MA4L3 Large Deviation Theory
Example Sheet 4

2023, term 2
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Students should hand in solutions to the questions by noon 12 pm Thursday week 10 (16.03.2023) online via the moodle page.

Exercise 13: [**Finite state Markov chains**] Let E be finite set and $\mathsf{P} = (p(x,x))_{x,y\in E}$ be a stochastic matrix. Recall that the empirical measure of the Markov chain $(Y_k)_{k\in\mathbb{N}}$ associated with P satisfies the large deviation principle with rate function given by the Perron-Frobenius eigenvalue, i.e., $I(q) = \sup_{\lambda\in\mathbb{R}^{|E|}}\{\langle\lambda, q\rangle - \log\rho(\mathsf{P}_\lambda)\}$. Recall that this rate function equals the function $\mathcal{J}$ in the lecture, that is,

$$\mathcal{J}(q) = \begin{cases} \sup_{u\gg 0}\left\{\sum_{x\in E} q_x \log\frac{u_x}{(u\mathsf{P})_x}\right\} & , \quad q\in\mathcal{M}_1(E), \\ +\infty & , \quad q\notin\mathcal{M}_1(E), \end{cases}$$

where $u \gg 0$ means that the supremum runs over all $u\colon E\to(0,\infty)$.

(a) [10 points] Find a maximiser of the variational problem defining $\mathcal{J}$.

(b) [15 points] Show that the function $\mathcal{J}$ is a rate function and that $\mathcal{J}$ is strictly convex and continuous on $\mathcal{M}_1(E)$.

Exercise 14: [**Computation - Finite state Markov chains**]
[25 points] Let the stochastic matrix

$$\mathsf{P} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

be given. Compute $\mathcal{J}(\mu)$ for $\mu\in\mathcal{M}_1(E)$.

Exercise 15: [**Mogulskii's theorem - concentration bound**] In the setting of Mogulskii's theorem consider an i.i.d. sequence $(X_i)_{i\in\mathbb{N}}$ of $\mathbb{R}^d$-valued random variables X_i with law $\mu\in\mathcal{M}_1(\mathbb{R}^d)$ such that

$$\Lambda(\lambda) = \log\mathbb{E}_\mu\big[\mathrm{e}^{\langle\lambda,X_1\rangle}\big] < \infty\,, \qquad \text{for all } \lambda\in\mathbb{R}^d\,.$$

(a) [15 points] Show that

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(|X|_1\geq N\big) = -\infty\,.$$

(b) [5 points] Decide if (a) holds when $\Lambda(\lambda)=\infty$ for some $\lambda\in\mathbb{R}^d$.

(c) [10 points] Show that (a) fails in $d=1$ for independent, exponentially distributed numbers, $X_1\sim\mathsf{Exp}(\alpha)$.

Exercise 16: [**Mogulskii's theorem**]
[20 points] Show that Mogulskii's theorem from the lecture can be extended to the laws $\nu_\varepsilon, \varepsilon > 0$, of

$$Y_\varepsilon(t) := \varepsilon \sum_{i=1}^{\lfloor \frac{t}{\varepsilon} \rfloor} X_i \,, \quad 0 \le t \le 1 \,,$$

where the law μ_N (and $Z_N(t)$) in the lecture correspond to the special case of $\varepsilon = N^{-1}$. The rate of the corresponding large deviation principle for the family $(\nu_\varepsilon)_{\varepsilon>0}$ is $\varepsilon \to 0$.

Hand-in times (submission on moodle page) – assignment sheets:

Sheet	Week	Date submission	Discussion in Support class/lecture
1	3	26.01.2023 at noon 12 pm	Week 4, Wednesday 01.02.2023
2	5	09.02.2023 at noon 12 pm	Week 6, Wednesday 15.02.2023
3	8	02.03.2023 at noon 12 pm	Week 9, Wednesday 08.03.2023
4	10	16.03.2023 at noon 12 pm	Week 10, Friday 17.03.2023