MA3K0 High-Dimensional Probability
Example Sheet 5

2020, term 2
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Exercise 17:

(a) Let $X_1, \ldots, X_N$ be real-valued random variables and suppose that there exists $\sigma > 0$ such that

$$\mathbb{E}\big[\exp\big(tX_i\big)\big] \le \exp\big(t\sigma^2/2\big) \quad \text{for all } t > 0 \text{ and } i = 1, \ldots, N\,.$$

Show that then

$$\mathbb{E}\Big(\max_{1\le i\le N}\{X_i\}\Big) \le \sigma\sqrt{2\log N}\,.$$

(b) State *Slepian's inequality* for two mean-zero Gaussian random vectors in $\mathbb{R}^N$.

(c) Define a *Gaussian process* for some index set $T \subset \mathbb{R}^n$ and the *Gaussian width* $\mathrm{w}(T)$ of a subset $T \subset \mathbb{R}^n$.

(d) The unit ball of the ℓ_1 norm in $\mathbb{R}^n$ is the set $B_1^{(n)} := \{x \in \mathbb{R}^n \colon \|x\|_1 \le 1\}$, $\|x\|_1 = \sum_{i=1}^n |x_i|$. Show that the Gaussian width of the ℓ_1 ball is bounded, i.e.,

$$\mathrm{w}(B_1^{(n)}) \le C\sqrt{\log n} \quad \text{for some absolute constant } C > 0.$$

Exercise 18:

(a) A real-valued random variable X with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in (0,\infty)$ satisfies the *Bernstein condition* with parameter $b > 0$ if

$$|\mathbb{E}\big[\big(X-\mu\big)^k\big]| \le \frac{1}{2}k!\sigma^2 b^{k-2} \quad k = 2,3,4,\ldots.$$

Show that a bounded random variable X with $|X-\mu| \le b$ satisfies the *Bernstein condition*.

(b) Show that the bounded random variable X in (a) is sub-exponential and derive a bound on the centred moment generating function

$$\mathbb{E}[\exp(\lambda(X - \mathbb{E}[X]))]\,.$$

(c) Give the parameter regimes for $t \ge 0$ for which the concentration inequality in (b) resembles sub-Gaussian and when it resembles sub-exponential behaviour.

Exercise 19:

(a) Prove the following generalisation of the symmetrisation Lemma. Let $F\colon \mathbb{R}_+ \to \mathbb{R}$ be an increasing, convex function. Show that

$$\mathbb{E}\left[F\left(\frac{1}{2}\Big\|\sum_{i=1}^N \varepsilon_i X_i\Big\|\right)\right] \le \mathbb{E}\left[F\left(\Big\|\sum_{i=1}^N X_i\Big\|\right)\right] \le \mathbb{E}\left[F\left(2\Big\|\sum_{i=1}^N \varepsilon_i X_i\Big\|\right)\right].$$

(b) Let $X_1, \dots, X_N$ be independent $\mathbb{R}$-valued random variables and let $\varepsilon_1, \dots, \varepsilon_N$ independent Rademacher functions (symmetric Bernoulli random variables). Show the following equivalence,

$$\sum_{i=1}^N X_i \ \text{ is sub-Gaussian } \Leftrightarrow \sum_{i=1}^N \varepsilon_i X_i \ \text{ is sub-Gaussian},$$

and show that there are absolute constants $C, c > 0$ such that

$$c\|\sum_{i=1}^N \varepsilon_i X_i\|_{\psi_2} \le \|\sum_{i=1}^N X_i\|_{\psi_2} \le C\|\sum_{i=1}^N \varepsilon_i X_i\|_{\psi_2}.$$

Exercise 20:

Let A be an $m \times n$ matrix with independent $\mathsf{N}(0,1)$ distributed entries A_{ij}. Then, one has that

$$\mathbb{E}\big[\|A\|\big] \le \sqrt{m} + \sqrt{n}.$$

Show that, for every $t \ge 0$, one has

$$\mathbb{P}\big(\|A\| \ge \sqrt{m} + \sqrt{n} + t\big) \le 2\exp(-Ct^2)$$

for some absolute constant $C > 0$.

Exercise 21:

(a) Let X be a random vector uniformly distributed on the Euclidean sphere in $\mathbb{R}^n$ with centre at the origin and radius $\sqrt{n}$, that is, $X \sim \mathsf{Unif}(\sqrt{n}\,\mathsf{S}^{n-1})$. Show that X is sub-Gaussian with $\|X\|_{\psi_2} \le C$ for some absolute constant $C > 0$. Prove this in an alternative, geometric way as follows: Define

$$Z := \frac{X}{\sqrt{n}} \sim \mathsf{Unif}(\mathsf{S}^{n-1}),$$

and note that it suffices to show that $\|Z\|_{\psi_2} \le C/\sqrt{n}$ ($Z = (Z_1, \dots, Z_n)$). Then use the fact that $\mathbb{P}(Z_1 \ge t) = \mathbb{P}(Z \in \mathcal{C}_t)$ with the spherical cap $\mathcal{C}_t = \{z \in \mathsf{S}^{n-1} \colon z_1 \ge t\}$.
(*Hint: Draw a figure for the two-dimensional case.*)

(b) Let A be an $m \times n$ random matrix whose entries $A_{ij}, i, j = 1, \dots, n$, are independent mean-zero sub-Gaussian real-valued random variables. Then, for any $t > 0$ one has

$$\|A\| \le CK\big(\sqrt{m} + \sqrt{n} + t\big)$$

with probability at least $1 - 2\exp(-t^2)$, where $K := \max_{1 \le i,j \le n}\big\{\|A_{ij}\|_{\psi_2}\big\}$, where $\|A\|$ is operator norm of the matrix A. Show that

$$\mathbb{E}\big[\|A\|\big] \le C'K\big(\sqrt{m} + \sqrt{n}\big)$$

for some absolute constant $C' > 0$.