MA3H2 Markov Processes and Percolation theory Example Sheet 6

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Question 1:

(a) Define the simple random walk on $\mathbb{Z}^d$, the n-step transition function, and express the corresponding transition function for the first visit to some lattice site in terms of the transition function.

(b) Give a mathematical criterion when a random walk given by a transition function $P\colon \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ is transient and recurrent respectively.

(c) Let T be the time which elapses before a simple random walk (on $\mathbb{Z}$) is absorbed at either of the absorbing barriers at 0 and N, having started at k where $0 \le k \le N$. Show that $\mathbb{P}(T < \infty) = 1$ and $\mathbb{E}(T^k) < \infty$ for all $k \ge 1$.

(d) Consider the simple Bernoulli random walk on $\mathbb{Z}$ with $p \in [0,1]$. Prove that it is recurrent if and only if $p = \frac{1}{2}$.

Question 2:

(a) (i) Define the jump chain (and its transition matrix) of a continuous-time Markov process $X = (X_t)_{t\ge 0}$ on a countable state space I.

(ii) Let $X = (X_t)_{t\ge 0}$ be a minimal right-continuous process with Q-matrix Q on a countable state space I. State two different conditions which ensure that $X = (X_t)_{t\ge 0}$ is a Markov process.

(b) (i) Give the definition of positive recurrence and null-recurrence of a state for a Markov process $X = (X_t)_{t\ge 0}$ on a countable state space I.

(ii) State when a vector $\lambda = (\lambda(i))_{i\in I}$ is an invariant distribution of a Markov process.

(c) Between each pair of the cities A, B and C there is a telephone line which may be put out of action by snowstorms. Snowstorms happen according to a Poisson process, with rate 8 per unit time, and when one occurs, each telephone line is put out of action independently, with probability 1/2. When a line is out of action, it takes a random length of time to be repaired; the duration of this repair time has exponential distribution with mean 1/14 and all repairs are carried out independently. Let $(X_t)_{t\ge 0}$ be a continuous-time Markov process, where X_t represents the number of lines out of action at time t. Determine the expected holding times in each state and hence, or otherwise, determine the Q-matrix of $(X_t)_{t\ge 0}$. Determine the long-run proportion of time when all pairs of cities may communicate, assuming that messages may be passed through the third city, if necessary.

Question 3:

(a) Let a continuous-time Markov process $X = (X_t)_{t\geq 0}$ with countable state space I be given.

(i) Define the hitting time and the hitting probability of a subset $A \subset I$ and give the system of equations for which the hitting probability (vector) is the minimal non-negative solution.

(ii) State the system of equations for which the expected hitting times of a subset $A \subset I$ is the minimal non-negative solution. Justify that the hitting times solve these equations, i.e., prove the statement without proving the minimality and the positivity of the solution.

(b) Consider the Q-matrix on $I = \{1, 2, 3, 4\}$ given by

$$Q = \begin{pmatrix} -1 & 1/2 & 1/2 & 0 \\ 1/4 & -1/2 & 0 & 1/4 \\ 1/6 & 0 & -1/3 & 1/6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) Calculate the probability of hitting 3 when starting from 1.

(ii) Calculate the expected time to hit 4 starting from 1.

Question 4:

(a) Define increasing/decreasing events and state the FKG (Fortuin, Kasteleyn, Ginibre) inequality. State and prove Harris's Lemma for events depending only on finitely many bonds.

(b) Let A and B be events. Define the BK (van den Berg, Kesten) operation $A \Box B$ for bond percolation (on a finite set of bonds) and state the BK inequality. Give an example for $A \Box B$ and an example for an event in $A \cap B \setminus (A \Box B)$.

(d) Consider bond percolation on $\mathbb{Z}^2$.

(i) Denote by $R_{n,L}(p)$ the probability of an open horizontal crossing of an nL by L rectangle with $L > 1, n \in \mathbb{N}$. Pick $c = \frac{1}{16}$ and $\lambda \in (0, 1)$. Prove the following statement: If $R_{2,L}(p) \geq 1 - c\lambda$ then $R_{2,2L}(p) \geq 1 - c\lambda^2$.

(ii) Give the main ideas for the proof of Harris Theorem (you may assume an appropriate lower bound on probabilities of left-right crossings of rectangles (RSW - Theorem)).