MA3H2 Markov Processes and Percolation theory Example Sheet 5

2010, term 2
Stefan Adams

Students should hand in solutions by 3pm Tuesday of week 10 to the maths pigeonloft.

Exercise 17:

(a) Let $X = (X_t)_{t\ge 0}$ be an irreducible positive recurrent birth-death process. Show that $X = (X_t)_{t\ge 0}$ is reversible in equilibrium.

(b) If $X = (X_t)_{t\ge 0}$ and $Y = (Y_t)_{t\ge 0}$ are independent reversible processes, prove that $(X_t, Y_t)_{t\ge 0}$ is a reversible process.

(c) Customers arrive in a barber's shop according to a Poisson process of rate $\lambda > 0$. The shop has s barbers and N waiting places; each barber works (on a single customer) provided that there is a customer to serve, and any customer arriving when the shop is full (i.e. the numbers of customers present is $N+s$) is not admitted and never returns. Every admitted customer waits in the queue and is then served, on a first-come-first-served order, the service taking an exponential time of rate $\mu > 0$; the service times of admitted customers are independent. After completing the hair cut, the customer leaves the shop and never returns. Set up a Markov process model for the number X_t of customers in the shop at time $t \ge 0$. Calculate the equilibrium distribution π of this process and explain why it is unique. Show that $X = (X_t)_{t\ge 0}$ in equilibrium is reversible, i.e. for all $T > 0$, $(X_t\colon 0 \le t \le T)$ has the same distribution as $(Y_t\colon 0 \le t \le T)$ where $Y_t = X_{T-t}$, and $X_0 \sim \pi$.

Exercise 18: Bond percolation on $\mathbb{Z}^d$.

(a) Explain and justify the following facts (recall that $p_c(d) = p_c(\mathbb{Z}^d)$):
(i) $p_c(1) = 1$; (ii) $p_c(d+1) \le p_c(d)$ for all $d \ge 1$; (iii) $\lambda(d) \le 2d-1$; (iv) $p_c(d) \sim (2d)^{-1}$ as $d \to \infty$.

(b) Consider now $\mathbb{Z}^2$: Prove that $\mathbb{P}_{\frac{1}{2}}(H(Q)) \ge \frac{1}{2}$ and $\mathbb{P}_{\frac{1}{2}}(H(R)) = \frac{1}{2}$, where Q is any square and R is an $(n+1)\times n$ rectangle. Use the fact that exactly one of the events $H(R)$ (horizontal crossing using open bonds of R) and $V(R^{\rm h})$ (vertical crossing of the dual $R^{\rm h}$; where for a rectangle $R = [a,b]\times[c,d]$ in $\mathbb{Z}^2$ the horizontal dual is the rectangle $R^{\rm h} = [a+\frac{1}{2}, b-\frac{1}{2}]\times[c-\frac{1}{2}, d+\frac{1}{2}]$ in the dual lattice $(\mathbb{Z}^2)^*$) always hold.

(c) Formulate Harris Theorem.

Exercise 19: Bond percolation on rooted trees. Consider bond percolation on a rooted tree $\mathbb{T}_k$, $k \ge 2$, (i.e. a k-branching tree with a root in which each vertex has k children).

(a) Prove that $p_c(\mathbb{T}_k) = p_T(\mathbb{T}_k) = \frac{1}{k}$.

(b) Write and prove a formula (respectively an equation) for the probability $\theta(p)$ of infinite cluster containing the root for $p > p_c(\mathbb{T}_k) = \frac{1}{k}$, and the expectation $\chi(p)$ of the size of the cluster containing the root for $p < p_c(\mathbb{T}_k)$.

(c) Find an explicit formula for $\theta(p)$ for $k = 2$.

Exercise 20:

(a) Define the BK (van den Berg, Kesten) operation $A \Box B$ for bond percolation (on a finite set of bonds).

(b) Prove that $A \Box B \subset A \cap B$ always and $A \Box B = A \cap B$ whenever A is increasing and B decreasing.

(c) Consider bond percolation on $\mathbb{Z}^d$. Formulate Russo's Lemma and prove Russo's Lemma.