MA4L3 Large Deviation Theory
Example Sheet 2

2023, term 2
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Students should hand in solutions to the questions by noon 12 pm Thursday week 5 (09.02.2023) online via the moodle page.

Exercise 5: [**Sanov's theorem for finite sample spaces** E] [20 points] Compute $I_\mu(\nu) = H(\nu|\mu)$ when the probability measure μ is the uniform distribution on E and ν is the uniform distribution on a subset $E' \subset E$ of E.

Exercise 6: [**Contraction**] [25 points] Show that the rate function J of the contraction principle, that is, for $T\colon E \to Y$ continuous and linear,

$$J(y) = \inf_{x\colon T(x)=y}\{I(x)\}, \quad y \in Y\,,$$

is strictly convex if the rate $I\colon E \to [0,\infty]$ is strictly convex on its domain $\mathcal{D}_I$. Explain why the linearity of T is necessary.

Exercise 7: [**Tilted LDP**]
[25 points] Let (E, d) be some Polish space. Show that the function I^H defined as

$$I^H(x) = \sup_{y\in E}\{H(y) - I(y)\} - (H(x) - I(x))\,, \quad x \in E\,,$$

is a good rate function when I is a good rate function. The function I^H is the rate function of the tilted LDP for a continuous function $H\colon E \to \mathbb{R}$ which is bounded from above, that is, the sequence $(\nu_N)_{N\ge1}$ satisfies the LDP with rate N and rate function I^H. Here, the probability measures ν_N are defined by a change of measure,

$$\nu_N(\mathrm{d}x) = \frac{1}{Z_N}\mathrm{e}^{NH(x)}\,\mu_N(\mathrm{d}x)\,,$$

where $Z_N = \int_E \mathrm{e}^{NH(x)}\,\mu_N(\mathrm{d}x)$, and where the sequence $(\mu_N)_{N\ge1}$ satisfies the LDP on E with rate N and rate function I.

Exercise 8: [**Example for exponential tightness**]
[30 points] Consider a sequence of real-valued random variables $(Z_N)_{N\in\mathbb{N}}$, where $\mathbb{P}(Z_N = 0) = 1 - 2p_N, \mathbb{P}(Z_N = -m_N) = p_N$, and $\mathbb{P}(Z_N = m_N) = p_N$ with $p_N \in [0,1], N \in \mathbb{N}$. Prove that if

$$\lim_{N\to\infty}\frac{1}{N}\log p_N = -\infty\,,$$

then the laws of $(Z_N)_{N\in\mathbb{N}}$ are exponentially tight, and moreover they satisfy the LDP with the convex, good rate function

$$I(x) = \begin{cases} 0 & , x = 0, \\ \infty & , \text{otherwise}\,. \end{cases}$$