MA3K0 High-Dimensional Probability Example Sheet 1

2020, term 2
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Students should hand in solutions to the questions by noon 12 pm Friday week 3 (24.01.2020) to the maths drop off box outside the undergraduate office.

Exercise 1: Expectation and moments

(a) [10 points] Prove the following extension of the Integral Identity, which is valid for any random variable X (not necessarily non-negative):

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > t)\,\mathrm{d}t - \int_{-\infty}^0 \mathbb{P}(X < t)\,\mathrm{d}t.$$

(b) [15 points] Let X be a random variable and $p \in (0, \infty)$. Show that

$$\mathbb{E}\big[|X|^p\big] = \int_0^\infty p t^{p-1} \mathbb{P}(|X| > t)\,\mathrm{d}t$$

whenever the right-hand side is finite.

Exercise 2: Chernoff's inequality

(a) [12 points] Let $X_i, i \in \mathbb{N}$, be independent Bernoulli random variables with parameter $p_i \in [0,1]$. Let $S_N := \sum_{i=1}^N X_i$ be their sum and denote its mean by $\mu := \mathbb{E}[S_N]$. Show that for $\delta \in (0,1]$ we have

$$\mathbb{P}(|S_N - \mu| \geq \delta\mu) \leq 2\mathrm{e}^{-c\mu\delta^2}$$

where $c > 0$ is an absolute constant.

(b) [13 points] Let $X \sim \mathsf{Poi}(\lambda), \lambda > 0$. Show that for $t \in (0, \lambda]$, we have

$$\mathbb{P}(|X - \lambda| \geq t) \leq 2\mathrm{e}^{-\frac{ct^2}{\lambda}},$$

where $c > 0$ is an absolute constant.

Exercise 3: sub-Gaussian random variables

(a) [15points] Let X be a sub-Gaussian random variable and define

$$\|X\|_{\psi_2} := \inf\Big\{t > 0\colon \mathbb{E}\big[\exp(X^2/t^2)\big] \leq 2\Big\}.$$

Show that $\|\cdot\|_{\psi_2}$ is indeed a norm on the space of sub-Gaussian random variables.

(b) [10 points] Let X be a random variable with symmetric Bernoulli distribution (i.e., $p = 1/2$). Show that X is a sub-Gaussian random variable with

$$\|X\|_{\psi_2} = \frac{1}{\sqrt{\log 2}}.$$

Exercise 4: A bound on the moment generating function (MGF)

[25 points] Let X be a random variable such that $|X| \leq K, K > 0$. Prove the following bound on the MGF of X:

$$\mathbb{E}\big[\exp(\lambda X)\big] \leq \exp\big(g(\lambda)\mathbb{E}[X^2]\big) \quad \text{where } g(\lambda) = \frac{\lambda^2/2}{1 - |\lambda|K/3},$$

provided that $|\lambda| < 3/K$.

hand-in times – assignment sheets:

Sheet	week	date	discussion in Support class
1	3	24.01.2020 at noon 12 pm	week 4
2	5	07.02.2020 at noon 12 pm	week 6
3	7	21.02.2020 at noon 12 pm	week 8
4	9	06.03.2020 at noon 12 pm	week 10