MA3K0 High-Dimensional Probability Example Sheet 4

2020, term 2
Stefan Adams

Students should hand in solutions to the questions by noon 12 pm Friday week 9 (06.03.2020) to the maths drop off box outside the undergraduate office, or (extended deadline possibility) submit your homework to the lecturer before the lecture noon 12 pm Monday week 10 (09.03.2020).

Exercise 13: Lipschitz functions Prove the following statements. By default, in the lecture, we call a function Lipschitz continuous if it is *locally Lipschitz continuous* [1]. A function $f\colon X \to Y$ for metric spaces (X, d_X) and (Y, d_Y) is *globally Lipschitz continuous* if there exists $L \in \mathbb{R}$ such that

$$\mathrm{d}_Y(f(x), f(y)) \le L\mathrm{d}_X(x,y) \qquad \text{for all } x, y \in X\,.$$

In the following some statements hold for global Lipschitz continuity and some only for local Lipschitz continuity. Recall that global Lipschitz continuity implies local Lipschitz continuity.

(a) [5 points] Every Lipschitz function is uniformly continuous.

(b) [5 points] Every differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, and

$$\|f\|_{\mathrm{Lip}} \le \|\nabla f\|_\infty\,.$$

(c) [5 points] For a fixed vector $v \in \mathbb{R}^n$, the linear functional $f(x) = \langle x, v\rangle$ ($\langle\cdot,\cdot\rangle$ Euclidean scalar product) is a Lipschitz function on $\mathbb{R}^n$, and $\|f\|_{\mathrm{Lip}} = \|v\|_2$ ($\|\cdot\|_2$ Euclidean norm).

(d) [5 points] A $m \times n$ matrix A acting as a linear operator

$$A\colon (\mathbb{R}^n, \|\cdot\|_2) \to (\mathbb{R}^m, \|\cdot\|_2)$$

is a Lipschitz function, and $\|A\|_{\mathrm{Lip}} = \|A\|$ ($\|\cdot\|$ operator norm).

Exercise 14: Blow-up of small sets Let A be a subset of the sphere $\sqrt{n}\mathsf{S}^{n-1}$ such that

$$\sigma(A) > 2\exp(-cs^2) \qquad \text{for some } s > 0.$$

Here σ is the normalised area measure on the sphere, that is,

$$\sigma(A) = \frac{\sigma_{n-1}(A)}{\sigma_{n-1}(\sqrt{n}\mathsf{S}^{n-1})} \quad A \subset \sqrt{n}\mathsf{S}^{n-1} \text{ measurable}\,,$$

and $\sigma_{n-1}(A)$ denotes the (surface) area measure of A.

(a) [10 points] Prove that $\sigma(A_s) > 1/2$.

(b) [10 points] Deduce from (a) that, for any $t \ge s$,

$$\sigma(A_{2t}) \ge 1 - 2\exp(-ct^2).$$

Here $c > 0$ is the absolute constant from the Blow-up Lemma in class.

[1] A function $f\colon X \to Y$ for metric spaces (X, d_X) and (Y, d_Y) is *locally Lipschitz continuous* if for every $x \in X$ there exists a neighbourhood $U \subset X$ such that $f\big|_U$ is globally Lipschitz continuous. Here $f\big|_U\colon U \to Y$ is the restriction of f to U.

Exercise 15: Gaussian calculus

(a) [10 points] Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable. If $X \sim \mathsf{N}(0, \sigma^2), \sigma > 0$, show that

$$\mathbb{E}\big[Xf(X)\big] = \sigma^2 \mathbb{E}\big[f'(X)\big] .$$

(b) [10 points] Prove the following statement: Let $X \sim \mathsf{N}(0, \Sigma)$ be a random vector in $\mathbb{R}^n$, Σ positive symmetric $n \times n$ matrix. Then for any differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ we have

$$\mathbb{E}\big[Xf(X)\big] = \Sigma \mathbb{E}\big[\nabla f(X)\big] .$$

(c) [15 points] Consider two independent Gaussian random vectors $X \sim \mathsf{N}(0, \Sigma^{(X)})$ and $Y \sim \mathsf{N}(0, \Sigma^{(Y)})$, and define the Gaussian interpolation vector

$$Z(u) := \sqrt{u}\, X + \sqrt{1-u}\, Y \quad u \in [0,1] .$$

Prove that, for any twice differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}u} \mathbb{E}\big[f(Z(u))\big] = \frac{1}{2} \sum_{i,j=1}^{n} (\Sigma_{ij}^{(X)} - \Sigma_{ij}^{(Y)}) \mathbb{E}\Big[\frac{\partial^2 f}{\partial x_i \partial x_j}(Z(u))\Big] .$$

Exercise 16: Gaussian type of chaos Consider a mean-zero sub-Gaussian random vector X in $\mathbb{R}^n$ with $\|X\|_{\psi_2} \leq K$. Let B be a $m \times n$ matrix and $Y \sim \mathsf{N}(0, \mathbb{1}_m)$ a standard normal random vector in $\mathbb{R}^m$.

(a) [7 points] Prove the comparison inequality

$$\mathbb{E}\Big[\exp\Big(\lambda^2 \|BX\|_2^2\Big)\Big] \leq \mathbb{E}\Big[\exp\Big(CK^2\lambda^2 \|B^T Y\|_2^2\Big)\Big] \qquad \text{for every } \lambda \in \mathbb{R}.$$

(b) [8 points]

Check that

$$\mathbb{E}\Big[\exp\Big(\lambda^2 \|B^T Y\|_2^2\Big)\Big] \leq \exp\Big(C\lambda^2 \|B\|_{\mathrm{F}}^2\Big)$$

provided that $|\lambda| \leq c/\|B\|$. ($\|B\|_{\mathrm{F}}$ is the Frobenius norm of the matrix B, that is, the Euclidean norm of all its entries).

(c) [5 points] Let X be an isotropic random vector in $\mathbb{R}^n$ and B be a $m \times n$ matrix. Check that

$$\mathbb{E}\big[\|BX\|_2^2\big] = \|B\|_{\mathrm{F}}^2 .$$

(d) [5 points]

Prove that, for any $t \geq 0$, we have

$$\mathbb{P}\big(\|BX\|_2 \geq CK\|B\|_{\mathrm{F}} + t\big) \leq \exp\Big(-\frac{ct^2}{K^2\|B\|^2}\Big) .$$

(*Hint: (a), (b): note that $B^T B =: A$ is a $n \times n$ matrix and that $\|BY\|_2^2 = \langle Y, AY \rangle$. The statements are about sub-Gaussian respectively sub-exponential random variables. For example, one may show show (b) directly by properties of sub-exponential random variables relying on the fact that the square of normal distributed random variables is sub-exponential. Use the Lemma about the MGF of Gaussian chaos (either lecture of in the book it is Lemma 6.2.2. For (a) use the methods for the MGF from the proof of the Comparison Lemma (Lemma 6.2.3 in the book). For (c), simply perform the calculation, and for (d) proceed similarly as done in class for a similar problem.*)