Theorem (10')

A $m\times n$ matrix whose rows A_i are independent mean-zero sub-Gaussian isotropic random vectors in $\mathbb{R}^n$.

Then, for any $t \geq 0$,

$$\left\| \frac{1}{m} A^T A - \mathbb{1}_n \right\| \leq K^2 \max\{\delta, \delta^2\}$$

where $\delta = C\left(\sqrt{\frac{n}{m}} + \frac{t}{\sqrt{m}}\right)$

with probability at least $1 - 2\exp(-t^2)$

$K := \max_{1 \leq i \leq m} \|A_i\|_{\psi_2}$

Proof:

Step ①: $\varepsilon = 1/4$ net $\mathcal{N} \subset S^{n-1}$ with cardinality $|\mathcal{N}| \leq 9^n$

$$\left\| \frac{1}{m} A^T A - \mathbb{1}_n \right\| \leq 2 \sup_{x \in \mathcal{N}} \left| \left\langle \left(\frac{1}{m} A^T A - \mathbb{1}_n\right) x, x \right\rangle \right|$$

using Lemma in lecture
(see Lemma 4.4.1 in the book and
Exercise 10(b)(ii) - Example Sheet 3
as $A^T A$ is $n\times n$ matrix

$$= 2 \sup_{x \in \mathcal{N}} \left| \frac{1}{m} \|Ax\|_2^2 - 1 \right|$$

It suffices to show that $\sup_{x \in \mathcal{N}} \left| \frac{1}{m} \|Ax\|_2^2 - 1 \right| \leq \frac{\varepsilon}{2}$

w.p. at least $1 - 2\exp(-t^2)$ where $\varepsilon := K^2 \max\{\delta, \delta^2\}$.

Step ②: As in Theorem ⑩. Pick $x \in S^{n-1}$

$$\|Ax\|_2^2 = \sum_{i=1}^m \langle A_i, x\rangle^2 =: \sum_{i=1}^m X_i^2$$

X_i independent sub-Gaussian with $\mathbb{E}[X_i^2] = 1$

$\Rightarrow X_i^2 - 1$ independent, mean-zero, and sub-exponential r.v.s with $\|X_i^2 - 1\|_{\psi_1} \leq CK^2$

Thus (results for exponential tail estimates)

$$\mathbb{P}\left(\left|\tfrac{1}{m}\|Ax\|_2^2 - 1\right| \geq \tfrac{\varepsilon}{2}\right) = \mathbb{P}\left(\left|\tfrac{1}{m}\sum_{i=1}^m X_i^2 - 1\right| \geq \varepsilon/2\right)$$

$$\leq 2\exp\left(-c_1 \min\{\varepsilon^2/K^4, \varepsilon/K^2\}\, m\right)$$

$$= 2\exp(-c_1 \delta^2 m) \leq 2\exp(-c_1 C^2 (n + t^2))$$

Step ③: Union bound. Choose C large enough to obtain

$$\mathbb{P}\left(\sup_{x \in \mathcal{N}} \left|\tfrac{1}{m}\|Ax\|_2^2 - 1\right| \geq \tfrac{\varepsilon}{2}\right) \leq 9 \exp(-c_1 C^2 (n + t^2))$$
$$\leq 2\exp(-t^2)$$

∎