linear in z and v and hence complex differentiable. By the definition of Π_A we have $R(z, v) = 0$ if and only if $v = \Pi_{A_0+zA_1}\varphi$. Finally the derivative of R with respect to the second argument is given by the map L_z from $\mathcal{H}^+(Q)$ into its dual with $L_z(\dot{v})(\psi) = (\dot{v}, \psi)_{A_0+zA_1}$.

By the Lax-Milgram theorem, L_z is invertible for $z \in D$. Hence the map $z \mapsto \varPi_{A_0+zA_1}\varphi$ is complex differentiable in z. □

Note that for a real symmetric A the above definition of $\varPi_{A,x}$ agrees with the definition of $\varPi_x$ in Section 3. We define, as before,

$$\mathcal{T}_A := l^{-d} \sum_{x\in\mathbb{T}_N} \varPi_{A,x}, \quad \mathcal{R}_A = \mathrm{id} - \mathcal{T}_A. \tag{5.18}$$

Then the following weaker version of Lemma 3.4 holds.

Lemma 5.4. *Assume that A satisfies* (5.1)*,* (5.2)*, and* (5.3)*. Then*

$$\|\mathcal{T}_A\varphi\|_{A_0} \le 9\|\varphi\|_{A_0} \text{ for all } \varphi \in \boldsymbol{\mathcal{X}}_N. \tag{5.19}$$

Proof. This is an adaptation of the argument from [4] to the complex case. For the convenience, we include the details. We have

$$l^{2d}\|\mathcal{T}_A\varphi\|_{A_0}^2 \le 2\, l^{2d}\, |(\mathcal{T}_A\varphi, \mathcal{T}_A\varphi)_A| \le 2 \sum_{x,y\in\mathbb{T}_N} |(\varPi_{A,x}\varphi, \varPi_{A,y}\varphi)_A|. \tag{5.20}$$

Set $T_x := \nabla\varPi_{A,x}\varphi$. Then T_x vanishes outside $Q_- + x$ since $\varPi_{A,x}\varphi$ vanishes outside $Q + x$. Thus, in view of (5.6) and (5.9), we get, similarly as in (5.12),

$$\begin{aligned} \big|(\varPi_{A,x}\varphi, \varPi_{A,y}\varphi)_A\big| &= \big|\langle AT_x, T_y\rangle\big| = \big|\langle A\,\mathbb{1}_{Q_-+x}T_x,\, \mathbb{1}_{Q_-+y}T_y\rangle\big| = \big|\langle A\,\mathbb{1}_{Q_-+y}T_x,\, \mathbb{1}_{Q_-+x}T_y\rangle\big| \le \\ &\le \tfrac{3}{2}\langle A_0\,\mathbb{1}_{Q_-+y}T_x,\, \mathbb{1}_{Q_-+y}T_x\rangle^{1/2}\langle A_0\,\mathbb{1}_{Q_-+x}T_y,\, \mathbb{1}_{Q_-+x}T_y\rangle^{1/2} \le \\ &\le \tfrac{3}{4}\langle A_0\,\mathbb{1}_{Q_-+y}T_x,\, \mathbb{1}_{Q_-+y}T_x\rangle + \tfrac{3}{4}\langle A_0\,\mathbb{1}_{Q_-+x}T_y,\, \mathbb{1}_{Q_-+x}T_y\rangle = \\ &= \tfrac{3}{4}\langle A_0\,\mathbb{1}_{Q_-+y}T_x, T_x\rangle + \tfrac{3}{4}\langle A_0\,\mathbb{1}_{Q_-+x}T_y, T_y\rangle. \end{aligned} \tag{5.21}$$

Now $\sum_{y\in\mathbb{T}_N} 1_{Q_+y}$ is the constant function l^d and thus

$$\begin{aligned} \sum_{x,y\in\mathbb{T}_N} \big|(\varPi_{A,x}\varphi, \varPi_{A,y}\varphi)_A\big| &\le \tfrac{3}{2}l^d \sum_{x\in\mathbb{T}_N}\langle A_0T_x, T_x\rangle = \tfrac{3}{2}l^d \sum_{x\in\mathbb{T}_N}(\varPi_{A,x}\varphi, \varPi_{A,x}\varphi)_{A_0} \le \\ &\le 3l^d \sum_{x\in\mathbb{T}_N} \Re(\varPi_{A,x}\varphi, \varPi_{A,x}\varphi)_A = 3l^d \sum_{x\in T_N} \Re(\varphi, \varPi_{A,x}\varphi)_A = \\ &= 3l^{2d}\Re(\varphi, \mathcal{T}_A\varphi)_A \le \tfrac{9}{2}l^{2d}\|\varphi\|_{A_0}\|\mathcal{T}_A\varphi\|_{A_0}. \end{aligned}$$

Combined with (5.20), this yields the assertion. □

Next, we bound the Fourier multipliers of operators $\mathcal{T}_A$ and $\mathcal{R}_A$. Using the relation $\varPi_{A,x} = \tau_x\varPi_A\tau_{-x}$ we get, as before,

$$\mathcal{T}_A(af_p) = l^{-d} \sum_{z\in\mathbb{T}_N} \mathrm{e}^{-i\langle p,z\rangle}\varPi_A(af_p)(z)\; f_p \tag{5.22}$$

and thus, the Fourier multiplier $\widehat{\mathcal{T}}_A(p)$ is given by

$$\widehat{\mathcal{T}}_A(p)a = l^{-d} \sum_{z\in\mathbb{T}_N} \mathrm{e}^{-i\langle p,z\rangle}\varPi_A(af_p)(z). \tag{5.23}$$

Also, the operator $\mathcal{A} = \nabla^* A \nabla$ satisfies again the equation

$$\mathcal{A}(af_p) = (\widehat{\mathcal{A}}(p)a)f_p \tag{5.24}$$

with

$$\langle \widehat{\mathcal{A}}(p)a, b\rangle_{\mathbb{C}^m} = (A(a \otimes q(p)), b \otimes q(p))_{\mathbb{C}^{m\times d}}, \; q(p)_j = \mathrm{e}^{ip_j} - 1. \tag{5.25}$$

Hence,

$$\|\widehat{\mathcal{A}}(p)\| \le \tfrac{3}{2}\|A_0\|\,|p|^2 \text{ and } \|\widehat{\mathcal{A}}(p)^{-1}\| \le \frac{\pi^2}{2c_0|p|^2}. \tag{5.26}$$

Lemma 5.5. *Assume that A satisfies* (5.1), (5.2), *and* (5.3). *Then, there is a constant $c < \infty$ (depending only on $\|A_0\|$, c_0, and d) such that, for all $p \in \widehat{\mathbb{T}}_N \setminus \{0\}$,*

(i) $\|\widehat{\mathcal{A}}_0(p)^{1/2}\widehat{\mathcal{T}}_A(p)\widehat{\mathcal{A}}_0(p)^{-1/2}\| \le 9, \quad \|\widehat{\mathcal{A}}_0(p)^{1/2}\widehat{\mathcal{R}}_A(p)\widehat{\mathcal{A}}_0(p)^{-1/2}\| \le 10.$

(ii) $\|\widehat{\mathcal{T}}_A(p)\| \le c\min(1, (|p|l)^4).$

(iii) $\|\widehat{\mathcal{R}}_A(p)\| \le c\min(1, \frac{1}{l}(1 + \frac{1}{|p|})).$

Proof. The proof is largely parallel to the proof of Lemma 4.1 for real symmetric A, but the constants are slightly worse since $(\cdot,\cdot)_A$ is no longer a scalar product.
(i): The second bound follows from the first since $\widehat{\mathcal{R}}_A(p) = \mathbb{1} - \widehat{\mathcal{T}}_A(p)$. For the first estimate, we apply Lemma 5.4 with $\varphi = af_p$, $a \in \mathbb{C}^m$. This yields

$$\langle \widehat{\mathcal{A}}_0(p)\widehat{\mathcal{T}}_A(p)a, \widehat{\mathcal{T}}_A(p)a\rangle_{\mathbb{C}^m} \le 81\langle \widehat{\mathcal{A}}_0(p)a, a\rangle_{\mathbb{C}^m}. \tag{5.27}$$

Taking $a = \widehat{\mathcal{A}}_0(p)^{-1/2}b$, we deduce that

$$\langle \widehat{\mathcal{A}}_0(p)^{1/2}\widehat{\mathcal{T}}_A(p)\widehat{\mathcal{A}}_0(p)^{-1/2}b, \widehat{\mathcal{A}}_0(p)^{1/2}\widehat{\mathcal{T}}_A(p)\widehat{\mathcal{A}}_0(p)^{-1/2}b\rangle_{\mathbb{C}^m} \le 81\langle b, b\rangle_{\mathbb{C}^m} \tag{5.28}$$

and this finishes the proof of (i).
(ii): First, we assume that $|p|l \ge 1$. It follows from (i) that $\|\widehat{\mathcal{T}}_A(p)\| \le 9\|\widehat{\mathcal{A}}_0(p)^{-1/2}\|\|\widehat{\mathcal{A}}_0(p)^{1/2}\|$. By (4.20) we have $\|\widehat{\mathcal{A}}_0(p)^{1/2}\| \le \|A_0\|^{1/2}|p|$ and $\|\widehat{\mathcal{A}}_0(p)^{-1/2}\| \le \frac{\pi}{2\sqrt{c_0}}\frac{1}{|p|}$, yielding (ii).
Now, assume that $|p|l \le 1$. We first estimate the norm of $\widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}_A(p)$. Note that

$$l^d\langle \widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}_A(p)a, b\rangle_{\mathbb{C}^m} = \langle \widehat{\mathcal{A}}(p)\sum_{z\in\mathbb{T}_N} \Pi_A(af_p)(z), bf_p(z)\rangle_{\mathbb{C}^m} = \langle \Pi_A(af_p), \mathcal{A}^*(bf_p)\rangle = (\Pi_A(af_p), bf_p)_A. \tag{5.29}$$

Thus by Lemma 5.3 (iii), (v), and (5.9), we get

$$\begin{aligned} l^d|\langle \widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}_A(p)a, b\rangle_{\mathbb{C}^m}| &= |(\Pi_A\Pi_A(af_p), bf_p)_A| = |(\Pi_A(af_p), \Pi_{A^*}(bf_p))_A| \\ &\le \tfrac{3}{2}\|\Pi_A(af_p)\|_{A_0}\|\Pi_{A^*}(bf_p)\|_{A_0}. \end{aligned} \tag{5.30}$$

To estimate $\|\Pi_A(af_p)\|_{A_0}$ we use (5.8) and the fact that $(\Pi_A(\varphi), \Pi_A(\varphi))_A = (\varphi, \Pi_A(\varphi))_A$ according to (5.13) since $\Pi_A(\varphi) \in \boldsymbol{\mathcal{H}}_+(Q)$, yielding

$$\begin{aligned} \tfrac{1}{2}\|\Pi_A(af_p)\|_{A_0}^2 &\le |(\Pi_A(af_p), \Pi_A(af_p))_A| = |(af_p, \Pi_A(af_p))_A| = |\langle \mathcal{A}(af_p), \Pi_A(af_p)\rangle| \\ &= |\langle(\widehat{\mathcal{A}}(p)a)f_p, \Pi_A(af_p)\rangle| = |\langle(\widehat{\mathcal{A}}(p)a)(f_p - 1), \Pi_A(af_p)\rangle|. \end{aligned} \tag{5.31}$$

In the last step we used the fact that functions in $\boldsymbol{X}_N$ have average zero. Now $\Pi_A(af_p)$ is supported in Q and $|f_p(z)-1| \le \sqrt{d}\,|p|l$ for $z \in Q$. In combination with (5.26) this yields

$$\tfrac{1}{2}\|\Pi_A(af_p)\|_{A_0}^2 \le \|(\widehat{\mathcal{A}}(p)a)(f_p-1)1_Q\|\,\|\Pi_A(af_p)\| \le \tfrac{3}{2}\|A_0\||p|^2\sqrt{d}\,|a|(|p|l)l^{d/2}\|\Pi_A(af_p)\|. \quad (5.32)$$

The Poincaré inequality ([9, 10]) implies that

$$\|\Pi_A(af_p)\|_{\ell^2(Q)} \le \bar{c}l\|\nabla\Pi_A(af_p)\|_{\ell^2(Q)} \le \frac{\bar{c}}{c_0^{1/2}}l\|\Pi_A(af_p)\|_{A_0}. \quad (5.33)$$

Combining the inequalities above we get

$$\|\Pi_A(af_p)\|_{A_0} \le 3\sqrt{d}\,\bar{c}\frac{\|A_0\|}{c_0^{1/2}}|p|^3l^2\,l^{d/2}|a|. \quad (5.34)$$

The same estimate holds for Π_{A*}. Hence, from (5.30),

$$l^d|\langle\widehat{\mathcal{A}}(p)\widehat{\mathcal{T}}_A(p)a,b\rangle_{\mathbb{C}^m}| \le \tfrac{27}{2}d\frac{\bar{c}^2\|A_0\|^2}{c_0}|p|^6l^4\,l^d|a||b|. \quad (5.35)$$

With the help of (5.26), this yields the claim for a suitable constant c.
(iii): For $|p|l \le 1$ the estimate follows from (i). Thus, we assume $|p|l \ge 1$. Again we first estimate $\widehat{\mathcal{A}}(p)\widehat{\mathcal{R}}_A(p)$. Since $\widehat{\mathcal{R}}_A(p) = \mathbb{1} - \widehat{\mathcal{T}}_A(p)$ we get from (5.29)

$$l^d\langle\widehat{\mathcal{A}}(p)\widehat{\mathcal{R}}_A(p)a,b\rangle_{\mathbb{C}^m} = l^d\langle\widehat{\mathcal{A}}(p)a,b\rangle_{\mathbb{C}^m} - (\Pi_A(af_p),bf_p)_A. \quad (5.36)$$

Let ω be a cut-off function such that

$$\omega(z) = 1 \text{ if } z \in \overline{Q}\setminus Q, \quad \omega = 0 \text{ if } \operatorname{dist}(z,\overline{Q}\setminus Q) \ge 1 + \frac{1}{|p|}, \quad 0 \le \omega \le 1, \quad |\nabla\omega| \le \bar{c}|p|. \quad (5.37)$$

By Lemma 5.3 (iii), we have $\Pi_A(1-\omega)(af_p) = (1-\omega)1_Q af_p$. Hence

$$(\Pi_A(af_p),bf_p)_A = (\Pi_A(a\omega f_p),bf_p)_A + (a(1-\omega)1_Q f_p,bf_p)_A \quad (5.38)$$

and

$$\begin{aligned}(a(1-\omega)1_Q f_p,bf_p)_A &= \langle a(1-\omega)1_Q f_p,\mathcal{A}^*(bf_p))_A = \langle a(1-\omega)1_Q f_p,(\widehat{\mathcal{A}}(p)^*b)f_p\rangle \\ &= \sum_{z\in Q}(1-\omega)\langle\widehat{\mathcal{A}}(p)a,b\rangle_{\mathbb{C}^m} = \sum_{z\in Q_-}(1-\omega)\langle\widehat{\mathcal{A}}(p)a,b\rangle_{\mathbb{C}^m}. \quad (5.39)\end{aligned}$$

In the last step we used that $\omega = 1$ on $Q_-\setminus Q$. Since $|Q_-| = l^d$, this yields

$$l^d\langle\widehat{\mathcal{A}}(p)\widehat{\mathcal{R}}_A(p)a,b\rangle_{\mathbb{C}^m} = -(\Pi_A(a\omega f_p),bf_p)_A + \sum_{z\in Q_-}\omega\langle\mathcal{A}(p)a,b\rangle_{\mathbb{C}^m}. \quad (5.40)$$

Given that ω is supported in a neighbourhood of the order $1+1/|p|$ around $\overline{Q}\setminus Q$, the last term is bounded by

$$\Big|\sum_{z\in Q_-}\omega\langle\widehat{\mathcal{A}}(p)a,b\rangle_{\mathbb{C}^m}\Big| \le 4dl^{d-1}(1+\frac{1}{|p|})\|\widehat{\mathcal{A}}(p)\||a||b| \le 6dl^{d-1}(1+\frac{1}{|p|})\|A_0\||p|^2|a||b|. \quad (5.41)$$

To estimate the remaining term we introduce again an additional cut-off function $\tilde{\omega}$ for which

$$\tilde{\omega}(z) = 1 \text{ if } \operatorname{dist}(z, \overline{Q}\setminus Q) \le 2+\frac{2}{|p|}, \quad \tilde{\omega} = 0 \text{ if } \operatorname{dist}(z, \overline{Q}\setminus Q) \ge 3+\frac{3}{|p|}, \quad 0 \le \tilde{\omega} \le 1, \quad |\nabla\tilde{\omega}| \le \bar{c}|p|. \tag{5.42}$$

Then, taking into account that $1-\tilde{\omega}$ and $\mathcal{A}(a\omega f_p)$ have disjoint support,

$$\begin{aligned}(\varPi_A(a\omega f_p), bf_p)_A &= (a\omega f_p, \varPi_{A^*}(bf_p))_A = (a\omega f_p, \varPi_{A^*}(b\tilde{\omega} f_p))_A + (a\omega f_p, (1-\tilde{\omega})1_Q bf_p)_A \\ &= (a\omega f_p, \varPi_{A^*}(b\tilde{\omega} f_p))_A + \langle \mathcal{A}(a\omega f_p), (1-\tilde{\omega})1_Q bf_p\rangle \\ &= (a\omega f_p, \varPi_{A^*}(b\tilde{\omega} f_p))_A. \end{aligned} \tag{5.43}$$

Hence,

$$\begin{aligned}|(\varPi_A(a\omega f_p), bf_p)_A| &\le \tfrac{3}{2}\|a\omega f_p\|_{A_0}\, \|\varPi_{A^*}(b\tilde{\omega} f_p)\|_{A_0} \le \tfrac{9}{2}\|a\omega f_p\|_{A_0}\, \|b\tilde{\omega} f_p\|_{A_0} \\ &\le C\|A_0\||p|^2 l^{d-1}(1+\frac{1}{|p|})|a||b|, \end{aligned} \tag{5.44}$$

where we used that ω and $\tilde{\omega}$ are supported in a strip of order $1+1/|p|$ around $\bar{Q}\setminus Q$, that $1/|p| \le l$ and that the gradients of ω, $\tilde{\omega}$ and f_p are bounded by $\bar{c}|p|$. The combination of (5.40), (5.41) and (5.44) now yields the estimate

$$\|\widehat{\mathcal{A}}(p)\widehat{\mathcal{R}}_A(p)\| \le C\|A_0\||p|^2 \tfrac{1}{l}(1+\tfrac{1}{|p|}). \tag{5.45}$$

In view of (5.26) this finishes the proof of (iii). □

We now define and estimate the operators $\mathcal{C}_{A,k}$. Assuming again that $L \ge 16$, we use

$$Q_j = \{1,\dots,l_j-1\}^d \quad \text{with } l_j = \lfloor \tfrac{1}{8}L^j \rfloor + 1 \quad \text{for } j = 1,\dots,N, \tag{5.46}$$

to define $\mathcal{T}_{A,j}$, $\mathcal{T}'_{A,j}$, $\mathcal{R}_{A,j}$, and $\mathcal{R}'_{A,j}$, in the same way as before. Introducing

$$\mathcal{M}_{A,k} := \mathcal{R}_{A,1}\dots\mathcal{R}_{A,k-1}\mathcal{R}_{A,k}, \ \text{ for } \ k = 1,\dots,N, \ \text{ and } \ \mathcal{M}_{A,0} := \mathrm{id}, \tag{5.47}$$

we set

$$\mathcal{C}_{A,k} := \mathcal{M}_{A,k-1}\mathcal{C}_A\mathcal{M}'_{A,k-1} - \mathcal{M}_{A,k}\mathcal{C}_A\mathcal{M}'_{A,k}, \tag{5.48}$$

for $k = 1,\dots,N$ and and

$$\mathcal{C}_{A,N+1} := \mathcal{M}_{A,N}\mathcal{C}_A\mathcal{M}'_{A,N} \tag{5.49}$$

for $k = N+1$.

Considering the same annuli A_j, $j = 0,1,\dots,N$, introduced in (4.50) and (4.51), as well as the functions $M_{k,c,L}(p)$ and $\widetilde{M}_{k,c,L}(p)$ defined in (4.52) and (4.53), we have the following bound.

Lemma 5.6. *Assume that A satisfies* (5.1), (5.2), *and* (5.3). *Then there exists a constant c (depending only on $\|A_0\|$, c_0 and the dimension d) such that for all $L \ge 16$, all N, and all $j = 1,\dots N$, we have*

$$\|\widehat{\mathcal{M}}_{A,k}(p)\| \le cM_{k,c,L}(p) \tag{5.50}$$

for $k = 0,\dots,N$ and

$$\|\widehat{\mathcal{T}}_{A,k+1}(p)\widehat{\mathcal{M}}_{A,k}(p)\| \le c\widetilde{M}_{k,c,L}(p) \tag{5.51}$$

for $k = 0,\dots,N-1$.

Proof. This is similar to Lemma 4.2 for the case of real symmetric A. However, the bound $\|\widetilde{\mathcal{R}}(p)\| \leq 1$ for $k \leq j$ has to be replaced by

$$\|\widehat{\mathcal{R}}_{A,k}(p)\| \leq 1 + \|\widehat{\mathcal{T}}_{A,k}(p)\| \leq 1 + c(|p|L^k)^4. \tag{5.52}$$

This yields

$$\prod_{k=0}^{j} \|\widehat{\mathcal{R}}_{A,k}(p)\| \leq \prod_{k=0}^{j}(1 + c(|p|L^k)^4) \leq c, \tag{5.53}$$

resulting in the additional factor c in the claim. □

With the help of Lemma 4.3, we can now bound $\|\mathcal{C}_{A,k}\|$ in the same way as in the proof of Theorem 2.1, starting from (5.49) for $k = N + 1$ and from the equality

$$\mathcal{C}_{A,k} = (\mathcal{T}_{A,k}\mathcal{M}_{A,k-1})\mathcal{C}_A\mathcal{M}'_{A,k-1} + \mathcal{M}_{A,k-1}\mathcal{C}_A(\mathcal{T}_{A,k}\mathcal{M}_{A,k-1})' - \mathcal{T}_{A,k}\mathcal{M}_{A,k-1}\mathcal{C}_A(\mathcal{T}_{A,k}\mathcal{M}_{A,k-1})' \tag{5.54}$$

for $k = 1, \dots, N$. The latter follows from (5.48) using the equation $\mathcal{R}_{A,k} = \mathrm{id} - \mathcal{T}_{A,k}$.

We can now use the Cauchy integral formula to control the derivatives with respect to A.

Lemma 5.7. *Let $D = \{z \in \mathbb{C} : |z| < 1\}$.*

(i) Suppose that $f : D \to \mathbb{C}^{m\times m}$ is holomorphic and

$$\sup_{z\in D} \|f(z)\| \leq M. \tag{5.55}$$

Then the j-th derivative satisfies

$$\|f^{(j)}(0)\| \leq M j!\ . \tag{5.56}$$

(ii) Suppose that $f : D \to \mathbb{C}^{m\times m}$ and $g : D \to \mathbb{C}^{m\times m}$ are holomorphic and

$$\sup_{z\in D} \|f(z)\| \leq M_1, \quad \sup_{z\in D} \|g(z)\| \leq M_2. \tag{5.57}$$

Then the function $h(t) = f(t)g^*(t)$ is real-analytic in $(-1, 1)$ and

$$\|h^{(j)}(0)\| \leq M_1 M_2 j!\ . \tag{5.58}$$

Here $g^*(t)$ denotes the adjoint matrix of $g(t)$.

Proof. Assertion (i) follows directly from the Cauchy integral formula. To show (ii), we note that $g(z) = \sum_j a_j z^j$ with $a_j \in \mathbb{C}^{m\times m}$. Define $G(z) := \sum_j a_j^* z^j$. Then $G(z) = g(z^*)^*$. Hence $\|G(z)\| = \|g(z^*)\|$. Thus $H := fG$ is holomorphic in D and satisfies $\sup_D \|H\| \leq M_1 M_2$. Hence $H^{(k)}(0) \leq k!\, M_1 M_2$. For $t \in (-1, 1)$ we have $H(t) = h(t)$ and the assertion follows. □

Proof of Theorem 2.2. It only remains to show the claim (iii). Let A_0 and A_1 as before and assume in addition that A_0 and A_1 are real and symmetric. Set

$$A(z) := A_0 + zA_1 \tag{5.59}$$

Then the maps

$$\begin{aligned} z &\mapsto \widehat{\mathcal{M}}_{A(z),k}(p), \quad z \mapsto \widehat{\mathcal{T}}_{A(z),k}(p)\widehat{\mathcal{M}}_{A(z),k}(p), \quad z \mapsto \widehat{\mathcal{M}}_{A(z),k}(p)\widehat{\mathcal{C}}_{A(z)}(p), \\ z &\mapsto \widehat{\mathcal{T}}_{A(z),k}(p)\widehat{\mathcal{M}}_{A(z),k}(p)\widehat{\mathcal{C}}_{A(z)}(p) \end{aligned} \tag{5.60}$$

are holomorphic in D. Moreover

$$
\begin{aligned}
\|\widehat{\mathcal{M}}_{A(z),k}(p)\| &\leq cM_{k,c,L}(p),\\
\|\widehat{\mathcal{T}}_{A(z),k}(p)\widehat{\mathcal{M}}_{A(z),k}(p)\| &\leq c\widetilde{M}_{k,c,L}(p),\\
\|\widehat{\mathcal{M}}_{A(z),k}(p)\widehat{\mathcal{C}}_{A(z)}(p)\| &\leq \frac{c}{|p|^2}M_{k,c,L}(p),\\
\|\widehat{\mathcal{T}}_{A(z),k}(p)\widehat{\mathcal{M}}_{A(z),k}(p)\widehat{\mathcal{C}}_{A(z)}(p)\| &\leq \frac{c}{|p|^2}\widetilde{M}_{k,c,L}(p),
\end{aligned}
\tag{5.61}
$$

again with the functions $M_{k,C,L}$ and $\widetilde{M}_{k,C,L}$ defined in (4.52) and (4.53), respectively. Hence it follows from Lemma 5.7 that

$$
\Big\|\frac{\mathrm{d}^j}{\mathrm{d}t^j}_{|t=0}\widehat{\mathcal{C}}_{A_0+tA_1,k}(p)\Big\| \leq \frac{cj!}{|p|^2}\,(2M_{k-1,c,L}(p)\widetilde{M}_{k-1,c,L}(p)+\widetilde{M}^2_{k-1,c,L}(p)) \tag{5.62}
$$

for $k = 1,\dots,N$ (with the obvious modification for $k = N+1$). Thus Lemma 4.3 and the estimate (4.49) for the inverse Fourier transform yield

$$
\sup_{x\in\mathbb{T}_N}\Big\|(\nabla^\alpha D^j_{\dot A}\mathcal{C}_{A_0,k}(x)(A_1,\dots,A_1)\Big\| \leq C_\alpha(d)j!L^{-(k-1)(d-2+|\alpha|)}L^{\eta(\alpha,d)}. \tag{5.63}
$$

Finally suppose that $\|\dot A\| \leq 1$ and set $A_0 = \frac{c_0}{2}\dot A$. Then the desired estimate (2.24) follows from (5.63). □

References

[1] R.A. Adams and J.J.F. Fournier, *Sobolev Spaces*, Academic Press 2nd ed., Elsevier, (2005)

[2] S. Adams, R. Kotecký and S. Müller, Strict convexity of the surface tension for non-convex potentials, in preparation.

[3] D.C. Brydges, Lectures on Renormalisation group, in *Statistical Mechanics*, editors S. Sheffield and T. Spencer, IAS/Park City Mathematics Series Vol. **16**, American Mathematical Society, pages 7–93, (2009).

[4] D.C. Brydges and A. Talarczyk, Finite range decompositions of positive-definite functions, Journal of Functional Analysis **236**, 682-711 (2006).

[5] D.C. Brydges, G. Guadagni and P.K. Mitter, Finite Range Decomposition of Gaussian Processes, Journal of Statistical Physics **115**, Nos. 1/2, 415-449 (2004).

[6] D.C. Brydges and H.T. Yau, Grad φ Perturbations of Massless Gaussian Fields, Commun. Math. Phys. **129**, 351-392 (1990).

[7] T. Funaki and H. Spohn, Motion by Mean Curvature from the Ginzburg-Landau $\nabla\varphi$ Interface Model, Communications in Mathematical Physics **185**, 1-36 (1997).

[8] C. Hainzl and R. Seiringer, General decomposition of radial functions on $\mathbb{R}^d$ and applications to N-body quantum systems, Lett. Math. Phys. **61**, 75-84 (2002).

[9] B.G. Pachapatte, , *On discrete inequalities of the Poincaré type*, Periodica Mathematica Hungarica **19**, 227–233 (1988)

[10] W.S. Cheung, *Some discrete Poincaré-type inequalities*, International Journal of Mathematics and Mathematical Sciences **25**, 479–488 (2001).

[11] K.G. Wilson, The renormalization group and critical phenomena, Rev. Modern Phys. **53** (3), 583-600, (1983).