MA231 Vector Analysis

2010, term 1
Stefan Adams

Example Sheet 2: Hints and partial solutions

A1 (a) $4x$ (b) $y(1+x^2y^2z^2)^{-1/2}$ (c) xye^z.

A2 (a) $\hat{N} = \frac{x}{R}$ on the surface $\mathcal{S}$ so that $v\cdot\hat{N} = \|x\|^2/R = R$ and $\int_{\mathcal{S}} v\cdot\hat{N}\,\mathrm{d}S = R\int_{\mathcal{S}}\mathrm{d}S = R$ (Surface area of $\mathcal{S}$) $= 4\pi R^3$. (b) Here $v\cdot\hat{N} = 0$ so the flux is zero. (c) Here $v\cdot\hat{N} = \frac{1}{R}(-x^2+y^2+z^2)$. But by symmetry $\int_{\mathcal{S}} x^2\,\mathrm{d}S = \int_{\mathcal{S}} y^2\,\mathrm{d}S = \int_{\mathcal{S}} z^2\,\mathrm{d}S$. So $\int_{\mathcal{S}} v\cdot\hat{N}\mathrm{d}S = \frac{1}{R}\int_{\mathcal{S}} z^2\,\mathrm{d}S = \frac{1}{3R}\int_{\mathcal{S}} x^2+y^2+z^2\,\mathrm{d}S = \frac{R}{3}\int_{\mathcal{S}}\mathrm{d}S = \frac{4\pi R^3}{3}$. Note that in each case the flux was calculated without ever having to start in on a parameterisation. Moral — look for symmetry tricks. The divergences in the three cases are $3, 0, 1$ and it is easy to confirm that each flux integral agrees with the corresponding volume integral given by the divergence theorem.

A3 The divergence of ∇f is $12(x^2+y^2+z^2)$. (a) 12 (b) $48\pi R^5/5$ (use polar coordinates).

A4 (a) $(0,0,2)$ (b) $(0,0,0)$ (c) $(2x,-2y,1)$.

A6 curl$(v) = 1$ so that

$$\text{Area}(\Omega) = \int_\Omega \text{Curl}(v)\,\mathrm{d}A = \int_{\partial\Omega} v\cdot\hat{T}\,\mathrm{d}s = \int_0^{2\pi}(0,\alpha\cos t)\cdot(-\alpha\sin t,\beta\cos t)\,\mathrm{d}t = \pi\alpha\beta.$$

A7 One possibility is to parametrise $\mathcal{S}$ by $x(u,v) = (u,v,2-u^2-v^2)$ for $(u,v)\in\Omega = \{u^2+v^2\le 2\}$. Then inward (non-unit) normal vector $\hat{N}(u,v) = (-2u,-2v,-1)$. By parameterising $\partial\mathcal{S}$ by $\alpha(t) = (\sqrt{2}\cos t,\sqrt{2}\sin t,0)$ for $t\in[0,2\pi]$ the tangent vector and normal vector are then suitably oriented for Stokes' theorem. Both sides of Stokes' identity give the value 2π, for example $\int_{\mathcal{S}}\nabla\times v\cdot\hat{N}\,\mathrm{d}S = \int_\Omega(2u+2v+1)\,\mathrm{d}u\mathrm{d}v = \int_\Omega\mathrm{d}u\mathrm{d}v =$ Area of Ω, (using symmetry to reduce the integrand from $2u+2v+1$ to 1).

B1 (a) A parametrisation is $\alpha(\theta,z) = (r\cos\theta, r\sin\theta, z)$ with $r = 4$. Result is zero. Why? Symmetry because of the given vector field (x-dependence and the symmetry of the surface, flux in and out the volume cancel each other out). (b) Summing the single parts (as in the lecture), result is $\frac{3}{2}$.

B2 (a) A simple calculation gives $\operatorname{div} f = 1$, integrating this over the disc with radius R gives the result πR^2. (b) Divergence $=$ 4. Flux $= 4\times$ Volume of the pyramid with base area 9 and height 2 $=$ 24.

B3 (a) Use the definitions of the cross product, of the curl and diligence. (b) Use the product rule (c) Chain rule. (d) Use the function $\phi(r) = \frac{1}{4\pi r^2}\int_{\partial B(a,r)} f$ and show that ϕ is constant. Why gives this the proof of the statement? Answer: take the limit $r\downarrow 0$.

B4 The expression for $\operatorname{div}(fv)$ follows from the product rule, the integration by parts formula can be derived by applying Gauss's theorem. Adding IBP for f, g and g, f gives Green's identity.

C3 (a) Stokes theorem for the vector field $\underline{v} = (f,0,0)$ becomes

$$\int_{\mathcal{C}} f\hat{T}_1\,\mathrm{d}s = \int_{\mathcal{S}}\left(\frac{\partial f}{\partial z}N_2 - \frac{\partial f}{\partial y}N_3\right)\mathrm{d}S.$$

Now repeat with $\underline{v} = (0,f,0)$ and $\underline{v} = (0,0,f)$. (b) Prove each co-ordinate of the identity separately, each is a case of the divergence theorem.

C4 Since f_1 and f_2 vanish on $\partial\Omega$, the boundary terms in Green's identity vanish and we get $\lambda_2\int_\Omega f_1f_2 = -\int_\Omega f_1\Delta f_2 = -\int_\Omega f_2\Delta f_1 = \lambda_1\int_\Omega f_1f_2$. Since $\lambda_1\ne\lambda_2$ this implies the result.

Marking scheme:

B 1: part (a).
B 2: part (b).
B 3: Part (c) and (d) with 3 points for part (d).
B 4: all parts.