MA231 Vector Analysis

2010, term 1
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Example Sheet 1: Hints and partial solutions

A3 (a) (i) $\nabla f = (2y, 2x+2y)$, (ii) $\nabla f = (y\cos(\pi y), x\cos(\pi y) - \pi xy\sin(\pi y))$. (b) $D_{(-1,5)}f(2,3) = (6,10)\cdot(-1,5) = 44$.

A4 (a) $\int_0^1 t\sqrt{4+9t^2}\,\mathrm{d}t$. (b) Use parameterisations of the two parts of the curve: $\alpha(t) = (t,0)$, for $t \in [-2,2]$, and $\beta(t) = (2\cos t, 2\sin t)$, for $t \in [0,\pi]$. The line integral along α is zero. The line integral along β gives the answer $-32/3$.

A5 (a) $x^2y + x^3 + C$. (b) $x^2yz + xz + y + C$. (c) Later in the course, when we have introduced curls, we see this immediately from the fact that the curl of v is non-zero. For now, argue by integration that it is impossible.

A6 (a) $(-2/3, -2/3, 1/3)$. (b) $\frac{(4s^2t-2t^3, -st^2, 2t^2-2s^2)}{\sqrt{(4s^2t-2t^3)^2+s^2t^4+(2t^2-2s^2)^2}}$.

A7 (a) $\partial r/\partial s = (1,0,2s)$ and $\partial r/\partial t = (0,1,1)$ gives $\left\|\partial r/\partial s \times \partial r/\partial t\right\| = \sqrt{4s^2+2}$. Thus $\int_S x\,dS = \int_0^1\int_{-1}^1 s\sqrt{4s^2+2}\,\mathrm{d}t\,\mathrm{d}s = \sqrt{6}-\sqrt{2}/3$. (b) For example parameterise by $x(s,t) = (s\cos t, s\sin t, s^2)$ over $s \in [0, L^{1/2}], t \in [0,2\pi]$. The surface area is $\frac{\pi}{6}((1+4L)^{3/2}-1)$.

B 1- B 4 see details of solutions in supervision classes or support class.

C1 $\nabla T = (2x, 2y, 1)$ and $D_{(\cos\theta,\sin\theta,1)}T(1,1,0) = (2,2,1)\cdot(\cos\theta,\sin\theta,1) = 1+2\cos\theta+2\sin\theta$. This is maximised at $\theta = \pi/4$.

C2 Let C be the straight line joining the origin to the point x_0. The FCT for gradient vector fields we have $f(x_0) - f(0) = \int_C v\cdot\hat{T}\,\mathrm{d}s = g(x_0) - g(0)$.

C3 (b) $u = \nabla\left(\frac{1}{2}\ln(1+g^2(x))\right)$. (c) $\nabla\phi(r^2) = 2\phi'(r^2)x$. Find ϕ so that $2\phi'(r^2) = r^m$, namely $\phi(r^2) = \frac{r^{m+2}}{m+2}$ if $m \neq -2$ and $\phi(r^2) = \log(r)$ if $m = -2$.

C4 The velocity of the tracer particle is $u(t, x(t))$. Differentiate in t using the chain rule to find the acceleration. To explain the notation, use ∇ as if it were the vector $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ so that the notation $u\cdot\nabla$ becomes $\sum_{i=1}^3 u_i\frac{\partial}{\partial x_i}$ which then acts on each component of u.

C5 (b) $\nabla f = m$ a constant. (c) $\nabla g = -3x/r^5$. (d) The dipole vector field is $-3r^{-5}(x\cdot m)x + r^{-3}m$.

C6 (a) The tangent vector is $(1, g'(t))$ which has length $\sqrt{1+g'(t)^2}$. (b) The unit normal vector is given by $\frac{(\partial g/\partial s, \partial g/\partial s, 1)}{\sqrt{1+(\partial g/\partial s)^2+(\partial g/\partial t)^2}}$. Hence the area of the desired part of the graph is $\int_0^1\int_0^1\sqrt{1+(\partial g/\partial s)^2+(\partial g/\partial t)^2}\,\mathrm{d}s\mathrm{d}t$.