MA231 Vector Analysis

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Example Sheet 3: Hints and partial solutions

A1 $\nabla \cdot v = 4r\sin\phi - \frac{1}{r^2}\tan\phi$ and $\nabla \times v = r\cos\phi\hat{e}_\theta$.

A2 (a) i: $r = 1, \theta = \pi/2$; $-i$: $r = 1, \theta = -\pi/2$; $1 + i$: $r = \sqrt{2}, \theta = \pi/4$; $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$: $r = 1, \theta = 2\pi/3$. (b) $\frac{3}{20} - \frac{1}{20}i$; $-i$; $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$. (c) $1+3i$ and $-1-3i$. $1+(3-\sqrt{8})i$ and $-1-(3+\sqrt{8})i$. (d) $\cos\theta + i\sin\theta$ for $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

A3 (a) $\bar{z} = x - iy$, $|\bar{z}| = |z|$ and $\arg(\bar{z}) = -\arg(z)$. (b) split into real and imaginary parts.

A4 Since $|z| = z\bar{z}$ the set is the unit circle and we can use $\phi(t) = \mathrm{e}^{2\pi i t}$.

A5 (a) (i) $R = \infty$, (ii) $R = e$ (recall $(1 + \frac{1}{n})^n \to e$). (b) The series converges on the set $\{z : |z - 4i| < 1/5\}$ and nowhere else. (c) $|\frac{i^n}{i+n^2}| = |\frac{1}{i+n^2}| \le \frac{1}{n^2}$ and the first series converges absolutely. The real part of $\frac{1}{i+n}$ is $\frac{n}{n^2+1}$ so the real part of the second series diverges by comparison with $\sum \frac{1}{2n}$.

A6 (a) Use the definition of $\exp(z)$ and $\sin(z)$ as power series. (b) Use part (a) or argue (after week 9) that the identity is true for real z and since both sides are holomorphic functions on C the identity must remain true on the whole of C (by Taylor's theorem for holomorphic functions and the uniqueness theorem for power series). (c) Use the representations for $\sin, \cos, \sinh, \cosh$ in terms of the exponential function.

B1 (a) We can choose any surface $\mathcal{S}$ with boundary $\mathcal{C}$, simplest is $r(u, v) = (u\cos v, u\sin v, 1 - u\cos v - u\sin v)$ for $u \in [0, 1], v \in [0, 2\pi]$. Surface normal: $N = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = (u, u, u)$ (using a picture we can see that this is the correct orientation of N). $\nabla \times v = (0, 0, 3u^2)$, $\int_{\mathcal{C}} \langle v, \hat{T}\rangle = \int_{\mathcal{S}} \langle \nabla \times v, \hat{N}\rangle = 3\pi/2$.

(b) The relations $\|\hat{e}_r\| = \|\hat{e}_\phi\| = \|\hat{e}_z\| = 1$ as well as $\hat{e}_r \cdot \hat{e}_\phi = \hat{e}_r \cdot \hat{e}_z = \hat{e}_\phi \cdot \hat{e}_z = 0$ and $\hat{e}_r \times \hat{e}_\phi = \hat{e}_z$ are easily checked by direct calculation.

(c) $\nabla T(r, \phi, \theta) = \frac{\partial T}{\partial r}\hat{e}_r$ and $\Delta T(r, \theta, \phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial T}{\partial r}\big)$.

B2 (a) Solution: all $z \in \mathbb{C}$ with $|z - (\frac{1}{3} + \frac{4}{3}i)| > \frac{\sqrt{8}}{3}$. (b) $z^7 = 8 - 8i$.

B3 (a) Convergence inside the circle of radius $1/6$ centered at $-i/3$ and divergence outside this circle (using the ratio test). Checking the Cauchy criterion gives convergence on the boundary. (b) $|\frac{1}{z+n^2}| \le \frac{1}{|Re(z+n^2)|} \le \frac{1}{n^2}$ when z has positive real part. So the series converges absolutley for all such z and, as the uniform limit of continuous functions, it is continuous. (c) (i) $z^2 = (z - i + i)^2 = -1(z - i)^0 + 2i(z - i)^1 + (z - i)^2$, $R = \infty$. (ii) Use $\frac{1}{z-1} = \frac{1}{i-1+z-i} = \frac{1}{i-1}(1 + \frac{z-i}{i-1})^{-1}$ and then the geometric series expansion of $(1 + w)^{-1}$. $R = \sqrt{2}$. (d) Write a quadratic equation in e^z and get z as the corresponding complex logarithm.

B4 (a) Use the rewriting of $\sin$ and $\cos$ via the exponential function. (b) Use properties of Möbius transformations (see lecture). In particular you may use the following fact: If f maps the circle $|z - a| = r$ onto the circle $|z - b| = \rho$, then f maps the closed disk $|z - a| \le r$ to either $|z - b| \le \rho$ or $|z - b| \ge \rho$.

C2 The components $\frac{m_i}{r}$ of u satisify $\Delta(m_i/r) = 0$. Also $\nabla \cdot \frac{m}{r} = \frac{1}{r}m \cdot x$. The identity then shows the result.

C3 (c) $|f_B(x)| \le \frac{C}{|x|-R}\mathrm{Vol}(B(0, R)) \to 0$ as $|x| \to \infty$. (d) $A = \lim_{|x|\to\infty} -C\int_{B(0,R)} \frac{|x|}{|x-y|}\,\mathrm{d}V(y)$. Now take the limit inside the integral.

C4 (a) Recall that $\overline{zw} = \bar{z}\bar{w}$ so that $\overline{z^n} = \bar{z}^n$. Since the coefficients are all real we have $0 = \overline{p(z)} = p(\bar{z})$. (c) Multiply out $(z - z_j)(z - \bar{z}_j)$ when $z_j = a_j + ib_j$.