# MA3K0 High-Dimensional Probability Example Sheet 2

2020, term 2
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Students should hand in solutions to the questions by noon 12 pm Friday week 5 (07.02.2020) to the maths drop off box outside the undergraduate office.

**Exercise 5: Khintchine inequality** [30 points] Let $X_1, \dots, X_N$ be independent sub-Gaussian random variables with zero mean and unit variance, and let $a \in \mathbb{R}^N$. Prove that for every $p \in [2, \infty)$ the following estimate holds

$$\|a\|_2 \le \|\sum_{i=1}^N a_i X_i\|_{L^p} \le CK\sqrt{p}\|a\|_2,$$

where $C > 0$ is an absolute constant and $K := \max_{1 \le i \le N}\{\|X_i\|_{\psi_2}\}$.

(*Hint: Use the General Hoeffding inequality from the class: Let $X_1, \dots, X_N$ be independent sub-Gaussian random variables with zero mean and unit variance, and let $a \in \mathbb{R}^N$. Then, for every $t \ge 0$, we have*

$$\mathbb{P}\Big(|\sum_{i=1}^N a_i X_i| \ge t\Big) \le 2\exp\Big(-\frac{ct^2}{K^2\|a\|_2^2}\Big),$$

*where $K := \max_{1 \le i \le N}\{\|X_i\|_{\psi_2}\}$ and where $c > 0$ is an absolute constant.*)

**Exercise 6: Expectation of the Euclidean norm**

Deduce from the '*Concentration of norm*'-Theorem [1] the following estimates for the expected Euclidean norm and the variance of a random vector $X = (X_1, \dots, X_n) \in \mathbb{R}^n$ with independent sub-Gaussian coordinates $X_i$ that satisfy $\mathbb{E}[X_i^2] = 1, i = 1, \dots, n$.

(a) [10 points] **Expectation:**

$$\sqrt{n} - CK^2 \le \mathbb{E}\big[\|X\|_2\big] \le \sqrt{n} + CK^2,$$

where $K := \max_{1 \le i \le N}\{\|X_i\|_{\psi_2}\}$ and $C >$ an absolute constant.

(b) [10 points] **Variance:**

$$\mathrm{var}(\|X\|_2) \le \widetilde{C}K^4,$$

where $K := \max_{1 \le i \le N}\{\|X_i\|_{\psi_2}\}$ and $\widetilde{C} >$ an absolute constant.

[1] Under the given assumptions the theorem states that

$$\|\|X\|_2 - \sqrt{n}\|_{\psi_2} \le CK^2$$

with $C > 0$ an absolute constant and $K = \max_{1 \le i \le N}\{\|X_i\|_{\psi_2}\}$.

**Exercise 7: Multivariate normal/Gaussian random variables** Using the rotation invariance of the normal distribution deduce the following statements:

(a) [10 points] Consider a $n$-dimensional Gaussian vector $Y \sim \mathsf{N}(0, \mathbb{1}_n)$ and a fixed vector $u \in \mathbb{R}^n$ ($\mathbb{1}_n$ is the identity $(n \times n)$-matrix). Then

$$\langle Y, u \rangle \sim \mathsf{N}(0, \|u\|_2^2),$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product.

(b) [15 points] Let $X_1, \ldots, X_n$ be independent Gaussian real-valued random variables $X_i \sim \mathsf{N}(0, \sigma_i^2)$, Then

$$\sum_{i=1}^n X_i \sim \mathsf{N}(0, \sigma^2) \qquad \text{where } \sigma^2 = \sum_{i=1}^n \sigma_i^2.$$

(*Hint: (b) is standard property of Gaussian random variables as discussed in the class, and for a proof use the convolution and the independence. It may be useful to do the proof for $n = 2$ first, and then argue with induction.*)

**Exercise 8: Normal and spherical distribution** Write $Y \sim \mathsf{N}(0, \mathbb{1}_n)$ as

$$Y = r\vec{Y},$$

where $r := \|Y\|_2$ is the length and $\vec{Y} = Y/\|Y\|_2$ is the direction of the $n$-dimensional random vector $Y$. Prove the following:

(a) [15 points] The length $r$ and the direction $\vec{Y}$ are independent random variables.

(b) [10 points] The direction $\vec{Y}$ is uniformly distributed on the unit sphere $\mathsf{S}^{n-1}$, where

$$\mathsf{S}^{n-1} = \{x \in \mathbb{R}^n \colon \sum_{i=1}^n x_i^2 = 1\}.$$

**hand-in times – assignment sheets:**

| Sheet | week | date | discussion in Support class |
|---|---|---|---|
| 1 | 3 | 24.01.2020 at noon 12 pm | week 4 |
| 2 | 5 | 07.02.2020 at noon 12 pm | week 6 |
| 3 | 7 | 21.02.2020 at noon 12 pm | week 8 |
| 4 | 9 | 06.03.2020 at noon 12 pm | week 10 |