MA4L3 Large Deviation Theory
Example Sheet 3

2023, term 2
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Students should hand in solutions to the questions by noon 12 pm Thursday week 8 (02.03.2023) online via the moodle page.

Exercise 9: [**I-continuity**] [20 points] Identify the I-continuous sets of the rate function I of the coin-tossing example, where

$$I(x) = \begin{cases} \log 2 + x\log x + (1-x)\log(1-x) & \text{if } x \in [0,1]\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

Exercise 10: [**Example - Gaussian process**] [30 points] Let $(\varphi_k)_{k\in\mathbb{Z}}$ be an $\mathbb{R}$-valued stationary mean-zero Gaussian process (also called Gaussian random field) with covariance function $C_k = \mathbb{E}[\varphi_0\varphi_k], k \in \mathbb{Z}$, satisfying

$$\sum_{k\in\mathbb{Z}} |C_k| < \infty\,.$$

Let $S_N = \frac{1}{N}\sum_{k=1}^N \varphi_k$ be the empirical average and denote its law by $\mu_N = \mathbb{P}\circ S_N^{-1}$. Show that the sequence $(\mu_N)_{N\ge 1}$ satisfies the LDP on $\mathbb{R}$ with rate N and with rate function

$$\Lambda^*(x) = \frac{x^2}{2C}\,, \qquad \text{with } C = \sum_{k\in\mathbb{Z}} C_k\,.$$

Hints: 1.) Recall that a family $(\varphi_x)_{x\in\mathbb{Z}^d}$ *is called* Gaussian process *or* Gaussian random field *if, for all finite subsets* $\Lambda \subset \mathbb{Z}^d$, *the random vector* $\varphi_\Lambda = (\varphi_x)_{x\in\Lambda}$ *is a Gaussian random vector in* $\mathbb{R}^\Lambda$, *or, equivalently, has a multivariate normal distribution.*

2.) You may assume under the stronger assumption $\sum_{k\in\mathbb{Z}} C_k^2 < \infty$ *that the process* $(\varphi_k)_{k\in\mathbb{Z}}$ *can be written as*

$$\varphi_k = \sum_{m\in\mathbb{Z}} a_m Y_{k-m}\,, \quad (Y_k)_{k\in\mathbb{Z}} \text{ i.i.d. standard normally distributed and } \sum_{k\in\mathbb{Z}} a_k^2 < \infty\,,$$

i.e., Y_k *real valued and* $Y_k \sim \mathsf{N}(0,1)$.

Exercise 11: [**Application of the Gärtner-Ellis theorem**] [20 points] Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. $\mathbb{R}^2$-valued random variables with marginal law that is bivariate normal, i.e.,

$$\mathbb{P}(X_1 \in \mathrm{d}x) = \frac{1}{2\pi}\mathrm{e}^{-\|x\|_2^2/2}\mathrm{d}x, \quad x \in \mathbb{R}^2,\ \|\cdot\|_2 \text{ Euclidean norm in } \mathbb{R}^2\,.$$

Let $(\mathsf{Poi}(t))_{t\ge 0}$ be a rate 1 Poisson process, independent of $(X_i)_{i\in\mathbb{N}}$. Let

$$Z_N = \frac{1}{N}\sum_{i=1}^{\mathsf{Poi}(N)} X_i\,,$$

and denote $\mu_N(\cdot) := \mathbb{P}(Z_N \in \cdot)$. Show that the sequence $(\mu_N)_{N\ge1}$ satisfies the LDP on $\mathbb{R}^2$ with rate N and rate function $\Lambda^*(x) = 1 + \|x\|_2(t^* + (t^*)^{-1})$, where $t^* = t^*(\|x\|_2)$ solves the equation $\|x\|_2 = t\mathrm{e}^{t^2/2}$.

Exercise 12: [**Markov chain version of Mogulskii's theorem**] [30 points] Suppose E is a finite set. Let $Y = (Y_k)_{k\in\mathbb{N}}$ be a finite state, irreducible Markov chain with state space E and transition matrix $\mathsf{P} = (p(x,y))_{x,y\in E}$. For a given deterministic function $f\colon E \to \mathbb{R}^d$ define $X_i = f(Y_i)$ for all $i \in \mathbb{N}$. Show Mogulskii's theorem for the $\mathbb{R}^d$-valued sequence $(X_i)_{i\in\mathbb{N}}$.

Hint: The Legendre transform Λ^* in Mogulskii's theorem is replaced by the rate function

$$I(z) = \sup_{\lambda\in\mathbb{R}^d} \big\{ \langle \lambda, z\rangle - \log \rho(\mathsf{P}_\lambda) \big\}, \quad z \in \mathbb{R}^d ,$$

where $\rho(\mathsf{P}_\lambda)$ is the Perron-Frobenius eigenvalue of the transition matrix $\mathsf{P}_\lambda = (p_\lambda(x,y))_{x,y\in E}$ with entries $p_\lambda(x,y) = p(x,y)\mathrm{e}^{\langle\lambda, f(y)\rangle}$, $\lambda \in \mathbb{R}^d, x, y \in E$.

Hand-in times (submission on moodle page) – assignment sheets:

Sheet	Week	Date submission	Discussion in Support class/lecture
1	3	26.01.2023 at noon 12 pm	Week 4, Wednesday 01.02.2023
2	5	09.02.2023 at noon 12 pm	Week 6, Wednesday 15.02.2023
3	8	02.03.2023 at noon 12 pm	Week 9, Wednesday 08.03.2023
4	10	16.03.2023 at noon 12 pm	Week 10, Friday 17.03.2023